Banff International Research Station
for Mathematical Innovation and Discovery

Constructing lattice integrable models from topological theories in one higher dimensions and their applications

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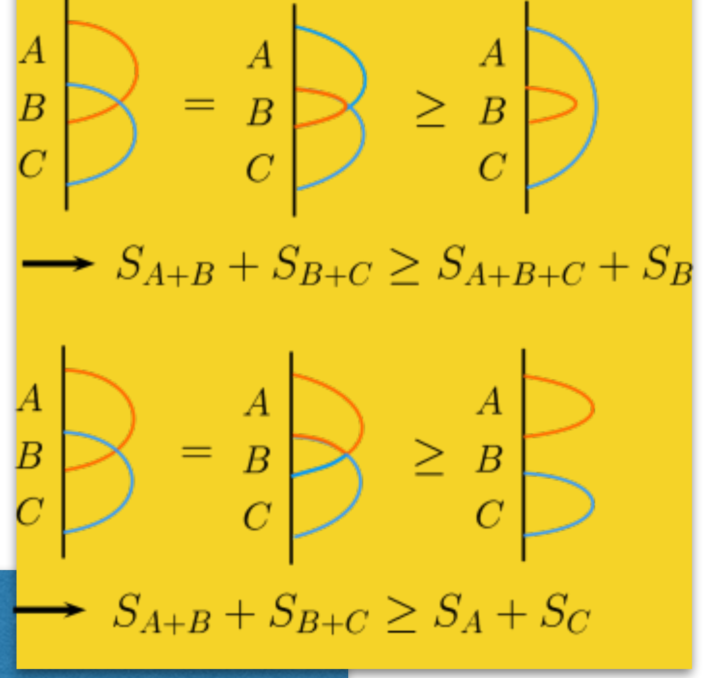
7th June, 2023,
Quantum Information Theory in Quantum Field Theory and Cosmology

Work done in collaboration with :
Lin Chen, Ruoshui Wang, Haochen Zhang, Kaixin Ji, Xiangdong Zeng,
Exact Holographic Networks From Topological Orders arXiv:2210.12127
+ Gong Cheng, Yikun Jiang, Bingxin Lao work in progress ~~~
A continuation of :
Arpan Bhattacharya, Yang Lei, Wei Li, Charles Melby-Thompson,
JHEP 04 (2019) 170, *JHEP* 05 (2019) 118, *JHEP* 01 (2018) 139, *JHEP* 08 (2016) 086
陈霖，刘希融
arXiv:2102.12022, 2102.12023, 2102.12024

清华大学 丘成桐数学科学中心
Yau Mathematical Sciences Center, Tsinghua University

Many body entanglement and holographic theories



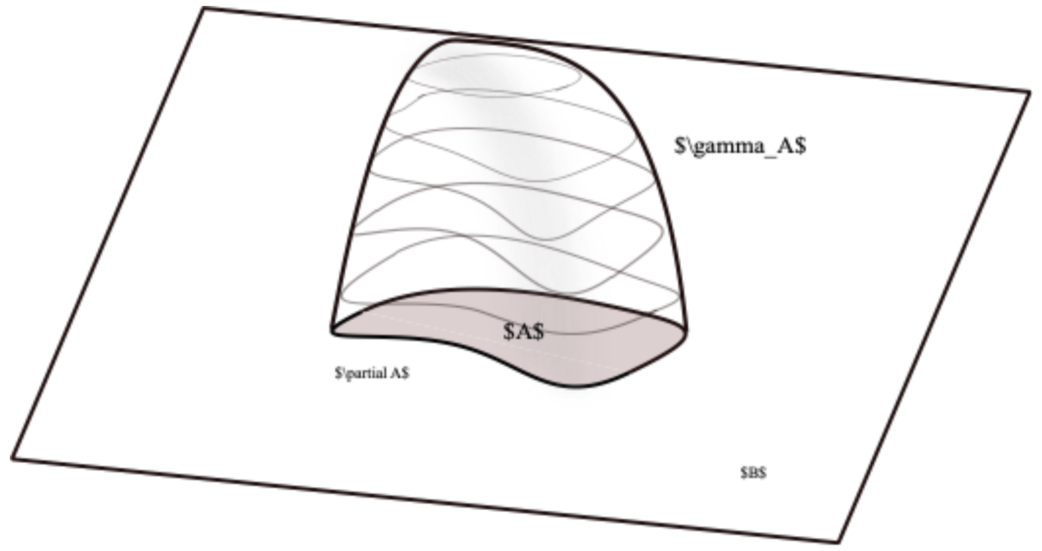
Quantum gravity …… algebra + geometry

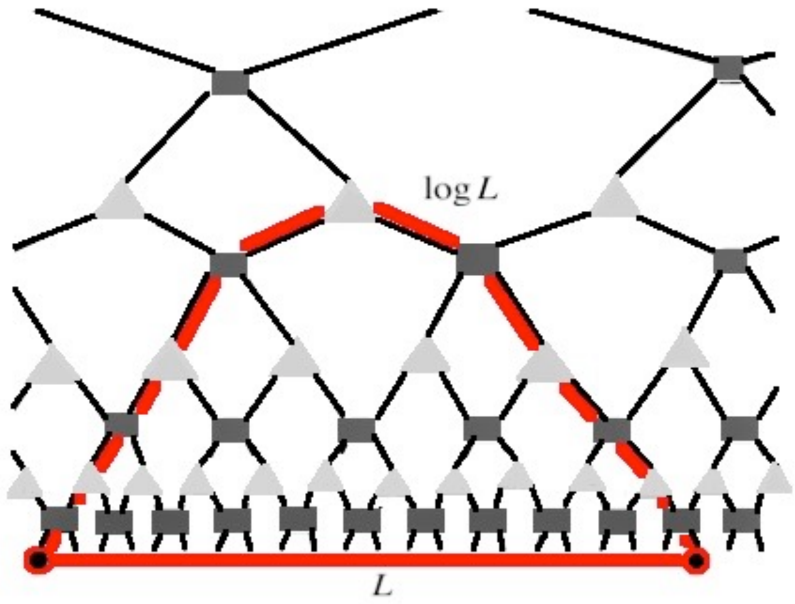
Entanglement satisfies e.g. strong sub-additivity that looks like triangle inequalities — somehow they fit well with geometric data

1. AdS/CFT says entanglement is geometry

Ryu-Takayanagi Formula:
(2006)

2. appropriate models that realise these ideas





Picture courtesy Orus

$$S_{EE} = \frac{A}{4G}$$

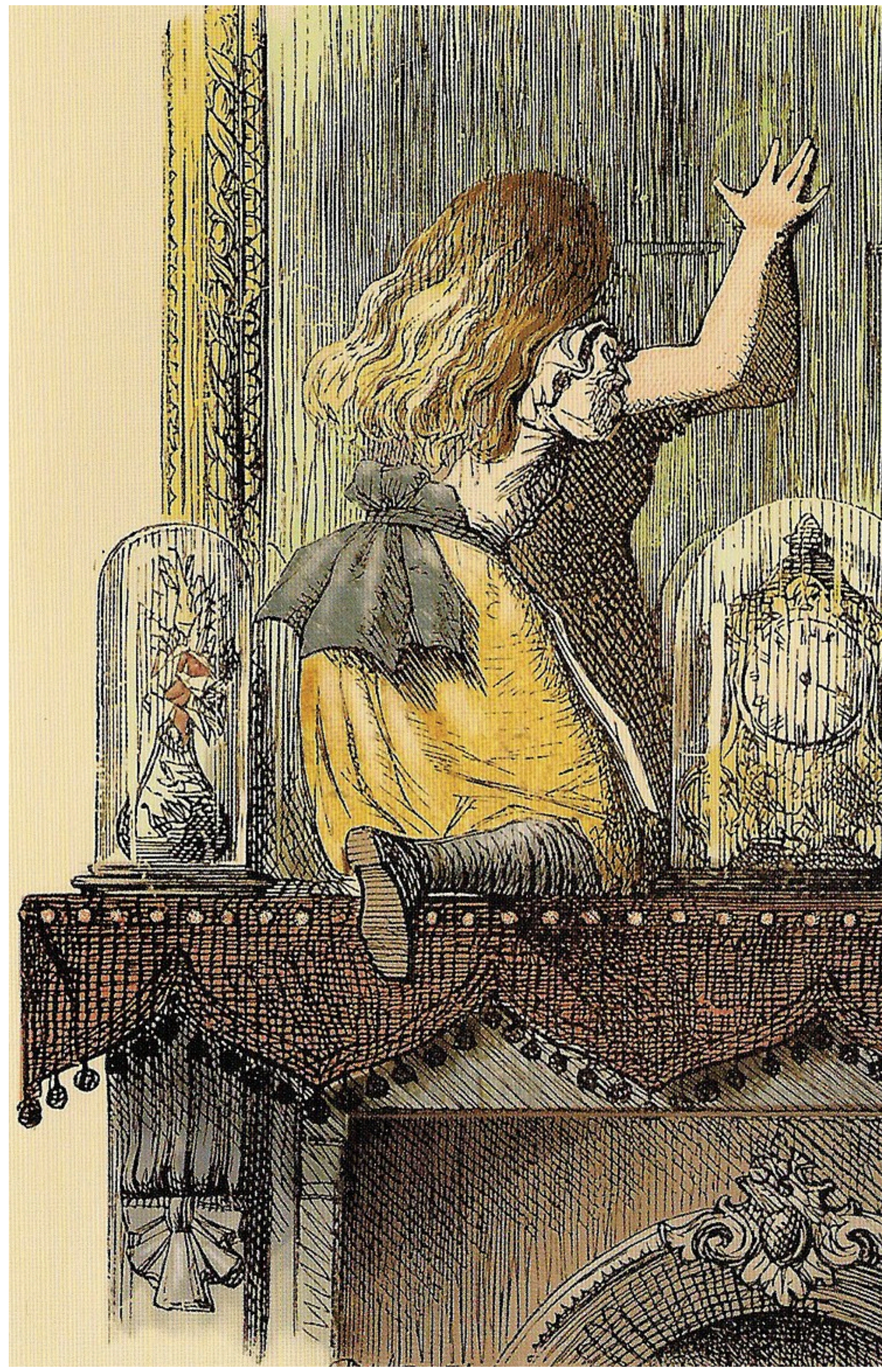
Tensor network is a geometrization of entanglement. It is explicitly local.

Brian Swingle (2012)

Overview

- Categorical Symmetry
- A holographic relation — symmetric QFT in d-dimensions and TQFT in d+1 dimensions
- Applications : Explicit constructions of models with given categorical symmetry in 1+1 d
    - RG operators
    - fixed point and factorisation of CFT
    - holographic tensor network vs AdS/CFT?
- Generalisations to higher dimensions
- Outlook

Categorical Symmetry



Courtesy: Wikipedia

- Traditionally, we understand symmetry through symmetry transformations of the system. A system is symmetric if an action leaves the system invariant. e.g. rotation, translation, reflection etc.

