Everything Everywhere All at Once:

Holographic Entropy Cone, Entanglement Wedge Nesting, Differential Entropy, Black Holes, Modular Chern Numbers, Toric Code, Cubohemioctahedron…

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Known Entropy Inequalities

Table 1: Representatives for each of the 8 inequality orbits of the holographic entropy cone $\mathcal{C}_5$ for 5 regions. Respectively, their orbit lengths are 15, 20, 45, 72, 10, 60, 60 and 90, thus defining 372 facets for $\mathcal{C}_5$ in a 31-dimensional entropy space.

Cuenca, 2019

1. $S_A + S_B \geq S_{AB}$	2. $S_{AB} + S_{AC} + S_{BC} \geq S_A + S_B + S_C + S_{ABC}$
3. $S_{ABC} + S_{ADE} + S_{BCDE} \geq S_A + S_{BC} + S_{DE} + S_{ABCDE}$	4. $S_{ABC} + S_{ABD} + S_{ACE} + S_{BDE} + S_{CDE} \geq S_{AB} + S_{AC} + S_{BD} + S_{CE} + S_{DE} + S_{ABCDE}$
5. $S_{ABC} + S_{ABD} + S_{ABE} + S_{ACD} + S_{ACE} + S_{ADE} + S_{BCE} + S_{BDE} + S_{CDE} \geq S_{AB} + S_{AC} + S_{AD} + S_{BE} + S_{CE} + S_{DE} + S_{BCD} + S_{ABCE} + S_{ABDE} + S_{ACDE}$	6. $3S_{ABC} + 3S_{ABD} + S_{ABE} + S_{ACD} + 3S_{ACE} + S_{ADE} + S_{BCD} + S_{BCE} + S_{BDE} + S_{CDE} \geq 2S_{AB} + 2S_{AC} + S_{AD} + S_{AE} + S_{BC} + 2S_{BD} + 2S_{CE} + S_{DE} + 2S_{ABCD} + 2S_{ABCE} + S_{ABDE} + S_{ACDE}$
7. $2S_{ABC} + S_{ABD} + S_{ABE} + S_{ACD} + S_{ADE} + S_{BCE} + S_{BDE} \geq S_{AB} + S_{AC} + S_{AD} + S_{BC} + S_{BE} + S_{DE} + S_{ABCD} + S_{ABCE} + S_{ABDE}$	8. $S_{AD} + S_{BC} + S_{ABE} + S_{ACE} + S_{ADE} + S_{BDE} + S_{CDE} \geq S_A + S_B + S_C + S_D + S_{AE} + S_{DE} + S_{BCE} + S_{ABDE} + S_{ACDE}$

$$
\begin{aligned}
&S_{ABDE} + S_{ABDF} + S_{ABEG} + S_{ADEF} + S_{ADEG} \\
&+ S_{ACDE} + S_{ACDF} + S_{ACEG} + S_{BDEF} + S_{BDEG} \\
&+ S_{BCDE} + S_{BCDF} + S_{BCEG} + S_{CDEF} + S_{CDEG} \\
&\geq \\
&S_{ABC} + S_{ADE} + S_{ADF} + S_{AEG} + S_{BDE} + S_{BDF} + S_{BEG} + S_{CDE} + S_{CDF} + S_{CEG} \\
&+ S_{ABDEF} + S_{ABDEG} + S_{ACDEF} + S_{ACDEG} + S_{BCDEF} + S_{BCDEG}
\end{aligned}
\tag{2.1}
$$

BC and Wang, 2022

New Inequalities

Toric inequalities are defined for m and n, which are both odd. They take the following form:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^+ B_j^-} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} S_{A_i^- B_j^-} + S_{A_1^{(m)}} \tag{1.6}$$

We characterize and explore these inequalities in Section 2.2, then prove them in Section 5.1. We exemplified how terms of (1.6) can be arranged on a discretized torus in inequality (1.3). As we explain in Section 2.2.1, that spatial arrangement has further, even more compelling features.

