Quantum (Un)complexity

A Resource for Quantum Computation

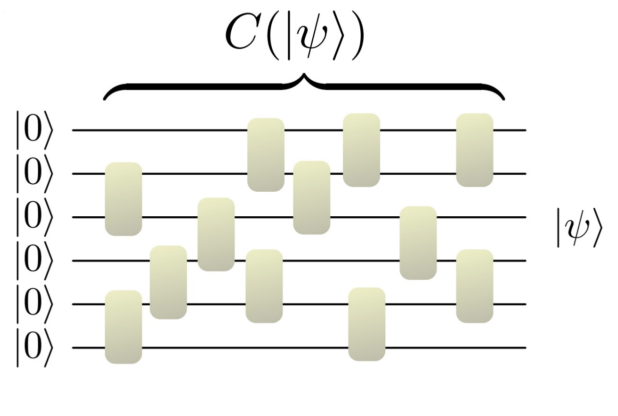
Anthony Munson

University of Maryland, College Park

What is quantum complexity?

Quantum complexity quantifies the difficulty of preparing the state from a simple, tensor-product state, e.g., the n-qubit all-zero state $|0^n\rangle$.

Example: on n qubits, the *exact unitary complexity* of a state $|\psi\rangle$ is the minimal number of 2-qubit gates in a circuit that implements a unitary U, with $|\psi\rangle = U|0^n\rangle$.



We denote the quantum complexity of a state $|\psi\rangle$ by $\mathcal{C}(|\psi\rangle)$.

Why quantum complexity?

Quantum information Complexity quantifies the difficulty of discriminating states and preparing superpositions [1-3].

Condensed matter Complexities that scale linearly with system size distinguish topological phases [4,5].

High-energy physics Conjecture in AdS/CFT: the complexity of the field-theoretic state dual to a wormhole connecting two black holes is proportional to the wormhole's length [6-11].

What is quantum uncomplexity?

- An n-qubit state ρ has maximal complexity $\mathcal{C}_{\max} \sim e^n$ [12].

- The uncomplexity of ρ is the difference between the state's complexity and the maximal complexity: $\mathcal{C}_{\max} - \mathcal{C}(\rho)$.

Uncomplexity as a resource

Complexity growth in systems with random dynamics [13]

Useful states in quantum computation are “blank” qubits, just as blank paper is useful in pencil writing

Conjecture that uncomplexity can be formally understood as a resource in quantum computation (Brown & Susskind) [9]

Resource theory of quantum uncomplexity

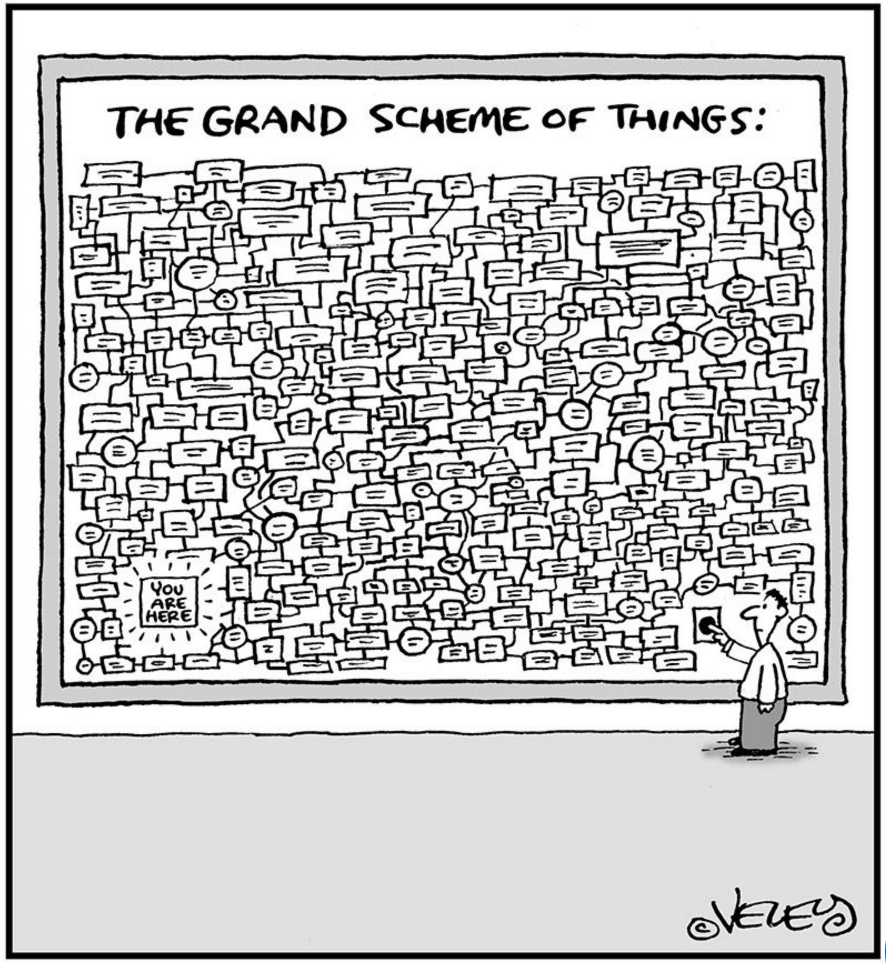
Nicole Yunger Halpern,[1,2,3,4,*] Naga B. T. Kothakonda,[5,6]
Jonas Haferkamp,[5,7] Anthony Munson,[1] Jens Eisert,[5,7] and Philippe Faist[5]