- These transformations form a group

- An important implication of symmetry is conservation laws. e.g. Noether theorem, and quantum mechanically, we get Ward identities etc that constrain the theory e.g. constraining the end point of RG flows etc

Categorical Symmetry

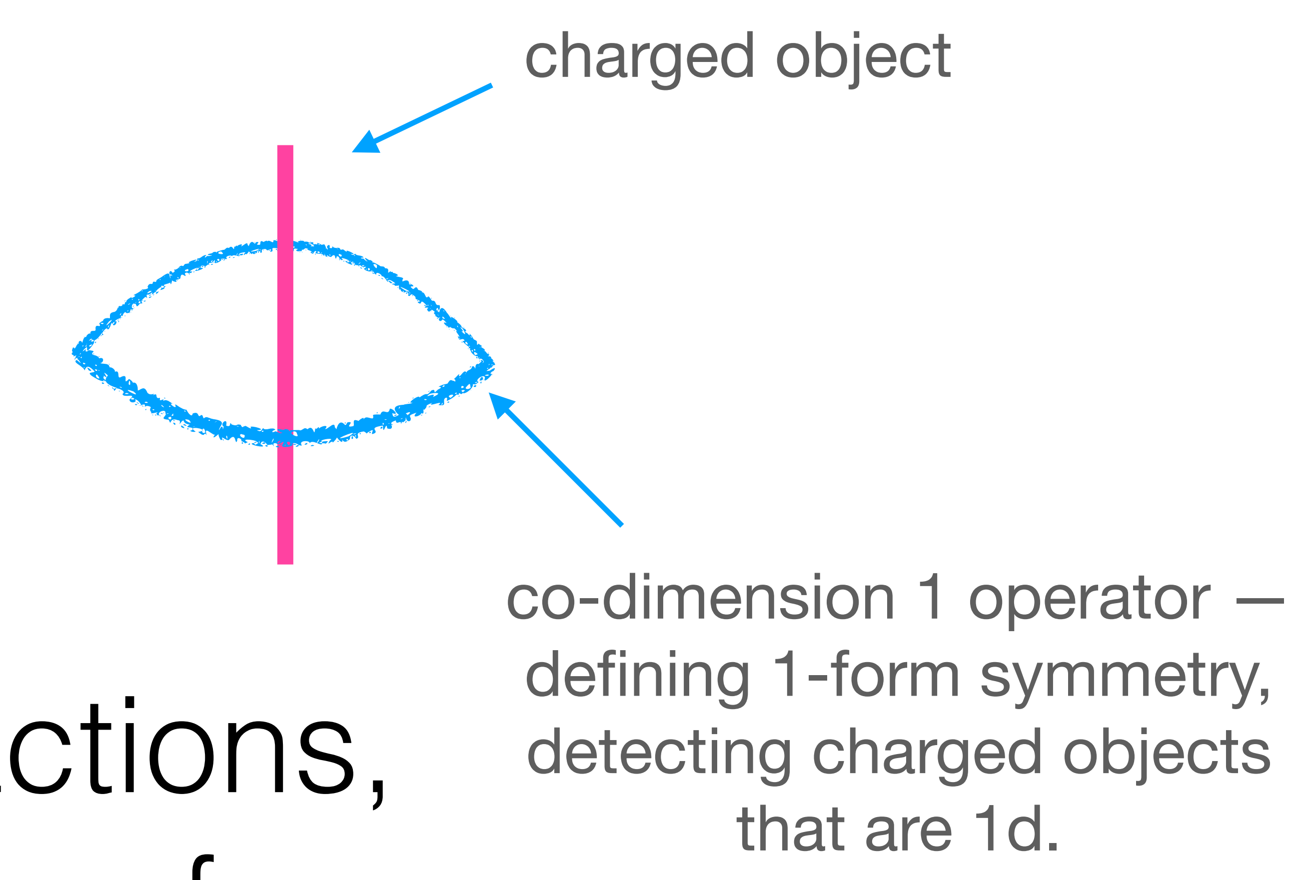
- in a continuous symmetry, the conserved current allows one to define charge operators — they are co-dimension one operators that commute with the Hamiltonian. i.e.

$$d * j = 0 \implies Q \equiv \int_{\Sigma_{d-1}} *j$$

- (Q is related to the symmetry generator by $U_{\Sigma_{d-1}}(\alpha) = \exp(i\alpha Q)$ say in a U(1) symmetry. The multiplication of $U_{\Sigma_{d-1}}(\alpha) = \exp(i\alpha Q)$ satisfies the group law.)

- Due to the symmetry, Q is a topological operator since the expectation value of Q is independent of the shape of Σ_{d-1} unless Σ_{d-1} crosses some operator with charges.

Categorical Symmetry



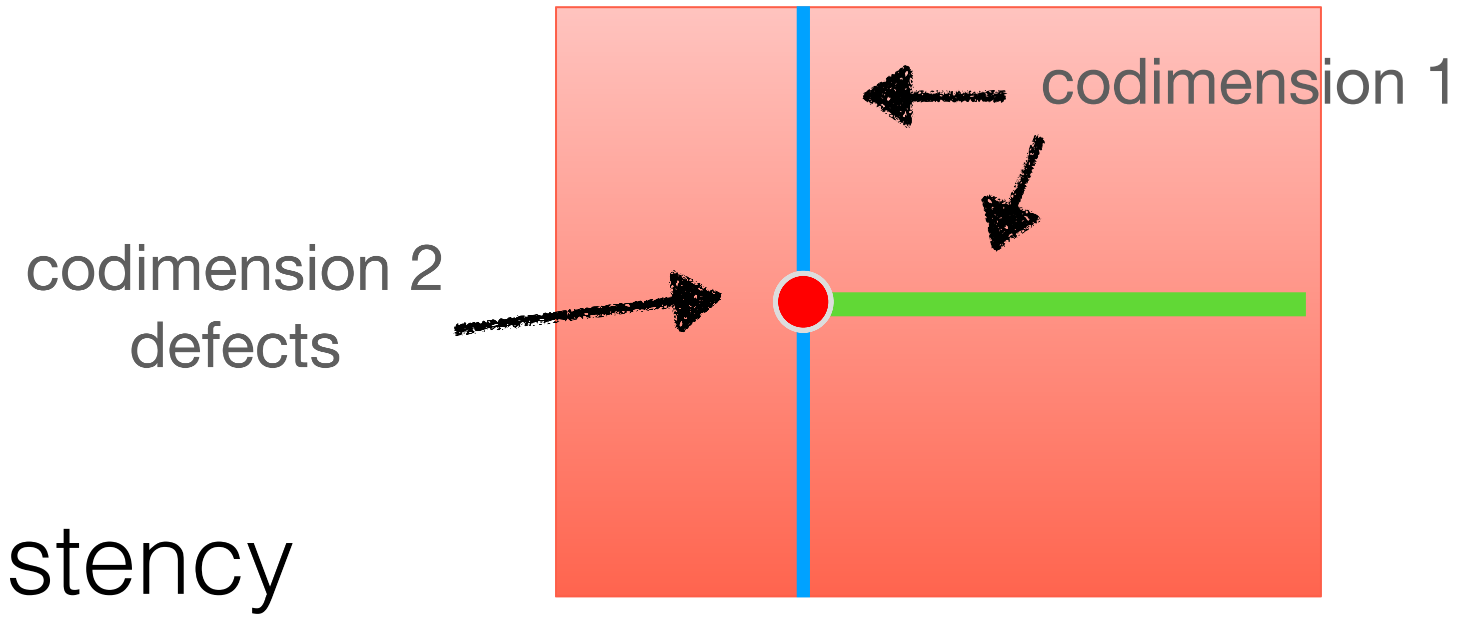
- Generalised symmetry — rather than looking for group actions, look for topological operators. If topological operators are of higher codimension, they correspond to "higher symmetries" — Gaiotto, Kapustin, Seiberg, Willett 2014

- e.g. in 1-form symmetry the charge operators are topological co-dimension1 operators

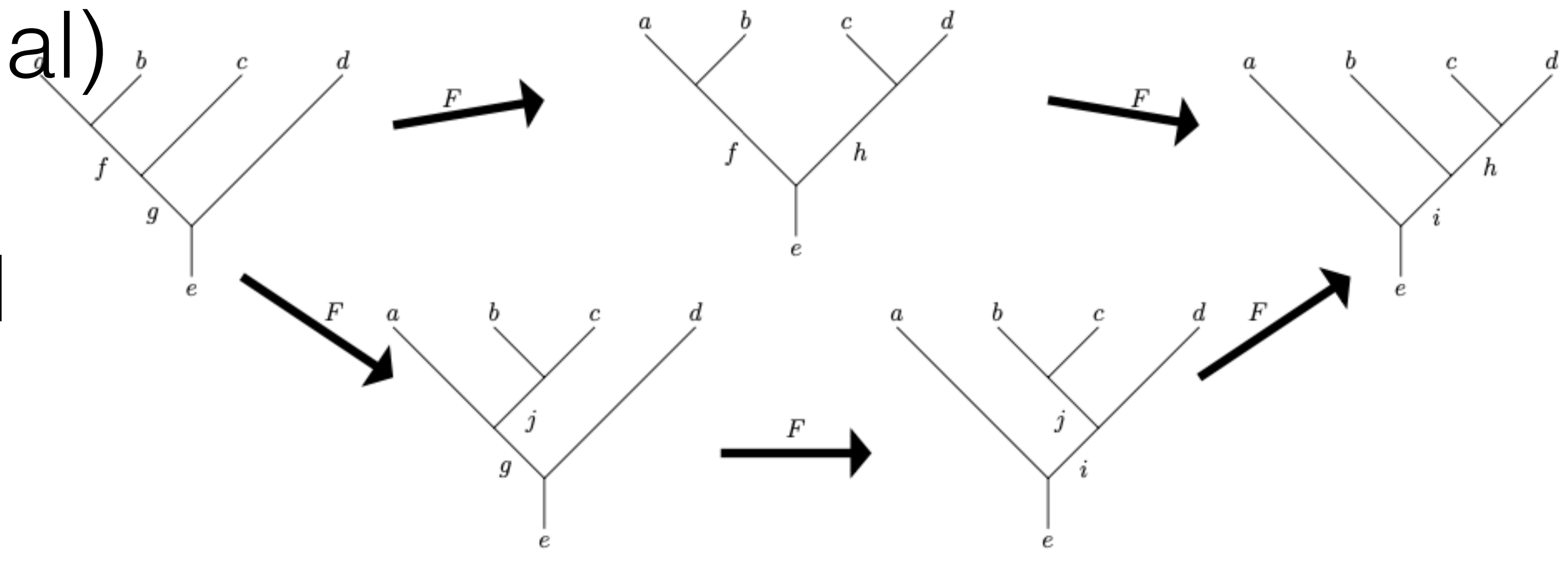
- The collection of these operators form a mathematical structure: a (higher) fusion category