Projective plane inequalities are defined for $m = n$. They read:

$$\frac{1}{2}\sum_{k=1}^{m}\sum_{i=1}^{m}\left(S_{A_i^{(k)} B_{i+k}^{(m-k)}} + S_{A_i^{(k)} B_{i+k+1}^{(m-k)}}\right) \geq \sum_{k=1}^{m}\sum_{i=1}^{m} S_{A_i^{(k-1)} B_{i+k}^{(m-k)}} + S_{A_1^{(m)}} \tag{1.7}$$

Notation:

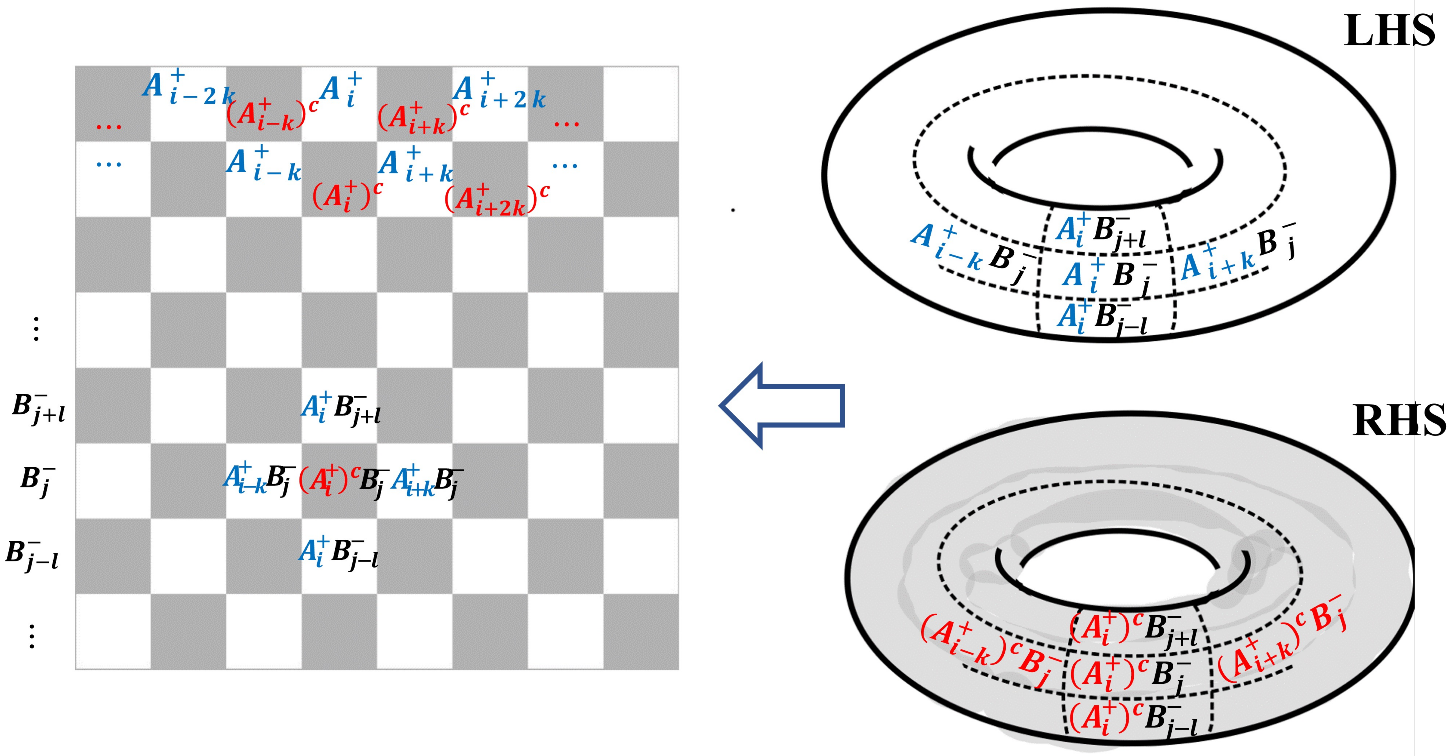
$$A_i^{(k)} = A_i A_{i+1} \dots A_{i+k-1} \quad \text{and} \quad B_j^{(l)} = B_j B_{j+1} \dots B_{j+l-1}$$

$$A_i^{\pm} \equiv A_i^{((m\pm 1)/2)} \quad \text{and} \quad B_j^{\pm} \equiv B_j^{((n\pm 1)/2)}$$