[1] *Joint Center for Quantum Information and Computer Science, NIST and University of Maryland, College Park, MD 20742, USA*
[2] *Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA*
[3] *ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138, USA*
[4] *Department of Physics, Harvard University, Cambridge, MA 02138, USA*
[5] *Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*
[6] *Institute for Theoretical Physics, University of Cologne, D-50937 Cologne, Germany*
[7] *Helmholtz-Zentrum Berlin für Materialien und Energie, 14109 Berlin, Germany*

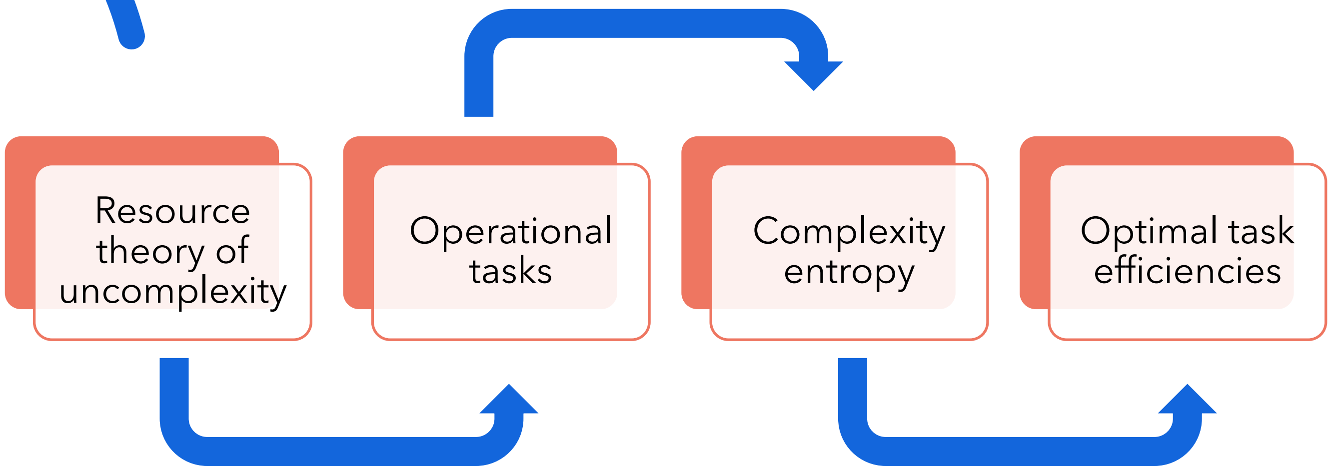
(Dated: October 26, 2021)