Categorical Symmetry

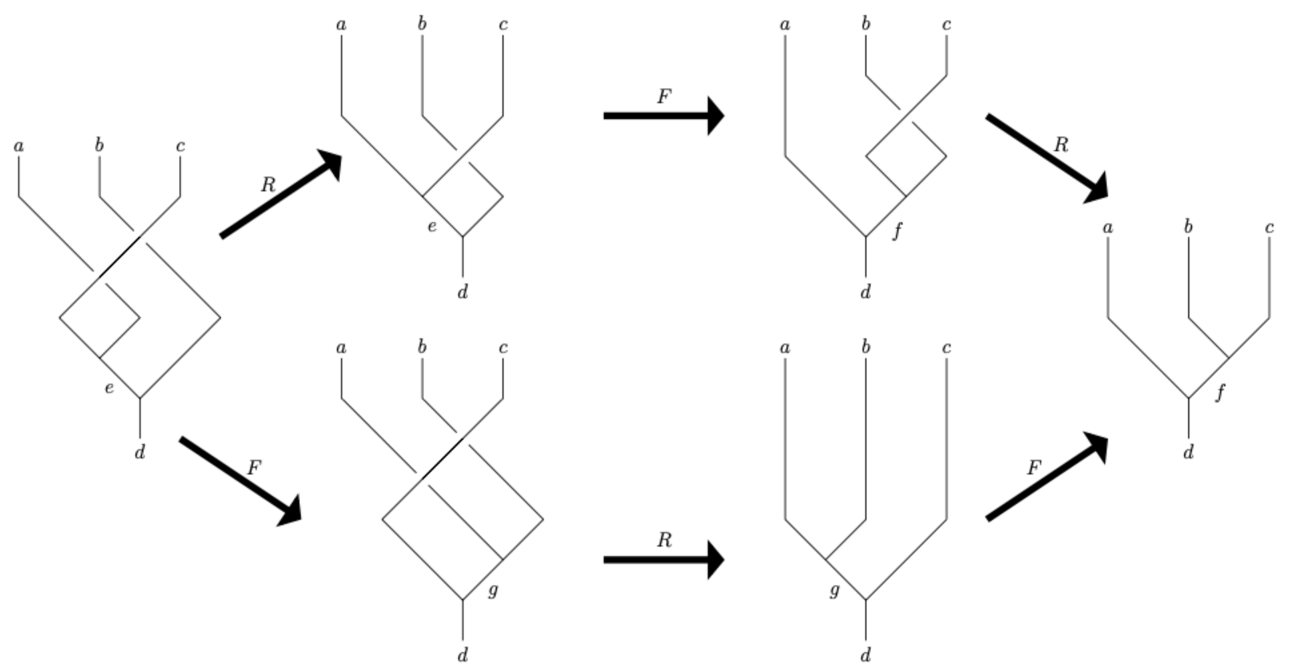
Bhardwaj, Tachikawa 2017;
Chang, Lin, Shao, Wang, Yin 2019;
Thorngren, Wang 2019; Ji, Wen 2020;
Kong, Lan, WenZhang, Zheng, 2020 ……



- codimension 1 operators — objects
- codimension 2 operators — 1 morphisms between objects etc
- These operators can fuse and the fusion should satisfy some consistency conditions e.g. associativity constraints
  Pentagon equation for 1-fusion category (where 2-morphisms are trivial)

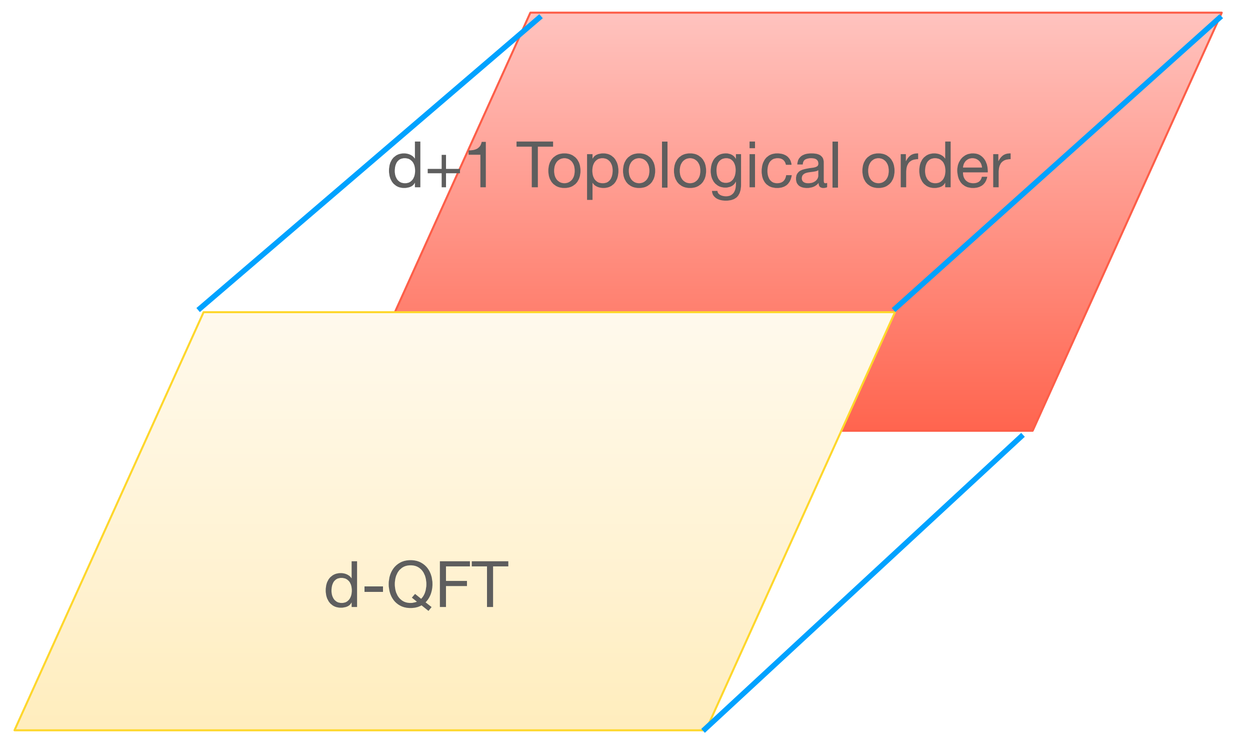


- They might also be able to braid and in that case they form a braided fusion category
- There are some very well-known examples well before the concept is formulated in this way — e.g. 2d CFT topological lines called the Verlinde lines have been studied in detail - they form a fusion category

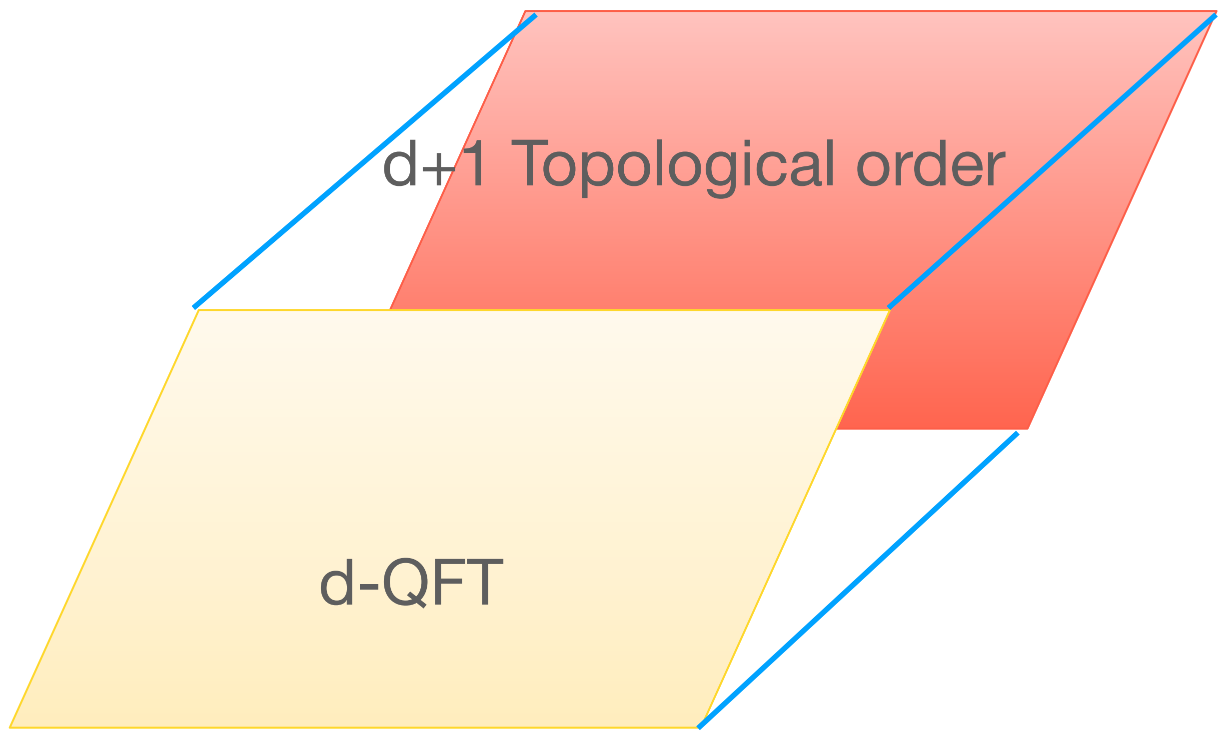


A Holographic relation

Decoupling symmetry and dynamics of a QFT



Holographic Relation



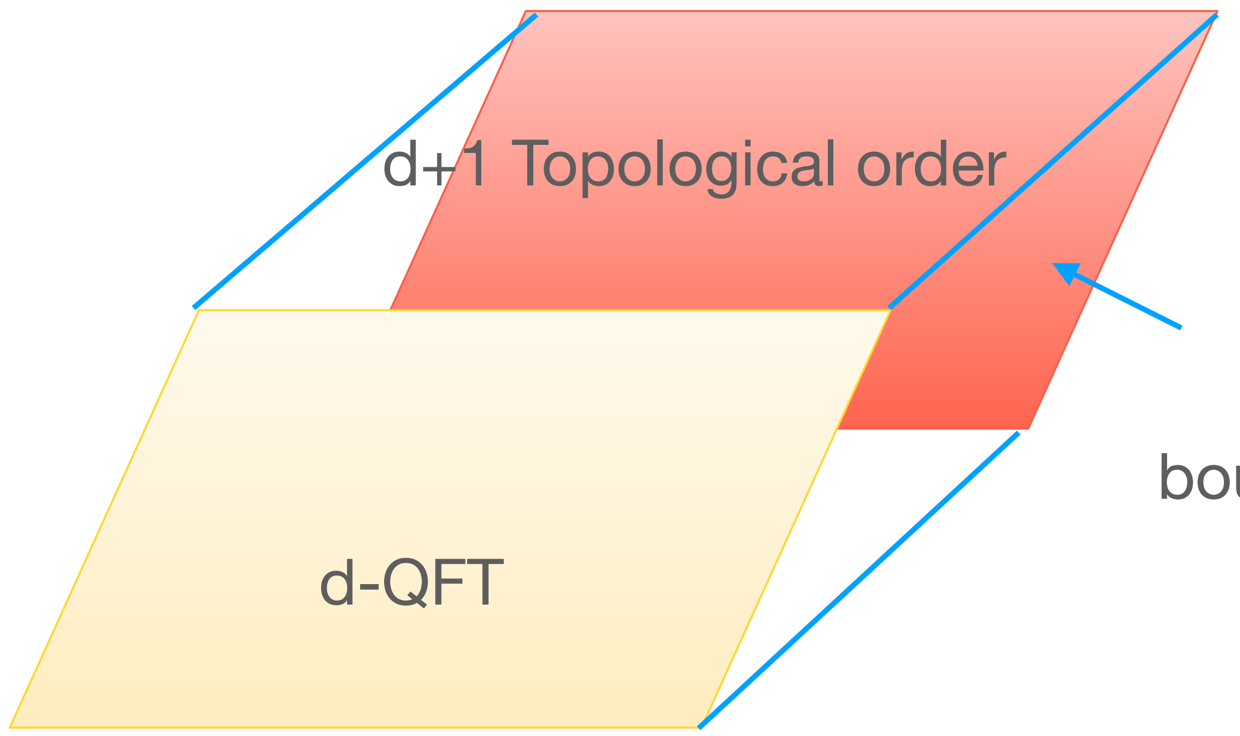
- It is observed in Gaiotto, Kulp 2020 for (discrete) group symmetries, that any QFT with symmetry G, can be formulated as the boundary condition of a TQFT (Dijkgraaf-Witten theory with gauge group G)

- This is part of a more general picture proposed in Kong, Lan, Wen, Zhang, and Zheng (2020) where the symmetry concerned is not restricted to group symmetry i.e. when the fusion rules of the defects are not groups or when they are not even unitary/ invertible (such as in the case of Verlinde lines), they can still be understood as boundaries of a TQFT in one higher dimension characterised by the (braided) tensor category formed by the defects

- * more precisely the TQFT in one higher dimension is the (Drinfeld) center Z(C) of the fusion category C, where C is formed from dropping the codimension 1 defects…)

- The (co-dimension 2 and lower dimensional) topological defects of the QFT corresponds to topological excitations in the TQFT. They are explicitly preserved in this picture.