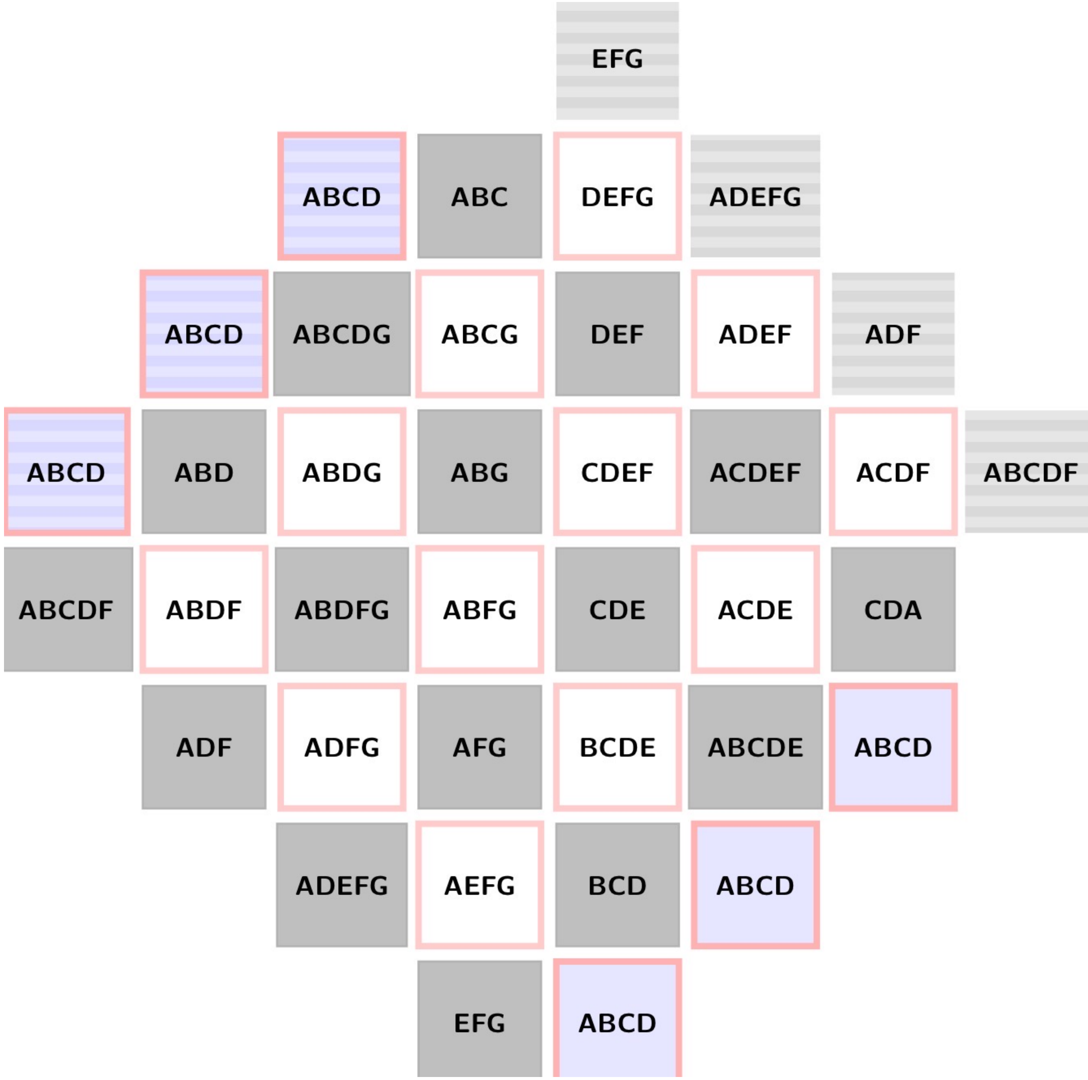
To give you a feeling:

$$
\begin{aligned}
& S_{A_1A_2A_3B_1} + S_{A_3A_4A_5B_1} + S_{A_5A_1A_2B_1} + S_{A_2A_3A_4B_1} + S_{A_4A_5A_1B_1} \\
+ & S_{A_1A_2A_3B_2} + S_{A_3A_4A_5B_2} + S_{A_5A_1A_2B_2} + S_{A_2A_3A_4B_2} + S_{A_4A_5A_1B_2} \\
+ & S_{A_1A_2A_3B_3} + S_{A_3A_4A_5B_3} + S_{A_5A_1A_2B_3} + S_{A_2A_3A_4B_3} + S_{A_4A_5A_1B_3} \\
& \geq \\
& S_{A_4A_5B_1} + S_{A_1A_2B_1} + S_{A_3A_4B_1} + S_{A_5A_1B_1} + S_{A_2A_3B_1} \\
+ & S_{A_4A_5B_2} + S_{A_1A_2B_2} + S_{A_3A_4B_2} + S_{A_5A_1B_2} + S_{A_2A_3B_2} \\
+ & S_{A_4A_5B_3} + S_{A_1A_2B_3} + S_{A_3A_4B_3} + S_{A_5A_1B_3} + S_{A_2A_3B_3} \\
& + S_{A_1A_2A_3A_4A_5}
\end{aligned}
\tag{2.9}
$$