Quantum complexity is emerging as a key property of many-body systems, including black holes, topological materials, and early quantum computers. A state's complexity quantifies the number of computational gates required to prepare the state from a simple tensor product. The greater a state's distance from maximal complexity, or "uncomplexity," the more useful the state is as input to a quantum computation. Separately, resource theories—simple models for agents subject to constraints—are burgeoning in quantum information theory. We unite the two domains, confirming Brown and Susskind's conjecture that a resource theory of uncomplexity can be defined. The allowed operations, *fuzzy operations*, are slightly random implementations of two-qubit gates chosen by an agent. We formalize two operational tasks, uncomplexity extraction and expenditure. Their optimal efficiencies depend on an entropy that we engineer to reflect complexity. We also present two monotones, uncomplexity measures that decline monotonically under fuzzy operations, in certain regimes. This work unleashes on many-body complexity the resource-theory toolkit from quantum information theory.

arXiv:2110.11371

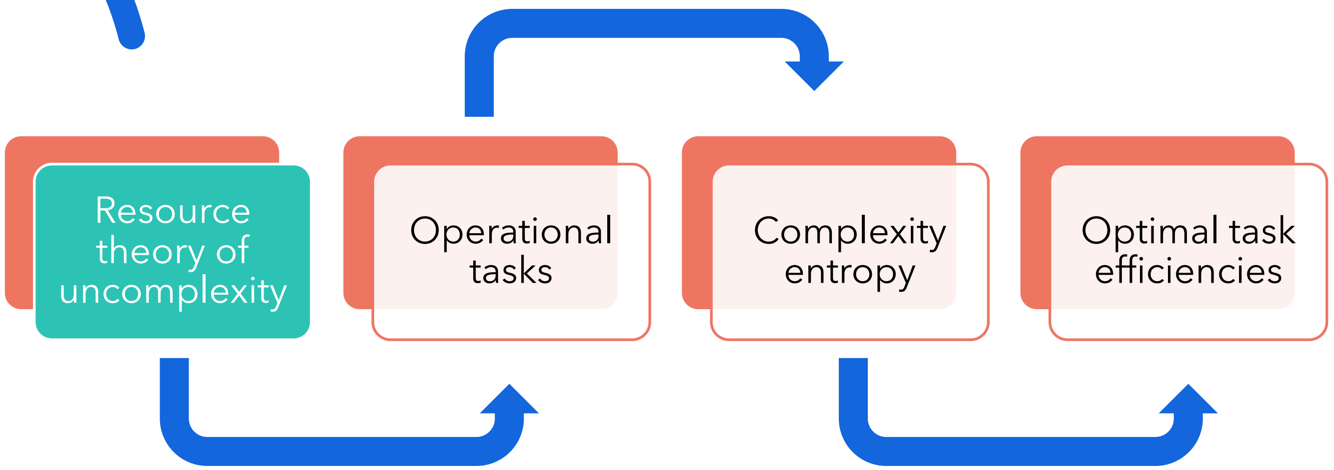

THE GRAND SCHEME OF THINGS:
YOU ARE HERE

Overview


Resource theory of uncomplexity
Operational tasks
Complexity entropy
Optimal task efficiencies


WE ARE BEING CONTROLLED BY THE RANDOM OUTCOMES OF A COMPLEX SYSTEM
Samuel

Resource theory of uncomplexity



What is a resource theory?

- An agent can perform any chosen operation that satisfies simple rules
- States difficult to prepare are scarce resources, which may facilitate operational tasks
- A theory is defined by its allowed operations on a set of states

Examples of resource theories

Resource theory of entanglement [14]

- Free states: separable states
- Free operations: local quantum operations and classical communication (LOCC)

Resource theory of athermality [15]

- States: pairs of density matrices and time-independent Hamiltonians
- Free states: thermal equilibrium states (Gibbs states)
- Free operations: processes that conserve total energy under system-bath heat exchanges

Resource theory of uncomplexity

A *fuzzy gate* is a gate $\tilde{U}$ implemented w.r.t. a probability distribution $p_{U,\epsilon}(\tilde{U})$ vanishing outside of the ϵ–ball of a desired gate U, where $\epsilon \geq 0$

- Distance between gates given by operator norm: $||\tilde{U} - U||_\infty \leq \epsilon$
- Physical interpretation: model of noise