Holographic Relation



- Explicit examples are studied in greater detail particularly in (gapped) 1+1 d theories and 2+1 d TQFT

  e.g. 1+1 d Symmetry Protected Topological (SPT) phases with symmetry G corresponds to topological boundary conditions to 2+1 d trivial Dijkgraaf Witten theories with gauge group G. Anomalous 1+1 d SPT phases with symmetry G corresponds to non-trivial Dijkgraaf Witten theories. Gapped boundaries correspond to spontaneous symmetry breaking of some of the defects that can be found in the bulk Z(C) Kong, Lan, Wen, Zhang, and Zheng (2020)

- One can show that CFT's (critical points) result from symmetry operators defined in Z(C) with non-trivial braiding properties

  Feiguin et al 2006; Kong, Zheng 2017, 2020 ; Levin 2019; Ji Wen 2019 ; Chatterjee, Wen 2022…..

- CFT as boundaries of Chern-Simons Theories.
  Wilson lines in the Chern-Simons theory <-> Verlinde lines in the boundary CFT.
  One needs to choose appropriate topological boundary condition on the other side to get modular invariant CFTs path-integrals.

Explicit Examples:

1+1 d Integrable models as boundaries of
2+1 d Turaev-Viro TQFT

Turaev Viro Formulation of TQFT in 2+1 d

Fusion Category

- List of simple objects: {a,b,c,…},
- 1-Morphisms: maps taking a to b, $f \in$ Hom(a,b); $a \xrightarrow{f} b$

  for simple objects: Hom(a,a)={Identity map}, Hom(a,b)=∅ when $a \neq b$.
- Fusion rules: $a \otimes b = \bigoplus_c N_{ab}^c c$,

  N_{ab}^c: non-negative integers,

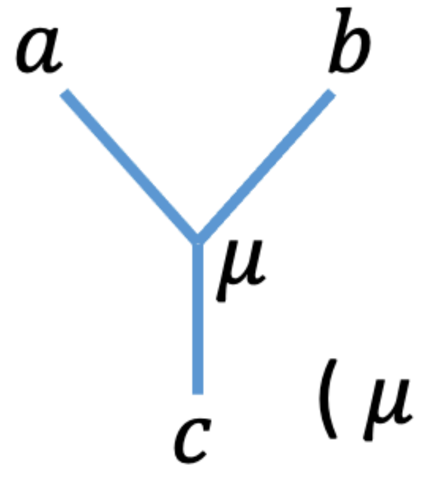
  quantum dimension: $d_a \stackrel{\text{def}}{=}$ max eigenvalue of N_a, [$(N_a)_b^c = N_{ab}^c$]

  Associativity of anyon fusion: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$,
- F-symbols

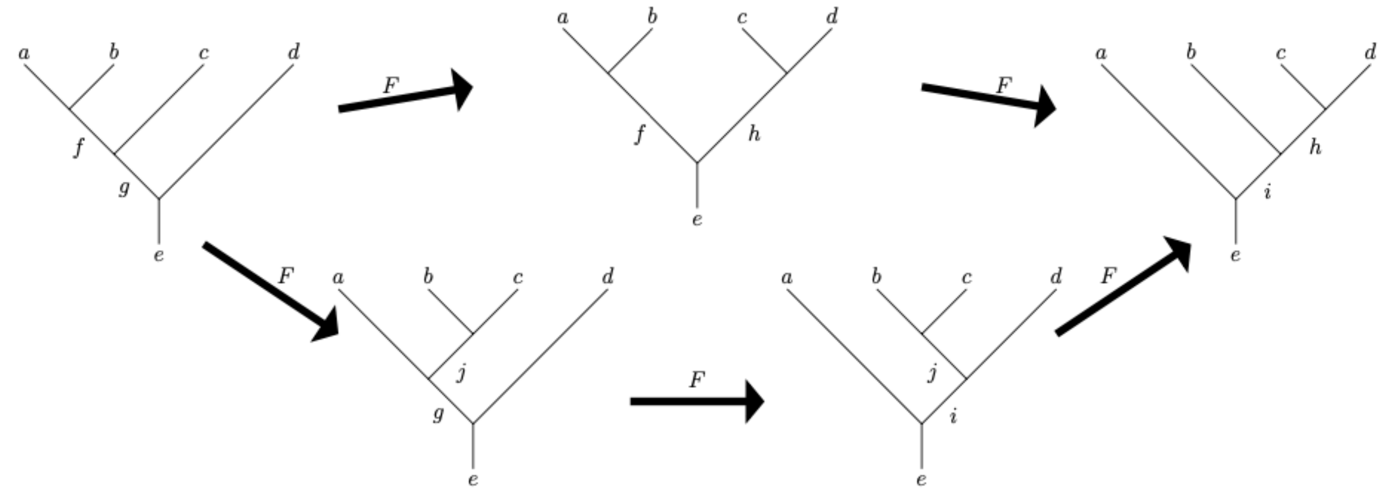
Turaev Viro Formulation of TQFT in 2+1 d

Turaev Viro Formulation

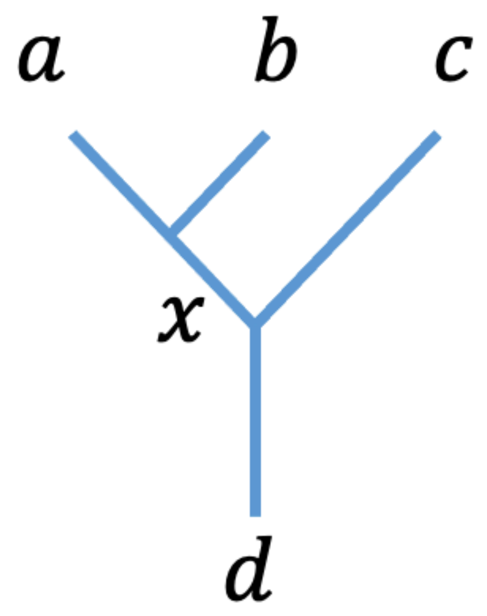
F-symbols

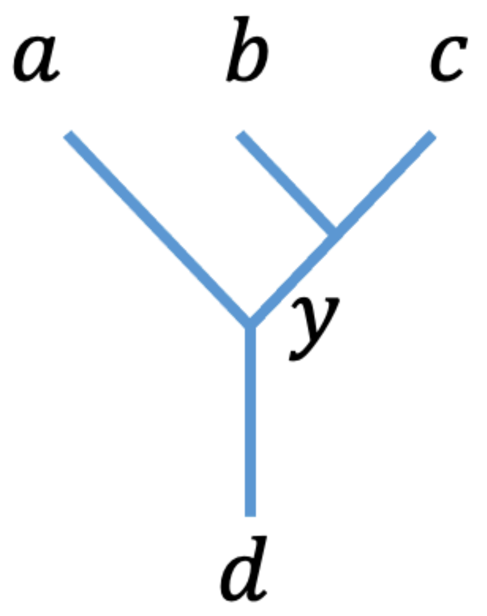
- Fusion diagrams; 

map $a \otimes b$ to c, and form a vector space V_c^{ab} $(\, \mu = 1, \dots, N_{ab}^{c}\,)$



- Consider the vector space V_d^{abc}, map $a \otimes b \otimes c$ to d,



$= \sum_y F^{abc}_{d;x,y}$


$(a \otimes b) \otimes c,\ V_d^{abc} \cong \bigoplus_x V_x^{ab} \otimes V_d^{xc}$ $\qquad$ $a \otimes (b \otimes c),\ V_d^{abc} \cong \bigoplus_y V_d^{ay} \otimes V_y^{bc}$

Turaev Viro Formulation of TQFT in 2+1 d

Turaev Viro Formulation

- A closed oriented 3D manifold $\mathcal{M}$,
- Consider a triangulation $\mathcal{K}$, i.e. split $\mathcal{M}$ into tetrahedrons,
- Give an order of vertices on $\mathcal{K}$, and a fusion category $\mathcal{C}$,
- Assign one object to each edge, one 1-morphism to each triangle, $\mu \in Hom(a \otimes b, c)$, $\mu = 1, \dots, N_{ab}^{c}$. If N_{ab}^{c}=0, this label is not allowed. In the following, consider N_{ab}^{c}=0,1.
- Assign a F symbol (tetrahedral symbol) F_{Δ_3} to each tetrahedron Δ_3 .
- The partition function of Turaev-Viro TQFT:

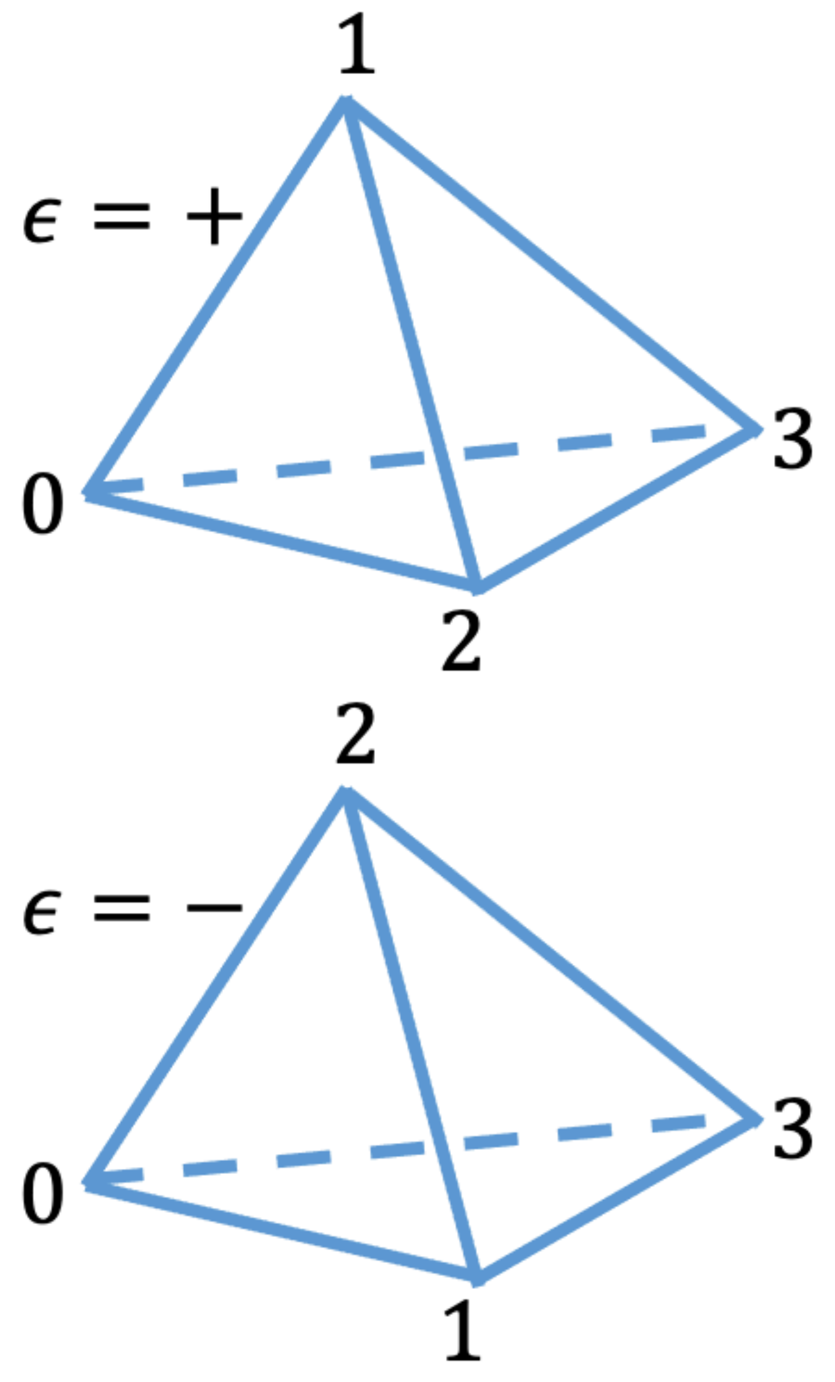
$$Z_{\mathcal{M},\mathcal{C}} = \frac{1}{|D_{\mathcal{C}}|^{N_v}} \sum_{\{a\} \in Obj(\mathcal{C})} \prod_{a} d_a \prod_{\Delta_3 \in \mathcal{K}} F_{\Delta_3}^{\epsilon(\Delta_3)}$$