The inequalities live on two-dimensional manifolds

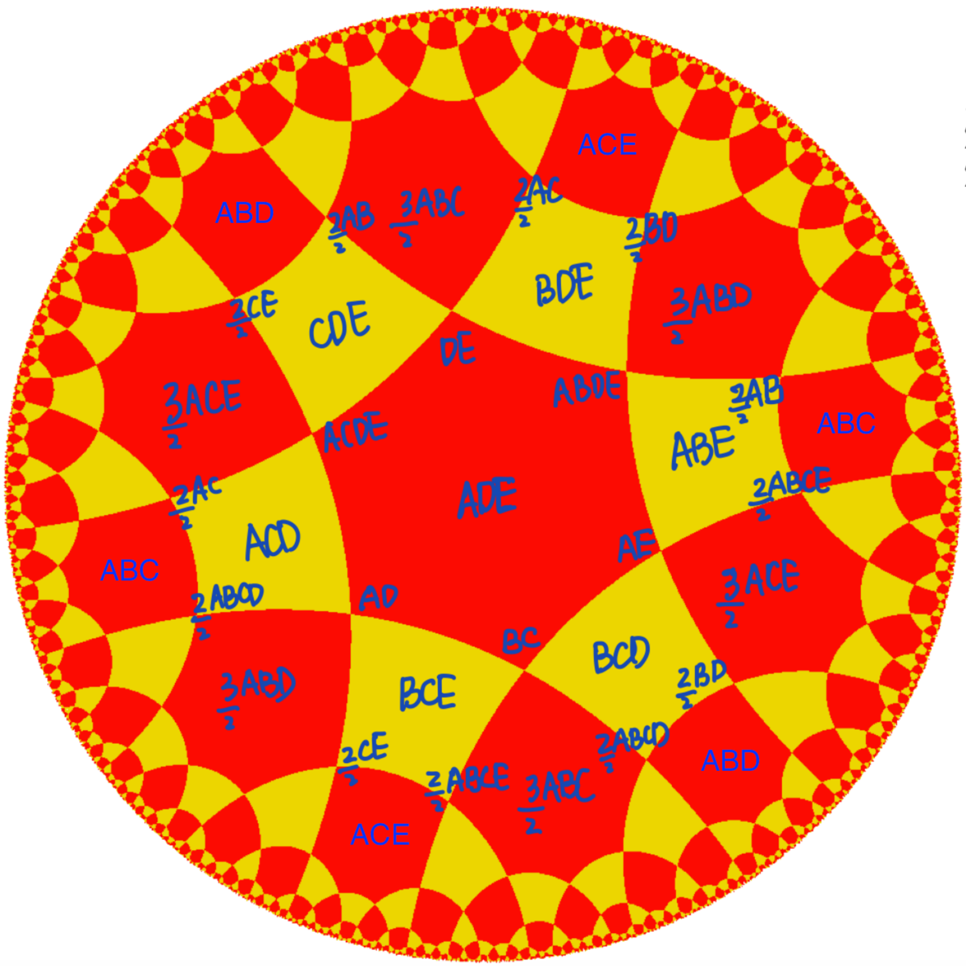


This inequality lives on the projective plane:



- Neighboring relations given by <u>entanglement wedge nesting</u>
- Neighboring four terms form a <u>discretized second derivative</u>

One last inequality: cubohemioctahedron



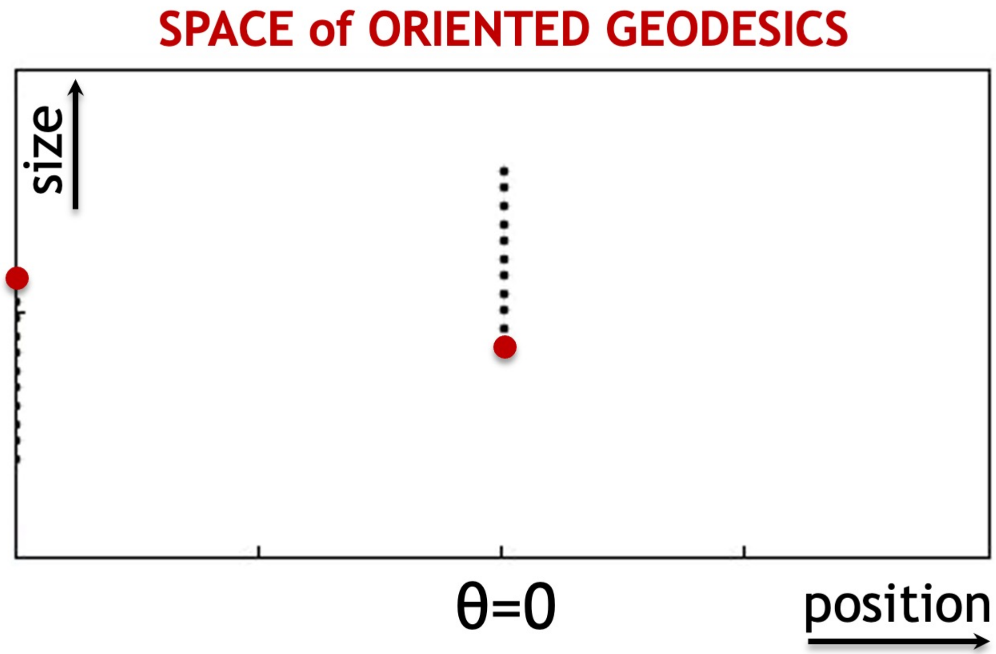
6. $3S_{ABC} + 3S_{ABD} + S_{ABE} + S_{ACD} + 3S_{ACE} + S_{ADE} + S_{BCD} + S_{BCE} + S_{BDE} + S_{CDE} \geq 2S_{AB} + 2S_{AC} + S_{AD} + S_{AE} + S_{BC} + 2S_{BD} + 2S_{CE} + S_{DE} + 2S_{ABCD} + 2S_{ABCE} + S_{ABDE} + S_{ACDE}$