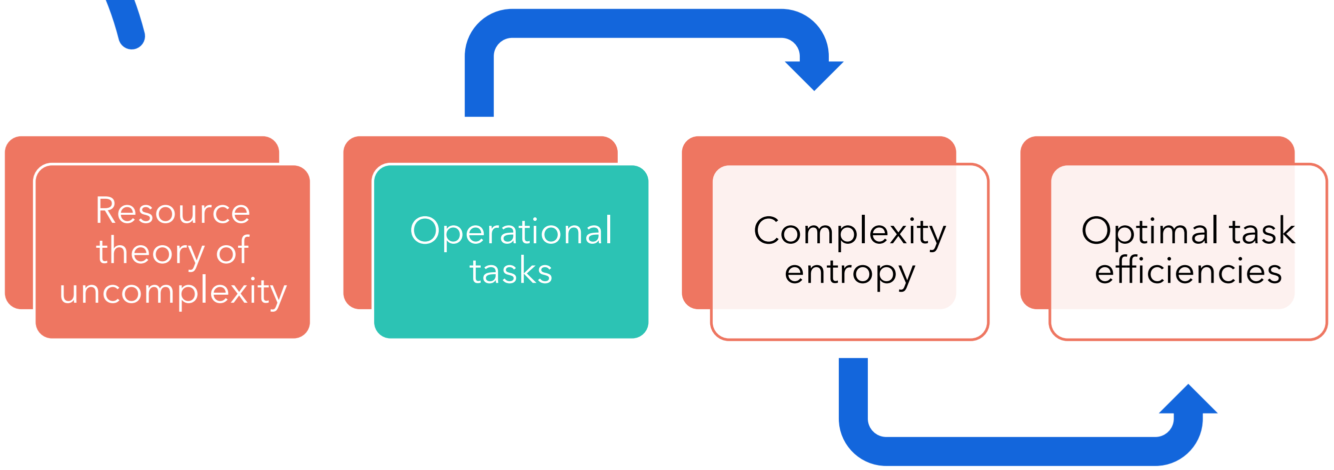
Allowed operations are *fuzzy operations*: compositions of fuzzy gates

No free states!

- Maximally complex $(n+m)$-qubit state has complexity $\sim e^{n+m}$, but tensoring together maximally complex n- and m-qubit states only gives complexity $\sim e^n + e^m$
- Therefore tensoring-on creates uncomplexity! [9,12, 16]


HOW ORIGAMI COMPLEXITY IS MEASURED
I STARTED WITH SIMPLE ORIGAMI PIECES. THE ONES I MAKE NOW REQUIRE MORE STEPS.
IT LOOKS LIKE A TENFOLD INCREASE!
©2020 Thaves/Dist. by Andrews McMeel Synd.
THAVES 9-26

Operational tasks



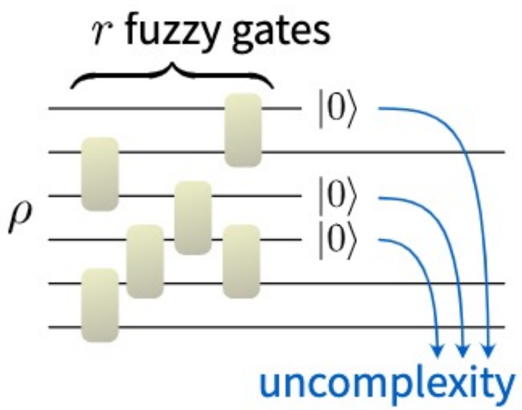
A tale of two tasks

Uncomplexity extraction

Uncomplexity expenditure

Uncomplexity extraction

Procedure
- Apply to ρ a circuit of $\leq r$ fuzzy gates
- Select some number w of the qubits



Task: Perform the above so that the selected qubits are δ-close to $|0^w\rangle$ in trace distance

Uncomplexity expenditure: setup

Let M_0 and M_r be the sets of 0- and r-complexity measurement operators, respectively:

$$M_0 := \{ \bigotimes_{j=1}^{n} ({}_j|0\rangle\langle 0|_j)^{\alpha_j} : \alpha_j = 0, 1 \}$$

$$M_r := \{ U_r^\dagger Q_0 U_r : Q_0 \in M_0 \}$$

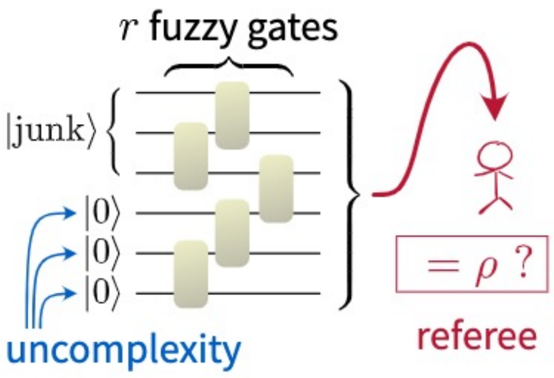
Setup:

- Computationally limited referee wants to distinguish ρ and $\mathbb{1}^{\otimes n}/2^n$ with $Q \in M_r$, guessing ρ with probability $\geq \eta$
- You, the agent, know Q and seek to fool the referee with a simulacrum $\tilde{\rho}$

Uncomplexity expenditure

Procedure:

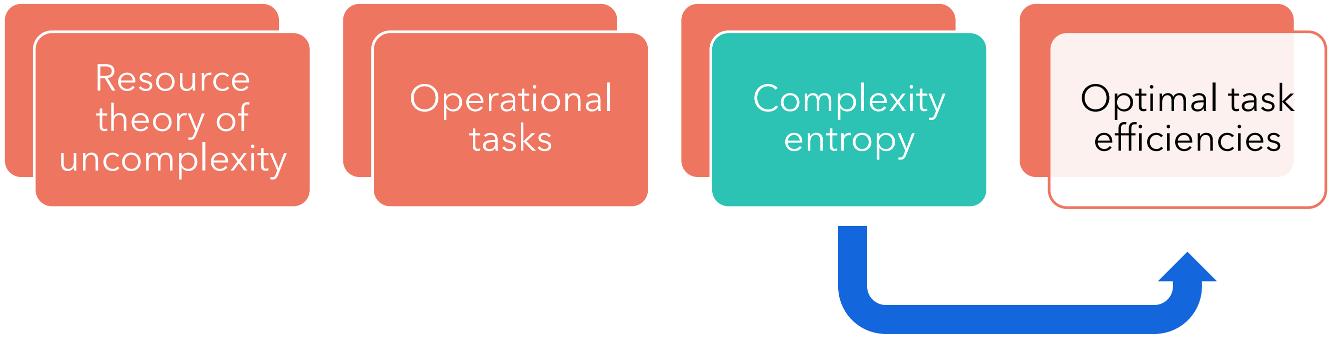
- Borrow w uncomplex $|0\rangle$'s from an "uncomplexity bank", along with an unknown $(n-w)$-qubit state
- Apply $\leq r$ gates to the joint state and yield $\tilde{\rho}$



Task: Have the referee, upon receiving $\tilde{\rho}$, guess ρ, with probability $\geq \eta$


1967
DEPT. OF ENTROPY

Complexity entropy



Motivation for complexity entropy

- Entropies are used to bound the efficiencies of operational tasks, e.g., Shannon entropy for data compression
- We want to quantify how uncertain a state *looks* to a computationally limited observer

Definition of complexity entropy

The *complexity entropy* of an n-qubit state ρ is, for $\eta \in (0, 1]$,

$$H_{\mathrm{c}}^{r,\eta}(\rho) := \min_{\substack{Q \in M_r, \\ \mathrm{Tr}(Q\rho) \geq \eta}} \left\{ \log_2 \left(\mathrm{Tr}(Q) \right) \right\}.$$

- Q is constrained to have complexity $\leq r$
- Type-I error: Q must successfully identify ρ with probability $\geq \eta$
- $H_{\mathrm{c}}^{r,\eta}$ gives the minimal possible uncertainty due to Type-II error