Turaev Viro Formulation of TQFT in 2+1 d

Turaev Viro Formulation

Partition function of TQFT

$$Z_{\mathcal{M},\mathcal{C}} = \frac{1}{|D_{\mathcal{C}}|^{N_v}} \sum_{\{a\}\in Obj(\mathcal{C})} \prod_a d_a \prod_{\Delta_3\in\mathcal{K}} F_{\Delta_3}^{\epsilon(\Delta_3)}$$



$D_{\mathcal{C}}$ is the dimension of $\mathcal{C}$.
N_v is the total number of vertices.
$\{a\}$ is a configuration of assigning objects to the edges in $\mathcal{K}$.
$\epsilon(\Delta_3) = \pm 1$ determined by the orientation of the tetrahedron.

The partition function $Z_{\mathcal{M},\mathcal{C}}$ is a topological invariant, i.e. it is independent of the choice of triangulation $\mathcal{K}$ and order of vertices.

Now we will take this TQFT and use it to construct (integrable) lattice models in one lower dimension.

RSOS integrable models and Minimal models and Levin Wen models

1. Tensor Categories can be used to construct Hamiltonians of CFT minimal models

Feiguin, Trebst, Ludwig, Troyes, Kitaev, Wang, Freedman PRL 2007;

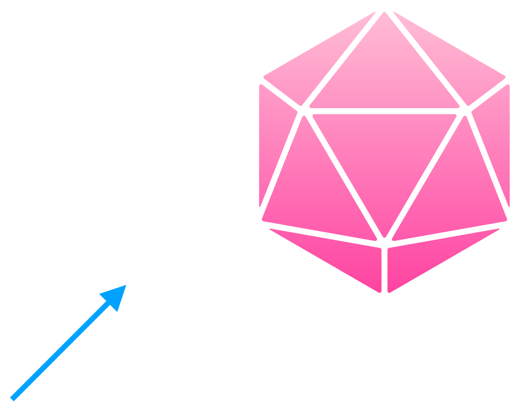
2. The partition functions of minimal models can be thought of as imposing boundary conditions on a corresponding topological model defined using these tensor categorical data. *Topological symmetry of CFT becomes explicit!*

Aasen, Fendley, Mong J. Phys. A; Math. Theor. 2016; 2020 ;

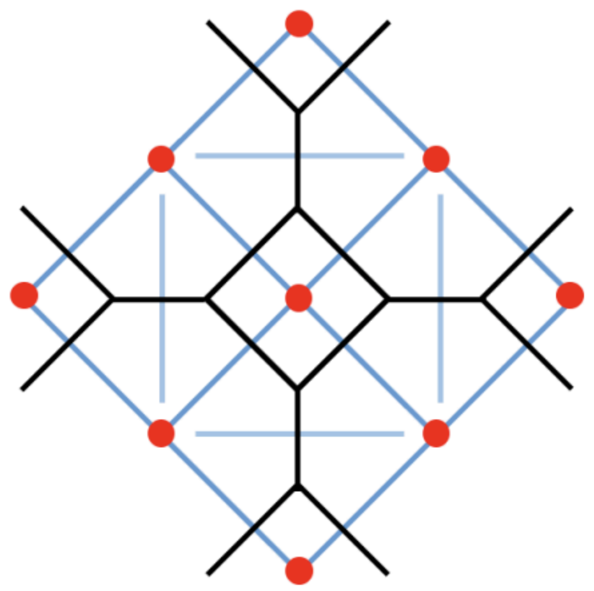
3. There is a strange correlator representation of these CFT partition functions — the overlap between a direct product state and a Levin - Wen wavefunction (Turaev-Viro formulation of TQFT)

Bal, Williamson, Vanhove, Bultinck, Haegeman, Verstraete PRL 2018;
Lootens, Vanhove, Verstraete PRL 2019

There are beautiful tensor network (PEPs) construction

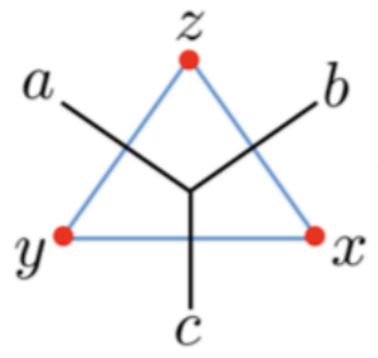


RSOS integrable models and Minimal models and Levin Wen models



- PEPS representation of Levin-Wen models ground state $|\Psi^{LW}\rangle$

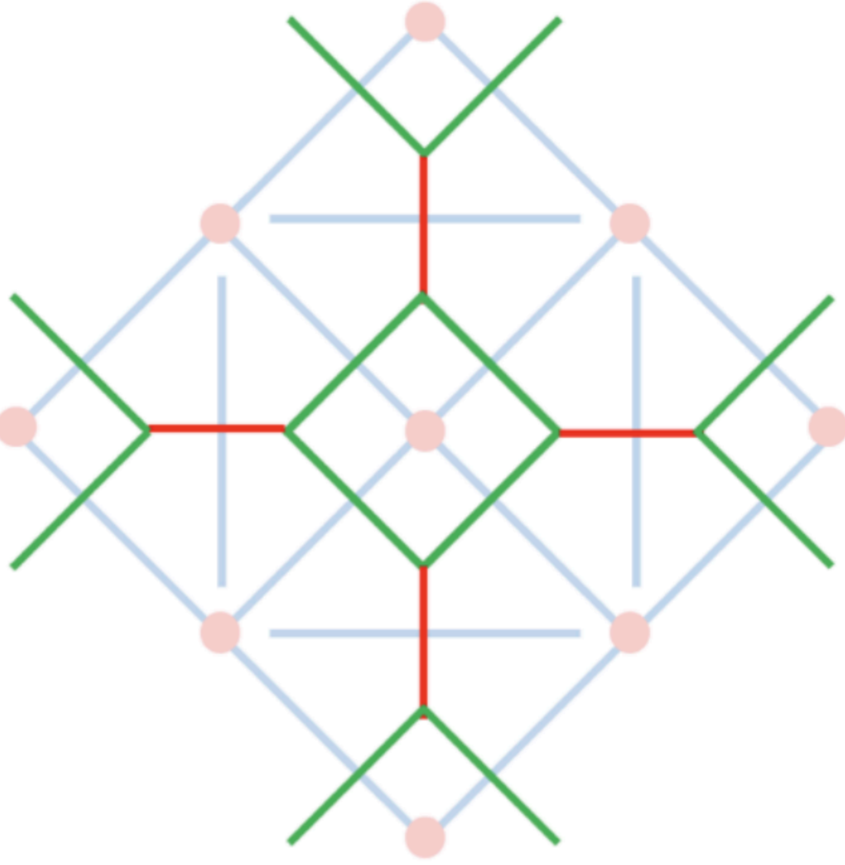
Gu, Levin, Swingle, Wen PRB 2009; Buershaper, Aguado, Vidal PRB 2009;
(More recently — the form we follow closely , is presented in
Bultinck, Marien, williamson, Sahinoglu, Haegeman, Verstraete Annals of physics 2017; Williamson, Bultinck, Verstraete 2017)



$$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F^{abx}_y)^*_{cz}$$

- Then pick some mysterious state $\langle\Omega_N|$ and take the overlap with $|\Psi^{LW}\rangle$ i.e. $\langle\Omega_N|\Psi^{LW}_a\rangle$

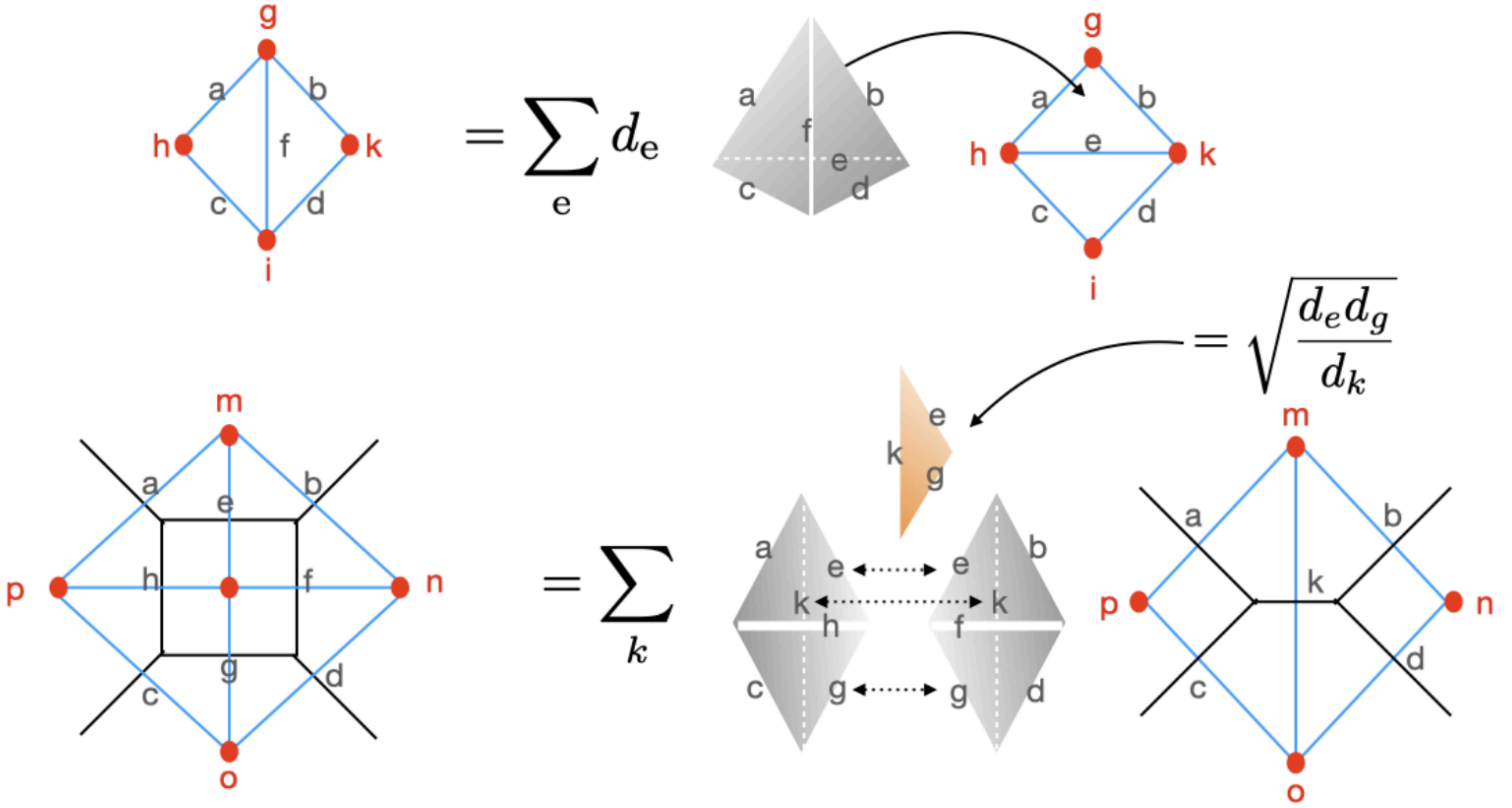
- Topological defects = string operators in the TQFT = explicitly can deform freely in the partition function i.e. Symmetry explicitly preserved.