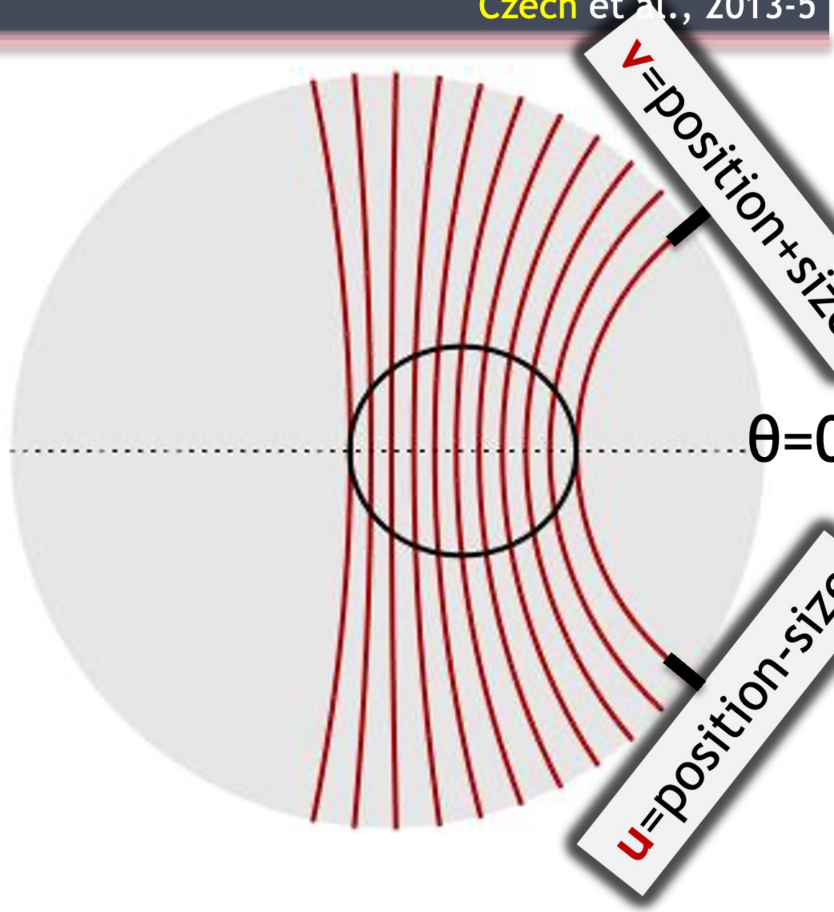
These two-dimensional manifolds are kinematic spaces

- What does that mean?
- A brief review…

How to describe a center of AdS_3?