Intuition for complexity entropy

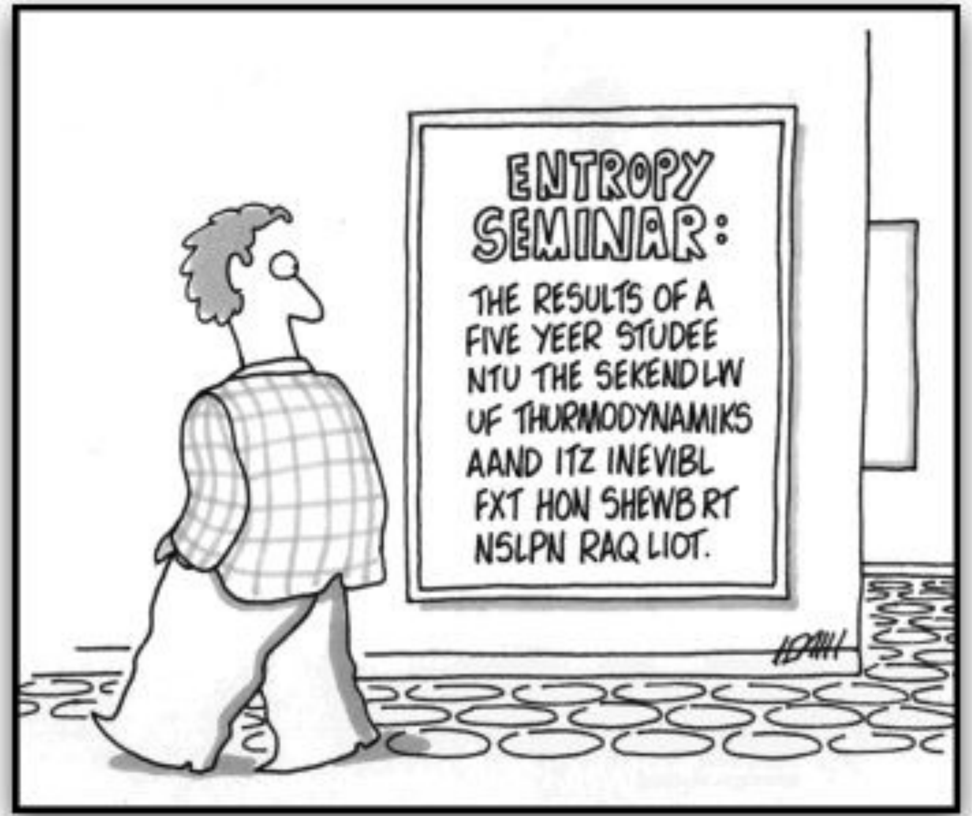
Limiting cases

- A low-complexity state, e.g., $\rho = |0^n\rangle\langle 0^n|$, may satisfy $\mathrm{Tr}(Q\rho) \geq \eta$ for some performable $Q = U_r|0^n\rangle\langle 0^n|U_r^\dagger$ and will yield $H_\mathrm{c}^{r,\eta}(\rho) = \log_2\left(\mathrm{Tr}(Q)\right) = \log_2(1) = 0$.
- A high-complexity state may only satisfy $\mathrm{Tr}(Q\rho) \geq \eta$ for some performable $Q = U_r \mathbb{1}^{\otimes n} U_r^\dagger = \mathbb{1}^{\otimes n}$ and will yield $H_\mathrm{c}^{r,\eta}(\rho) = \log_2\left(\mathrm{Tr}(Q)\right) = \log_2(2^n) = n$.

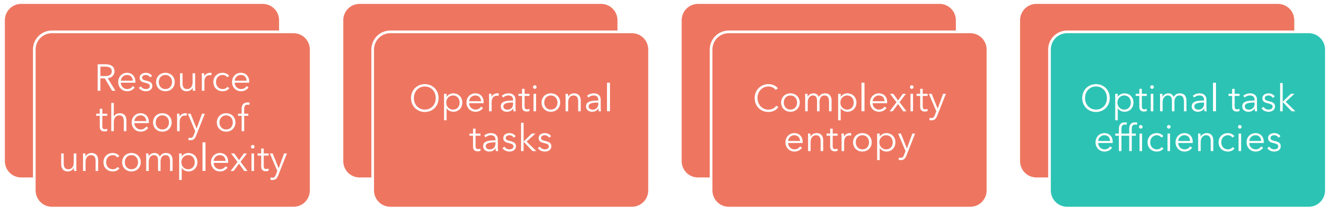
Relation to hypothesis-testing entropy

$$H_\mathrm{h}^\eta(\rho) := \min_{\substack{0 \leq Q \leq \mathbb{1}, \\ \mathrm{Tr}(Q\rho) \geq \eta}} \left\{ \log_2\left(\mathrm{Tr}(Q)\right)\right\}$$

- The hypothesis-testing entropy quantifies the uncertainty in a hypothesis test between ρ and $\mathbb{1}^{\otimes n}/2^n$.
- Like the complexity entropy but lacks computational restrictions


ENTROPY
SEMINAR:
THE RESULTS OF A
FIVE YEER STUDEE
NTU THE SEKEND LW
UF THURMODYNAMIKS
AAND ITZ INEVIBL
FXT HON SHEWB RT
NSLPN RAQ LIOT.