$: \langle\frac{1}{2}|$

$: \langle 0| + r\langle 1|$

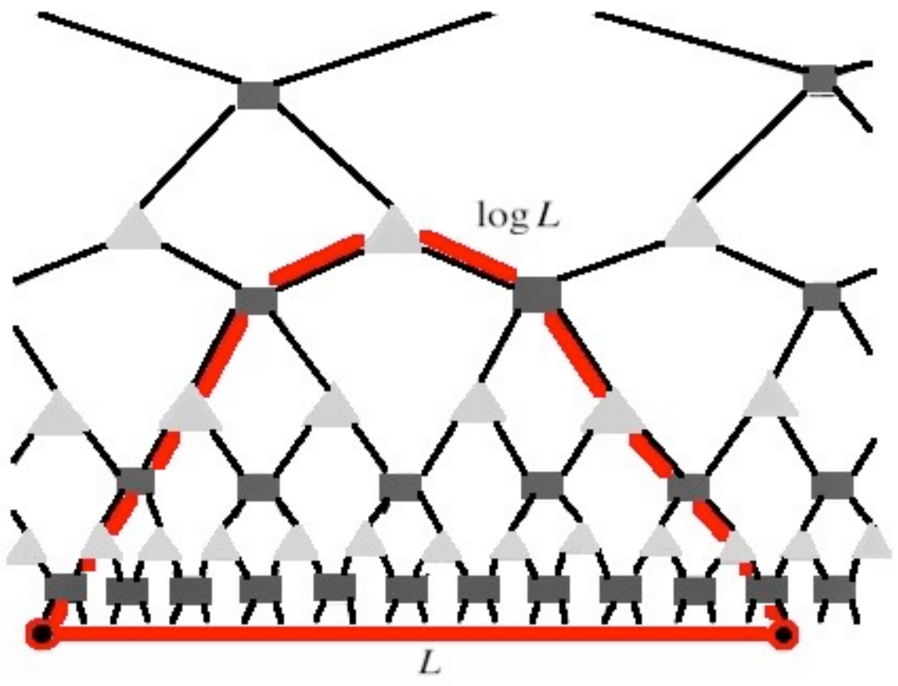
Minimal models and Levin Wen models — RG operators and holographic tensor networks



$$\langle \Omega_N | \Psi_a^{LW} \rangle$$

$$\langle \Omega_N | FF | \Psi_{ka}^{LW} \rangle = \langle \Omega_{N-1} | \Psi_{ka}^{LW} \rangle$$

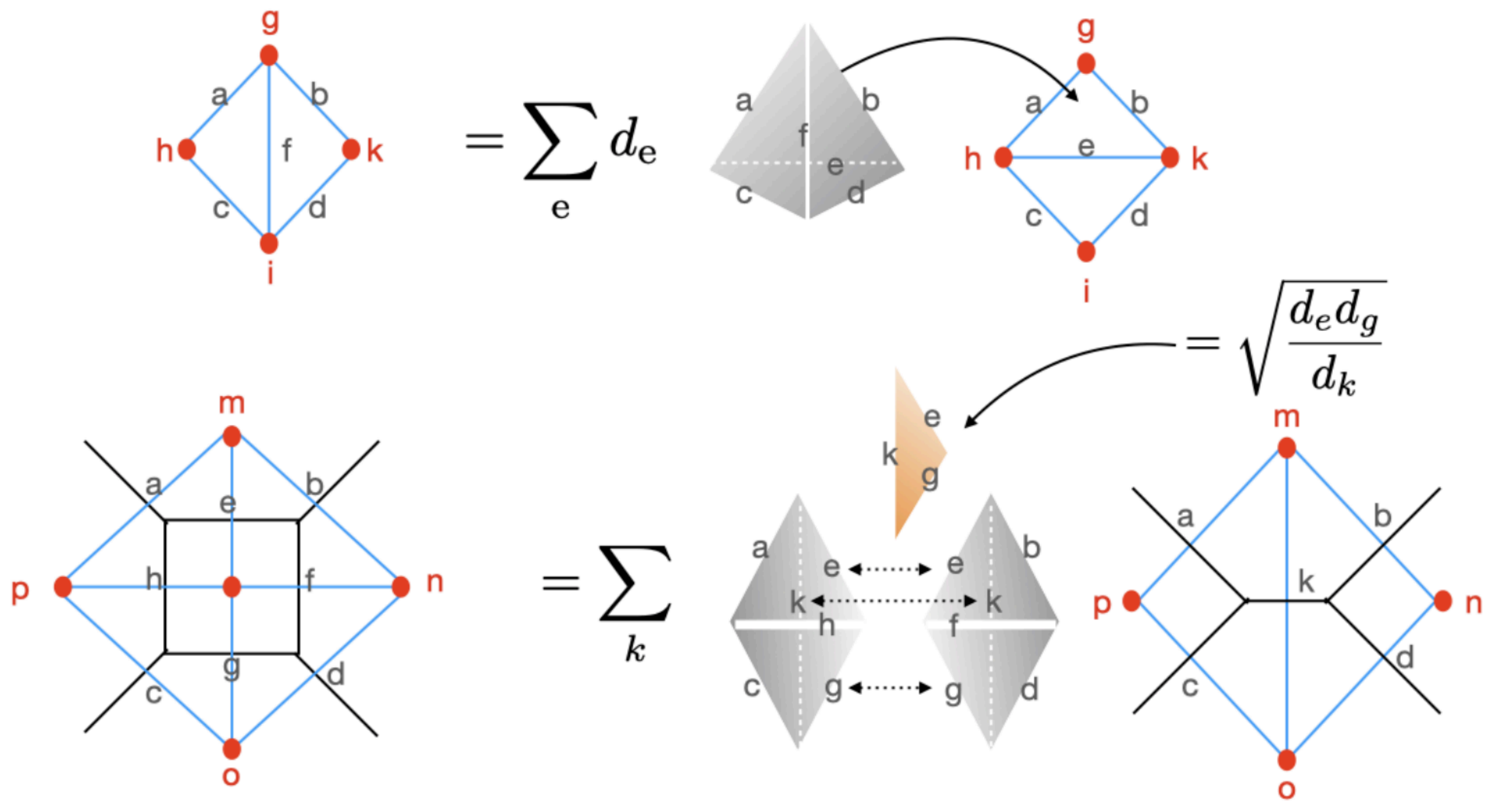
Here, we make the observation that $\langle \Omega_N | FFF \cdots$ looks like Euclidean AdS3. Isn't it in fact an analytic holographic tensor network !



Picture courtesy Orus

This map "FFFF…" is an RG map determined by topological symmetry.
Every CFT is given by an eigenstate of this map!

Minimal models and Levin Wen models — RG operators and holographic tensor networks



$$\langle \Omega_N | \Psi^{LW}_{a} \rangle$$

$$\langle \Omega_N | F F | \Psi^{LW}_{ka} \rangle = \langle \Omega_{N-1} | \Psi^{LW}_{ka} \rangle$$

We solved a bunch of trivial fixed points — they are classified by “Frobenius algebra” of the category — they are the known gapped boundary conditions of these 2+1 d topological models. CFT’s are the phase transitions between these gapped boundary conditions.

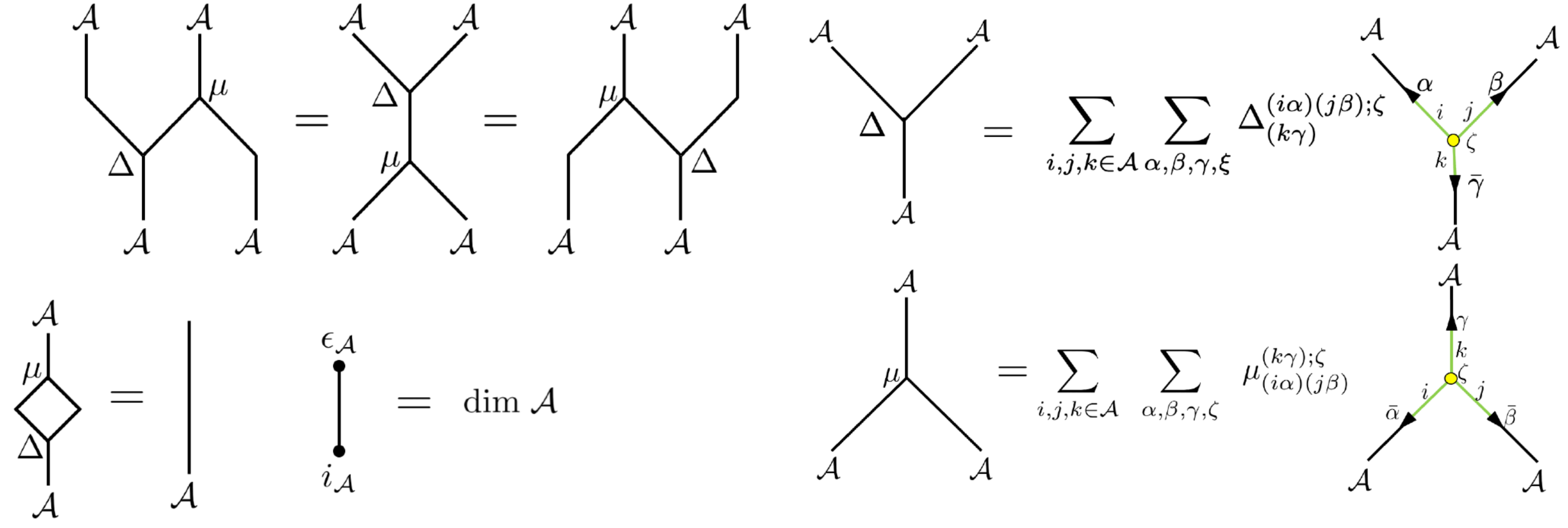
How to find fixed point?