Czech et al., 2013-5

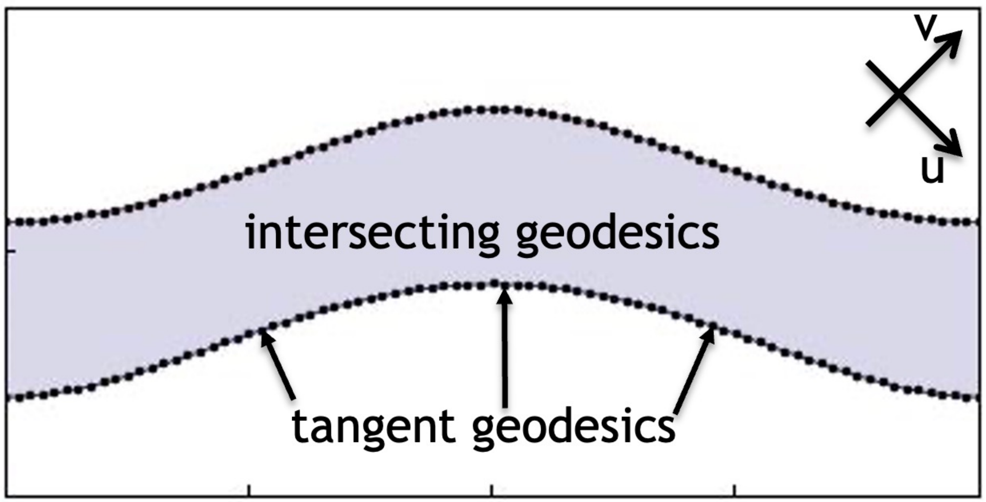


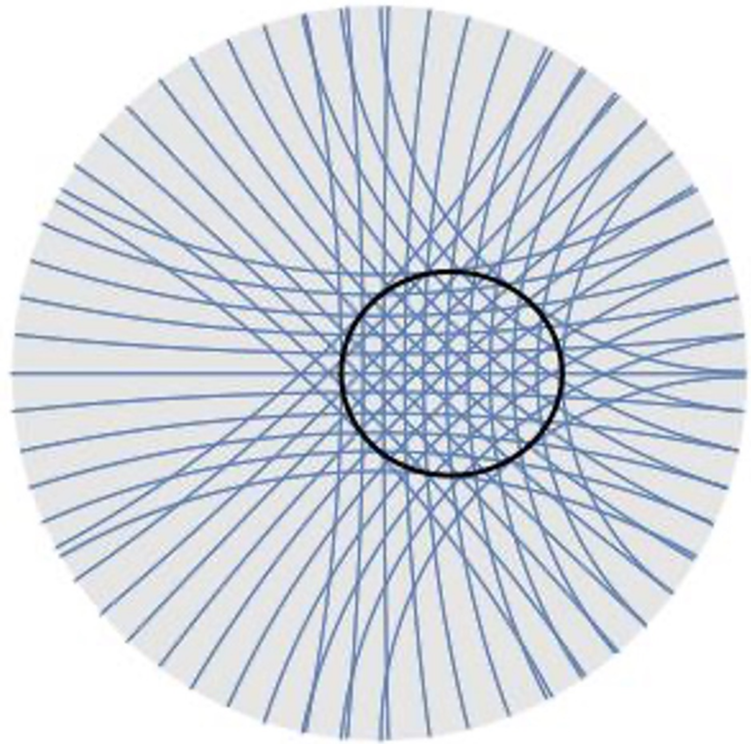


How to describe a center of AdS_3?

Czech et al., 2013-5

SPACE of ORIENTED GEODESICS





$$\frac{\text{circumference}}{4G} = \int_{\text{intersect}} \frac{\partial^2 S(u,v)}{\partial u\, \partial v}\, du\, dv$$

- All the new inequalities have the schematic form:

$$\int_{\text{intersect}} \frac{\partial^2 S(u,v)}{\partial u\, \partial v}\, du\, dv > \text{S(all the A-regions)}$$

New inequalities:

Length of some curve
in a two-sided black hole background

>

Entropy of the black hole

- LHS can also be understood as the Chern number of an "entanglement Berry parameter space"