Optimal task efficiencies



Two theorems, one entropy

Uncomplexity extraction

Uncomplexity expenditure

Each theorem establishes for one of the two tasks

- the existence of a protocol achieving the task
- the near-optimality of the protocol

Theorem 1:
Uncomplexity Extraction

Let ρ denote any n-qubit state, $r \in \mathbb{Z}_{\geq 0}$, and $\delta \geq 0$. Assume $\delta \geq r\epsilon$. For every $\eta \in [1-(\delta - r\epsilon)^2, 1]$, some protocol extracts $w = n - H_c^{r,\eta}(\rho)$ qubits δ-close to $|0^w\rangle$ in trace distance.

Conversely, every uncomplexity-extraction protocol obeys $w \leq n - H_c^{r,1-\delta}(\rho)$.

Low-complexity limit: some protocol extracts $w = n$ qubits, others extract $w \leq n$ qubits.

High-complexity limit: all protocols extract $w = 0$ qubits.

$$H_c^{r,\eta}(\rho) := \min_{\substack{Q \in M_r, \\ \mathrm{Tr}(Q\rho) \geq \eta}} \left\{ \log_2 \left(\mathrm{Tr}(Q) \right) \right\}$$

Theorem 2:
Uncomplexity Expenditure

Let ρ denote an arbitrary n-qubit state. Let $r \in \mathbb{Z}_{\geq 0}$ and $\delta \geq 0$, and assume that $\delta \geq 2r\epsilon$. For every $\eta \in (0, 1]$, and for every $(n - w)$-qubit state σ, ρ can be imitated with $w = n - H_{\text{c}}^{r,\eta}(\rho)$ uncomplex $|0\rangle$'s.

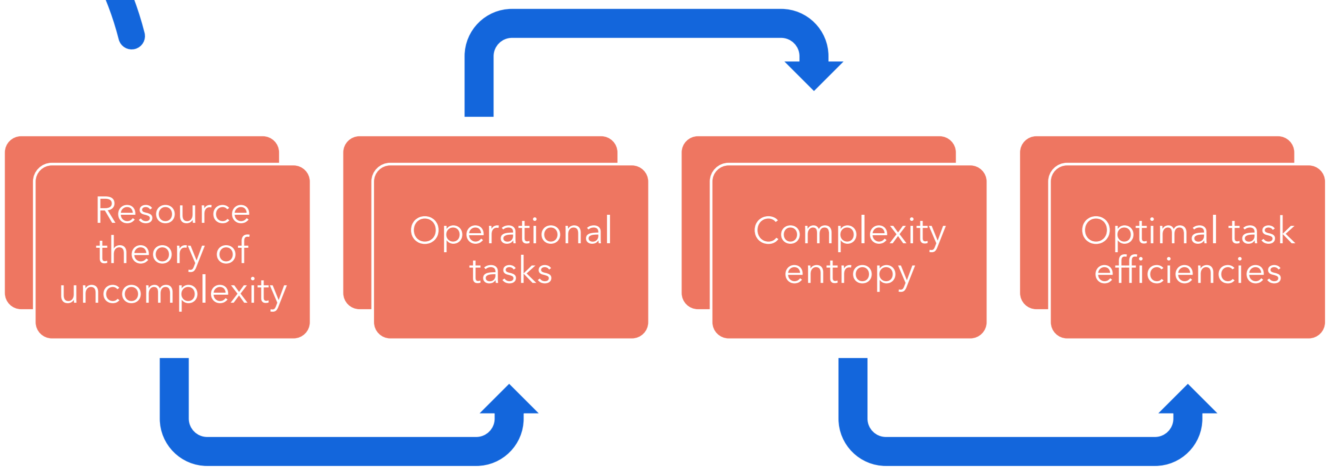
$$H_{\text{c}}^{r,\eta}(\rho) := \min_{\substack{Q \in M_r, \\ \text{Tr}(Q\rho) \geq \eta}} \left\{ \log_2 \left(\text{Tr}(Q) \right) \right\}$$

Low-complexity limit: ρ can be imitated with $w = n$ qubits.