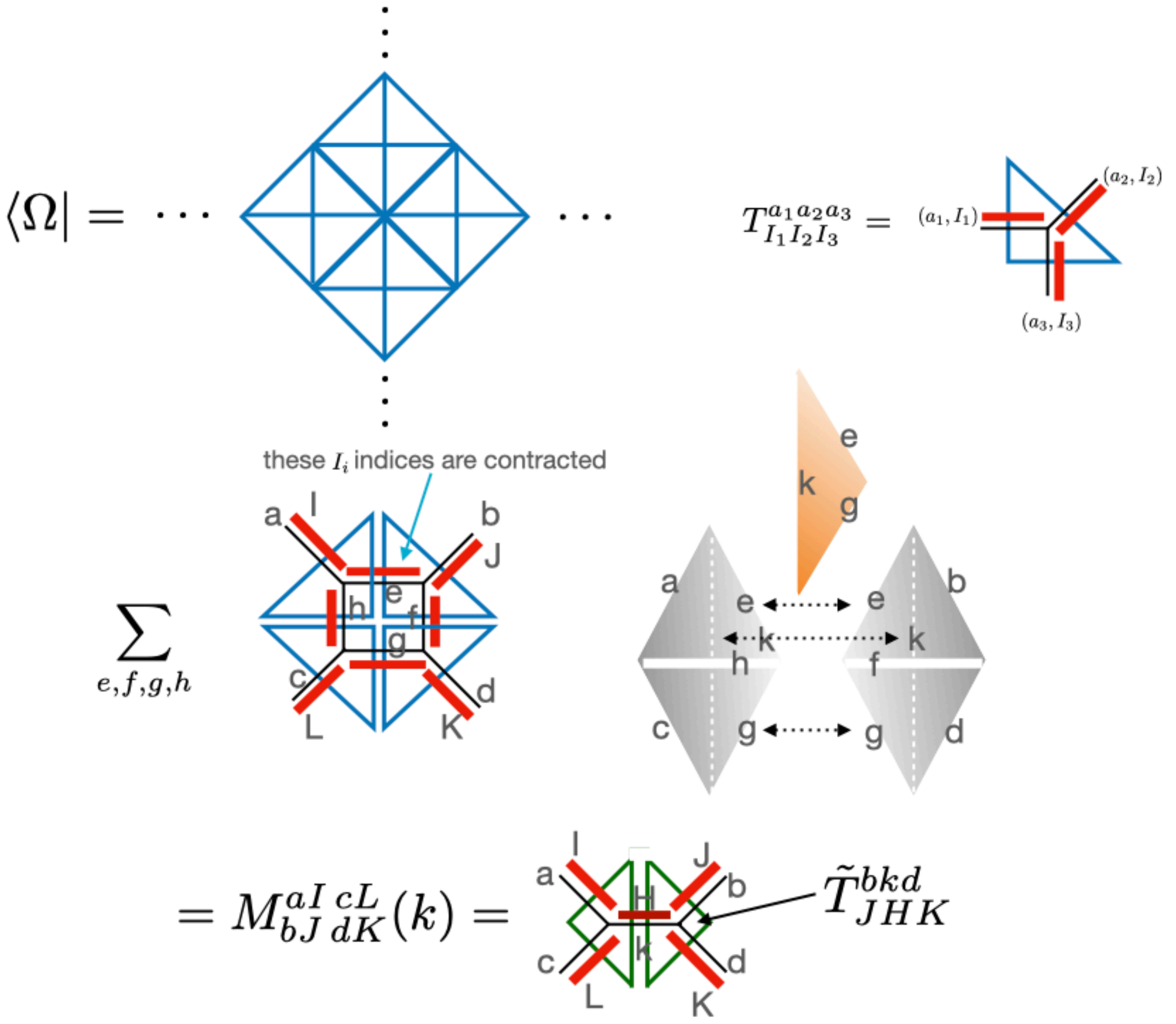
CFT's are between topological solutions

c.s. A Chatterjee, XG Wen
arXiv preprint arXiv:2205.06244

A set of topological fixed point to the RG operator —

Frobenius algebra





How to find fixed point?

CFT's are between topological solutions

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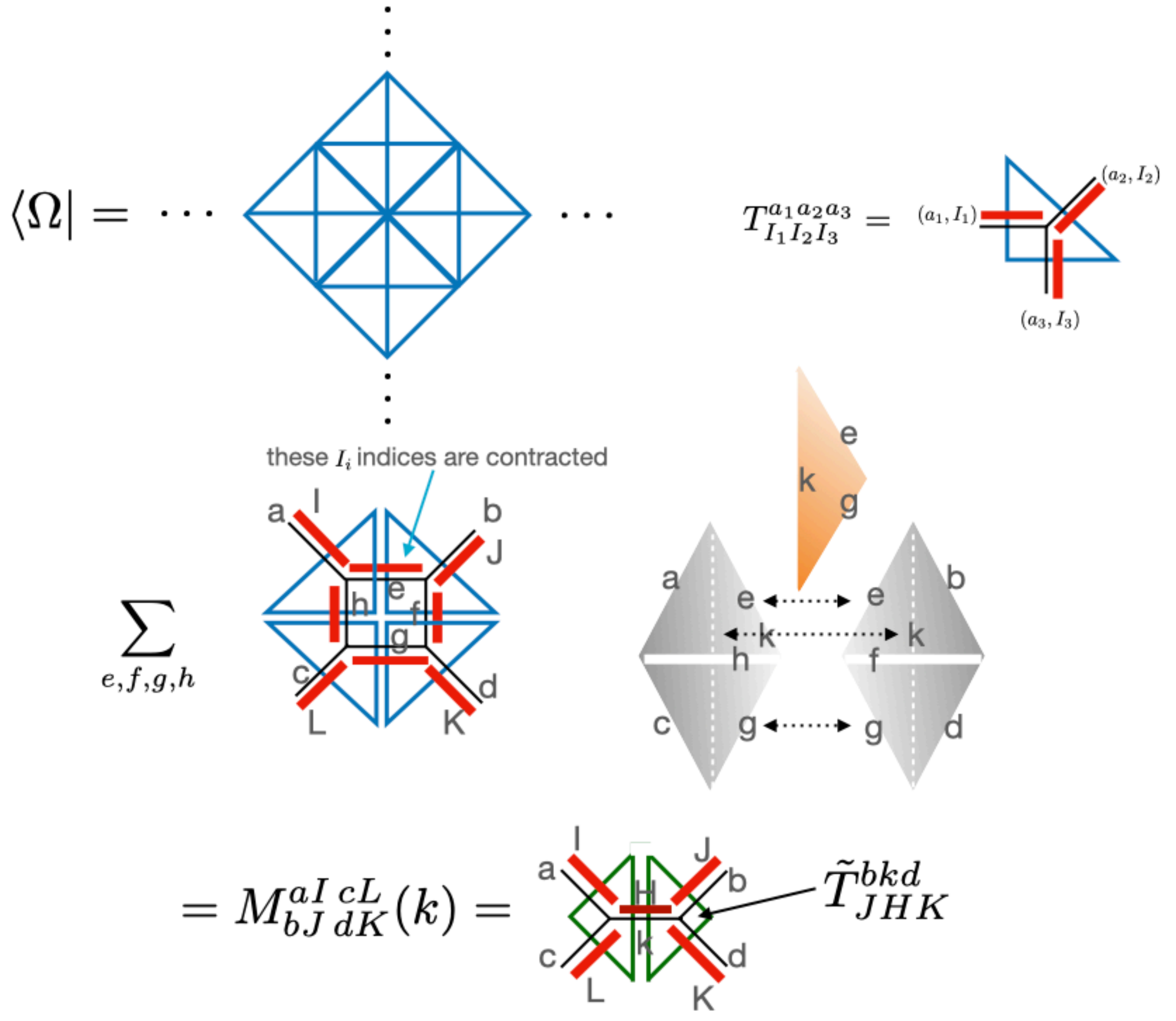
For fixed bond dimension— it would flow to a topological fixed point eventually. — one could get an approximate CFT when the bc is "confused"
Recover the critical point (to 1 significant figure with bond dimension just 1) for SU(2)_{k}

k	A1/A0—theoretical	Our numerics
2	0.643594	0.60-0.61
3	0.697043	0.67-0.68
4	0.719471	0.69-0.70
5	0.731426	0.71-0.72
6	0.738656	0.72-0.73



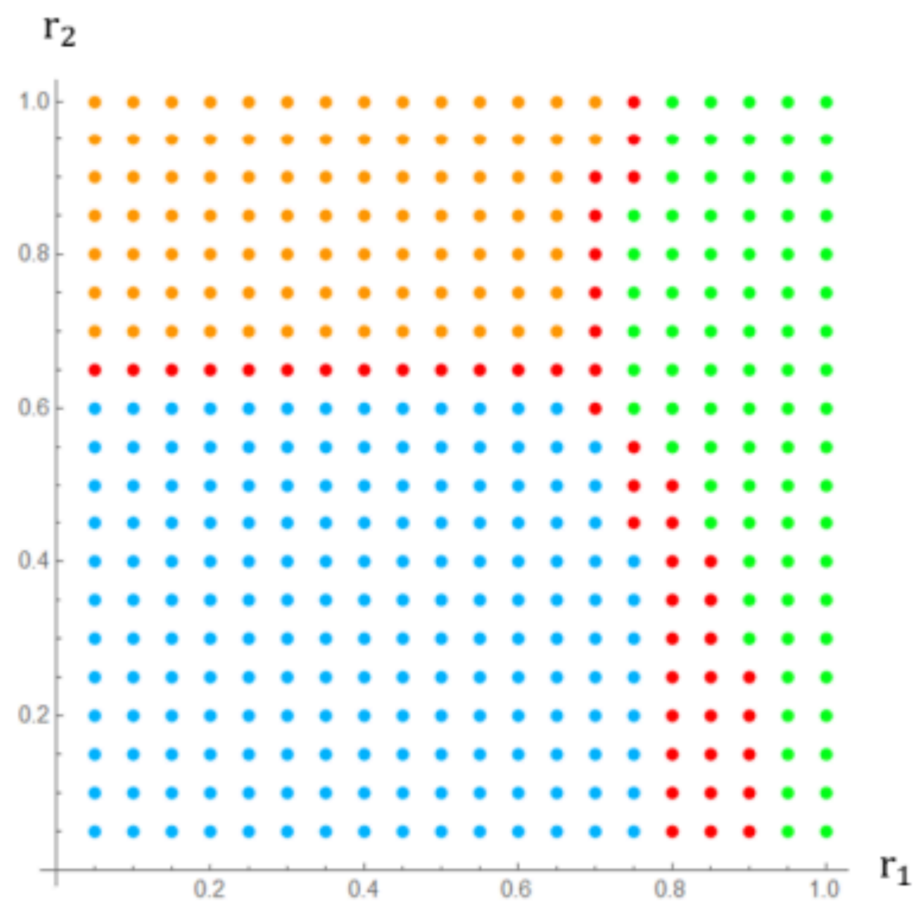
How to find fixed point?