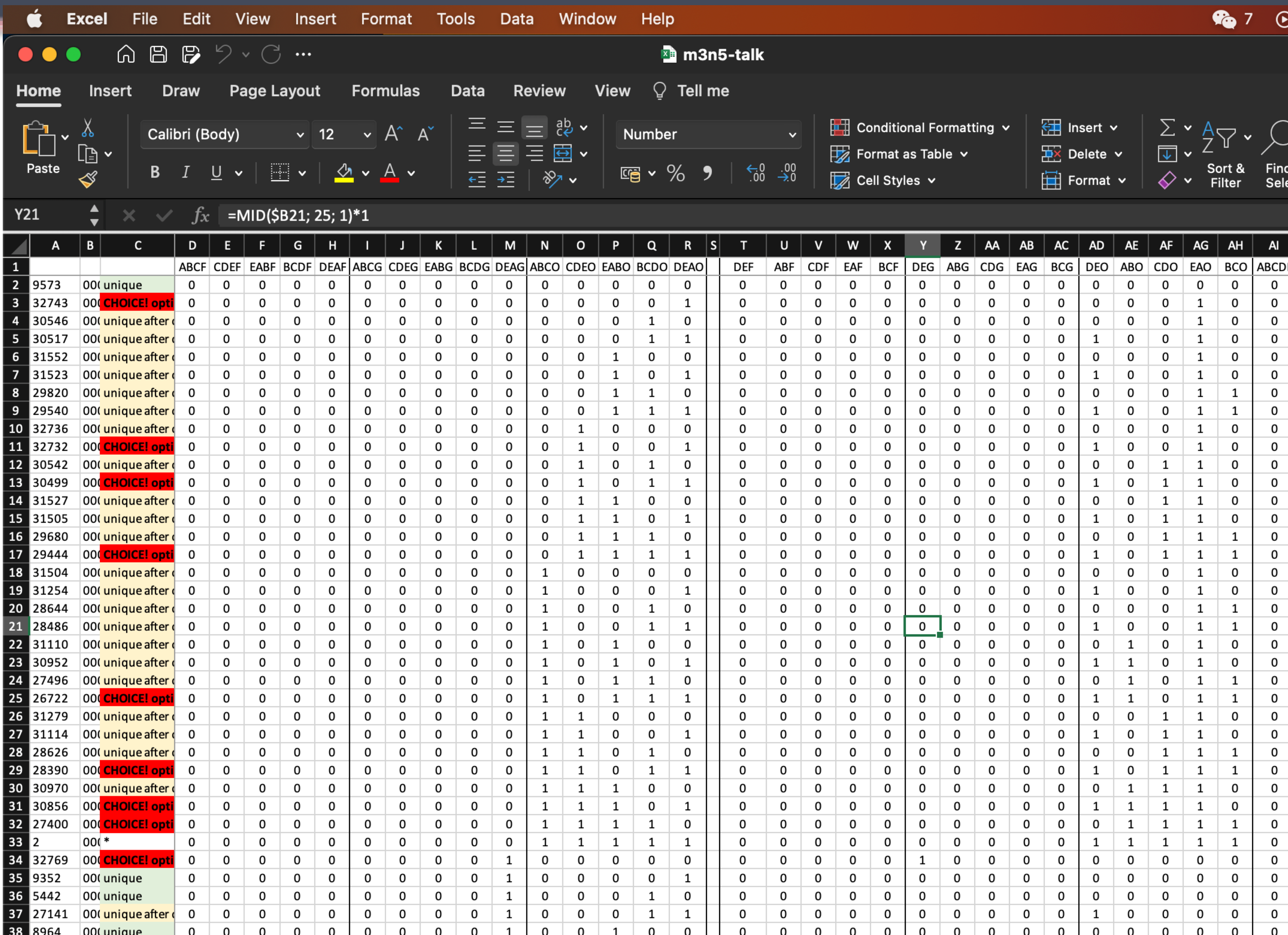
Why toric code?

- Proof involves constructing a table for LHS terms and RHS terms:

	x			$y = f(x)$			
	AB	BC	AC	A	B	C	ABC
O	0	0	0	0	0	0	0
	0	0	1	0	0	0	1
	0	1	0	0	0	0	1
C	0	1	1	0	0	1	1
	1	0	0	0	0	0	1
A	1	0	1	1	0	0	1
B	1	1	0	0	1	0	1
	1	1	1	0	0	0	1