High-complexity limit: ρ can be imitated with $w = 0$ qubits.


WHY IS IT ALWAYS "THEY'RE SO CUTE," AND NEVER "THEY'RE SO COMPLEX"?

Summary


Resource theory of uncomplexity
Operational tasks
Complexity entropy
Optimal task efficiencies

Future research

- Determine properties and applications of the complexity entropy
- Describe "phases" of uncomplexity extraction
- Explore connections to black hole physics

Thank you!

Resource theory of quantum uncomplexity

Nicole Yunger Halpern,[1,2,3,4,*] Naga B. T. Kothakonda,[5,6]
Jonas Haferkamp,[5,7] Anthony Munson,[1] Jens Eisert,[5,7] and Philippe Faist[5]

[1] *Joint Center for Quantum Information and Computer Science, NIST and University of Maryland, College Park, MD 20742, USA*
[2] *Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA*
[3] *ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138, USA*
[4] *Department of Physics, Harvard University, Cambridge, MA 02138, USA*
[5] *Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*
[6] *Institute for Theoretical Physics, University of Cologne, D-50937 Cologne, Germany*
[7] *Helmholtz-Zentrum Berlin für Materialien und Energie, 14109 Berlin, Germany*
(Dated: October 26, 2021)

Quantum complexity is emerging as a key property of many-body systems, including black holes, topological materials, and early quantum computers. A state's complexity quantifies the number of computational gates required to prepare the state from a simple tensor product. The greater a state's distance from maximal complexity, or "uncomplexity," the more useful the state is as input to a quantum computation. Separately, resource theories—simple models for agents subject to constraints—are burgeoning in quantum information theory. We unite the two domains, confirming Brown and Susskind's conjecture that a resource theory of uncomplexity can be defined. The allowed operations, *fuzzy operations*, are slightly random implementations of two-qubit gates chosen by an agent. We formalize two operational tasks, uncomplexity extraction and expenditure. Their optimal efficiencies depend on an entropy that we engineer to reflect complexity. We also present two monotones, uncomplexity measures that decline monotonically under fuzzy operations, in certain regimes. This work unleashes on many-body complexity the resource-theory toolkit from quantum information theory.

arXiv:2110.11371

References

[1] S. Aaronson, Y. Atia, and L. Susskind, arXiv e-prints , arXiv:2009.07450 (2020), 2009.07450.

[2] D. Girolami and F. Anzà, Phys. Rev. Lett. 126, 170502 (2021).

[3] D. Girolami, Phys. Rev. Lett. 122, 010505 (2019).

[4] M. Schwarz, K. Temme, F. Verstraete, D. Perez-Garcia, and T. S. Cubitt, Phys. Rev. A 88, 032321 (2013).

[5] M. B. Hastings, Phys. Rev. Lett. 107, 210501 (2011).

[6] L. Susskind, Fort. Phys. 64, 24 (2016).

[7] D. Stanford and L. Susskind, Phys. Rev. D 90, 126007 (2014).

[8] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, Phys. Rev. D 93, 086006 (2016).

References

[9] A. R. Brown and L. Susskind, Phys. Rev. D 97, 086015 (2018).

[10] A. Bouland, B. Fefferman, and U. Vazirani, arXiv:1910.14646 (2019).

[11] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, Phys. Rev. Lett. 116, 191301 (2016).

[12] L. Susskind, arXiv e-prints , arXiv:1810.11563 (2018), 1810.11563.

[13] J. Haferkamp, P. Faist, N. B. T. Kothakonda, J. Eisert, and N. Yunger Halpern, arXiv e-prints, arXiv:2106.05305 (2021), 2106.05305.

[14] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).

[15] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Phys. Rev. Lett. 111, 250404 (2013).

[16] E. Knill and R. Laflamme, Phys. Rev. Lett. 81, 5672 (1998).