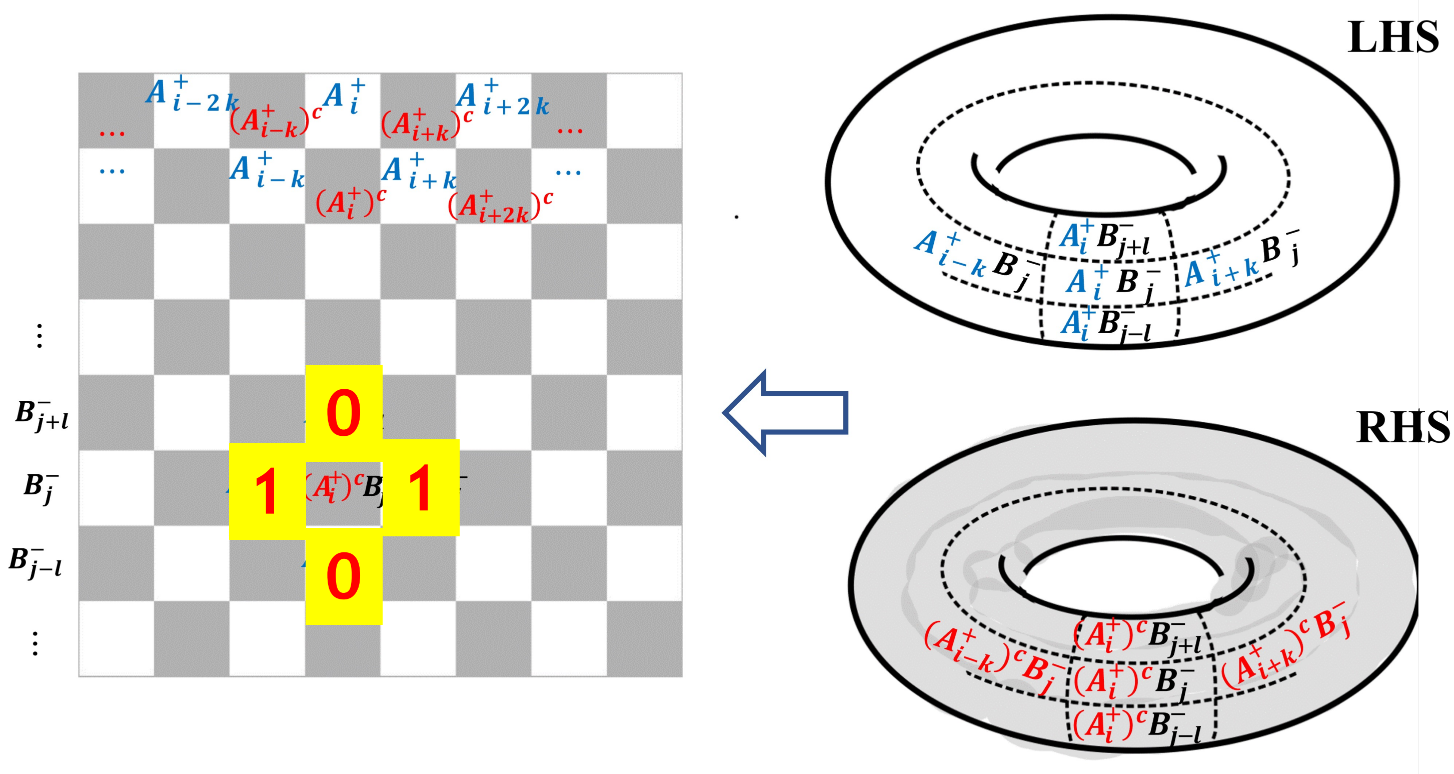
Table 2. Proof by contraction of monogamy of the holographic mutual information.

Another example of a “contraction map”



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI
1				ABCF	CDEF	EABF	BCDF	DEAF	ABCG	CDEG	EABG	BCDG	DEAG	ABCO	CDEO	EABO	BCDO	DEAO		DEF	ABF	CDF	EAF	BCF	DEG	ABG	CDG	EAG	BCG	DEO	ABO	CDO	EAO	BCO	ABCDE
2	9573	000	unique	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	32743	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1		0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
4	30546	000	unique after	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
5	30517	000	unique after	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0
6	31552	000	unique after	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
7	31523	000	unique after	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0
8	29820	000	unique after	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
9	29540	000	unique after	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0
10	32736	000	unique after	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
11	32732	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0
12	30542	000	unique after	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0		0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
13	30499	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1		0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0
14	31527	000	unique after	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0		0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
15	31505	000	unique after	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1		0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0
16	29680	000	unique after	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0		0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0
17	29444	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1		0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	0
18	31504	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
19	31254	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0
20	28644	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0		0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
21	28486	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1		0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0
22	31110	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0		0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
23	30952	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1		0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0
24	27496	000	unique after	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0		0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0
25	26722	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1		0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	0
26	31279	000	unique after	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
27	31114	000	unique after	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1		0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0
28	28626	000	unique after	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0		0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0
29	28390	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1		0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	0
30	30970	000	unique after	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0		0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0
31	30856	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1		0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0
32	27400	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0		0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0
33	2	000	*	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1		0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0
34	32769	000	CHOICE! opti	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0		0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
35	9352	000	unique	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	5442	000	unique	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	27141	000	unique after	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1		0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
38	8964	000	unique	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The assignment of 0’s and 1’s is based on the existence of certain loops on the two-d manifold



- This is just like local excitations and “logical bits” in the toric code. Still trying to understand why.
- Probably related to the toric code being a stabilizer state… The inequalities probably count stabilizer operators…

Everything Everywhere All at Once:

- Two new infinite families of holographic entropy inequalities.
- They naturally live on (discretized) two-dimensional manifolds. One of the resulting polytopes is the cubohemioctahedron.
- The spatial organization on the 2d manifolds is dictated by entanglement wedge nesting.
- These manifolds are kinematic spaces or modular Berry parameter spaces.
- The inequalities motivate a new type of differential entropy.
- Schematically, they read:
  "length of a curve" > "black hole horizon"
  "modular Chern number" > "black hole horizon"
- The proof involves manipulations, which are familiar from the toric code. This probably reflects properties of stabilizer states.

THANK YOU!