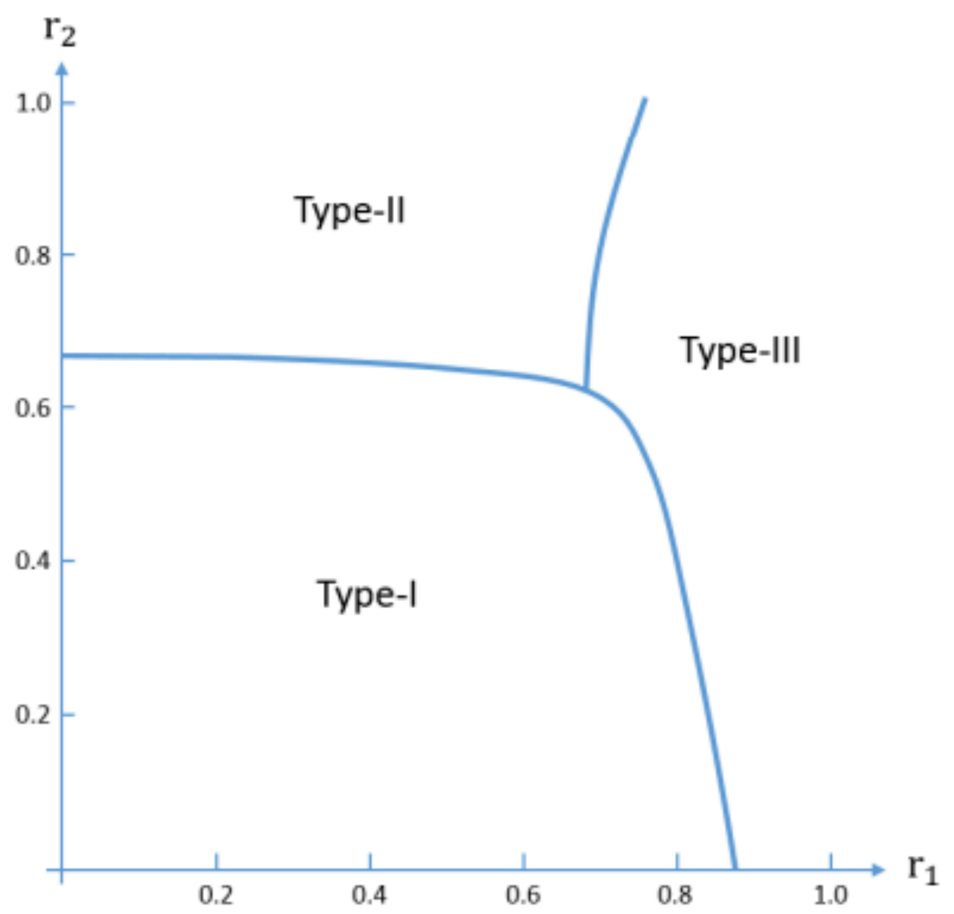
CFT's are between topological solutions

for k=4:

$$T^{a_1a_2a_3} = \begin{cases} 1, & (a_1 = a_3 = 1, a_2 = 0), \\ r_1, & (a_1 = a_3 = 1, a_2 = 1), \\ r_2, & (a_1 = a_3 = 1, a_2 = 2), \\ 0, & (\text{Otherwise}). \end{cases}$$

This class of models are related to the 19-vertex model.



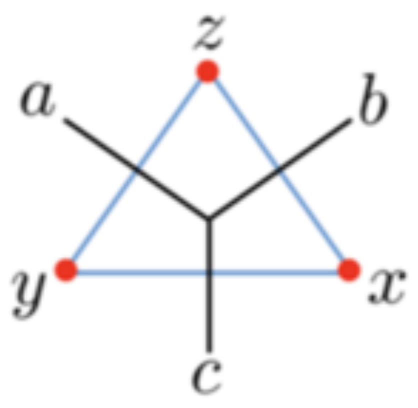


In the type I fixed point, $T^{a_1a_2a_3}$ only has one non-vanishing component T^{000}, i.e. all the boundary legs are projected to 0. In the type II fixed point, $T^{a_1a_2a_3}$ has following non-vanishing components $T^{000} = T^{022} = T^{202} = T^{220}$. In the type III fixed point, $T^{a_1a_2a_3}$ has following non-vanishing components $T^{000} = T^{011} = T^{101} = T^{110} = T^{022} = T^{202} = T^{220} = T^{112} = T^{121} = T^{211}$ (the component T^{111} is allowed by the fusion rules but is 0).

The three topological fixed points correspond to three different Frobenius algebra in the input category. Each of them correspond to a Lagrangian algebra of the topological order which determines the collection of anyons (or topological defects) that condenses at the 2d boundary.
We note that in the three topological condensates, it is clear that they definitely share the subset of $A_{sub} = (0 \boxtimes \bar{0}) \oplus (2 \boxtimes \bar{2})$. All the three boundary conditions can be considered as a sequential condensation, first con- densing A_{sub}. Therefore we conclude that the tri-critical point should also naturally have A_{sub} condensed.

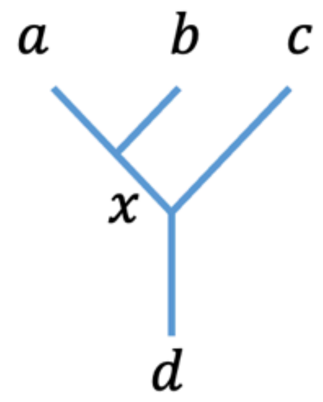
Fixed points corresponding to CFTs

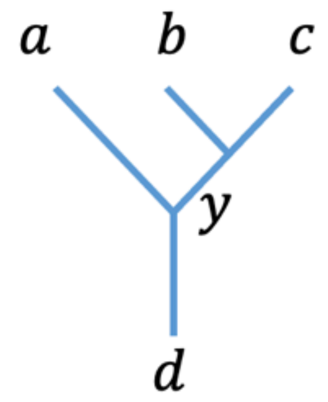
Fixed Point Boundary Corresponding to CFT?

- Note that 

$= \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \frac{1}{\sqrt{d_c d_z}} (F^{abx}_{y})^{*}_{cz}$

this is proportional to the open string (boundary operator) fusion coefficient in a diagonal rational eft characterised by the tensor category!

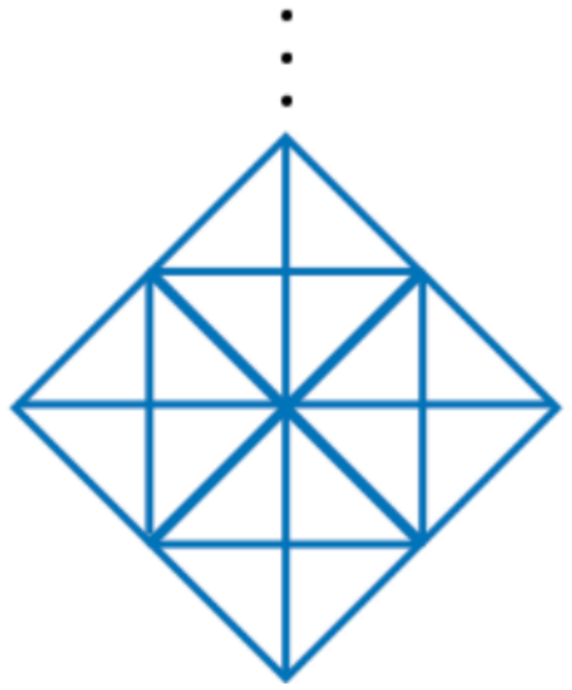
Also, it is well-known that conformal blocks transform as 

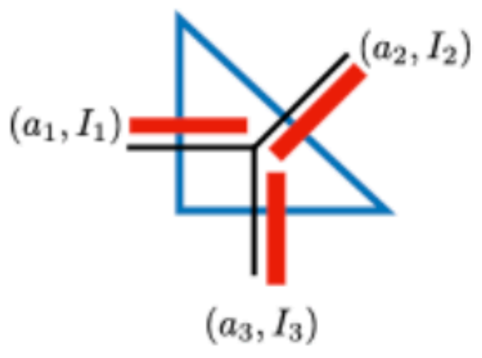
$= \sum_{y} F^{abc}_{d;x,y}$ 

(where a,b,c,d,x,y are the labels of the families of primary representations of the)

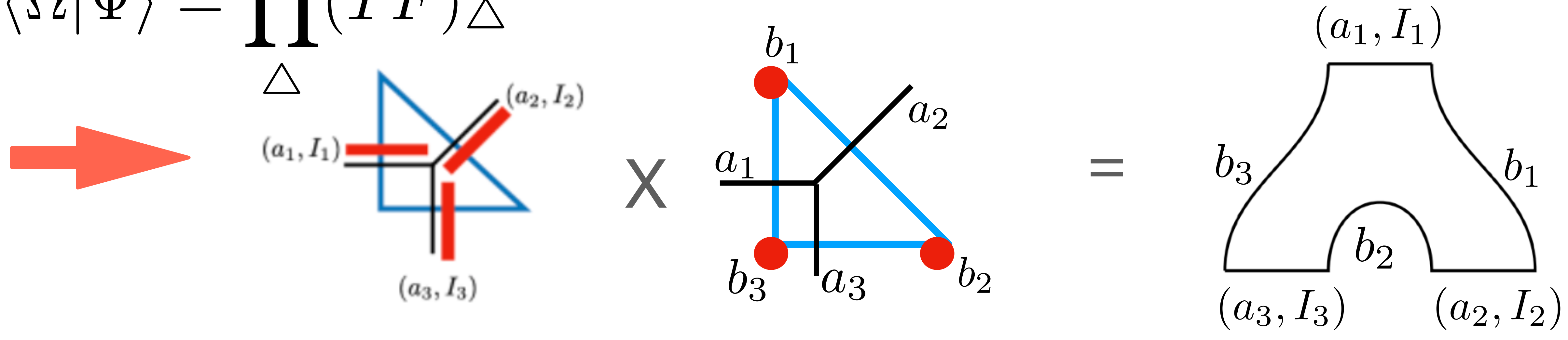
Therefore ~~~

Fixed Point Boundary Corresponding to CFT?

$\langle\Omega| = \cdots$  $\cdots$

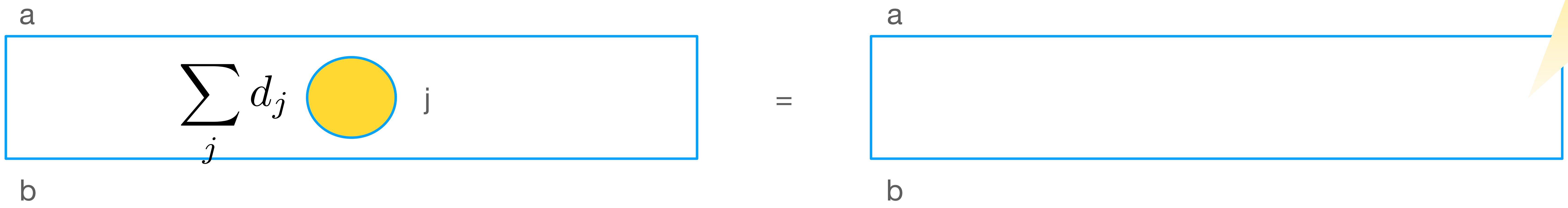
$T^{a_1 a_2 a_3}_{I_1 I_2 I_3} =$ 

$$\langle\Omega|\Psi\rangle = \prod_{\triangle} (TF)_{\triangle}$$



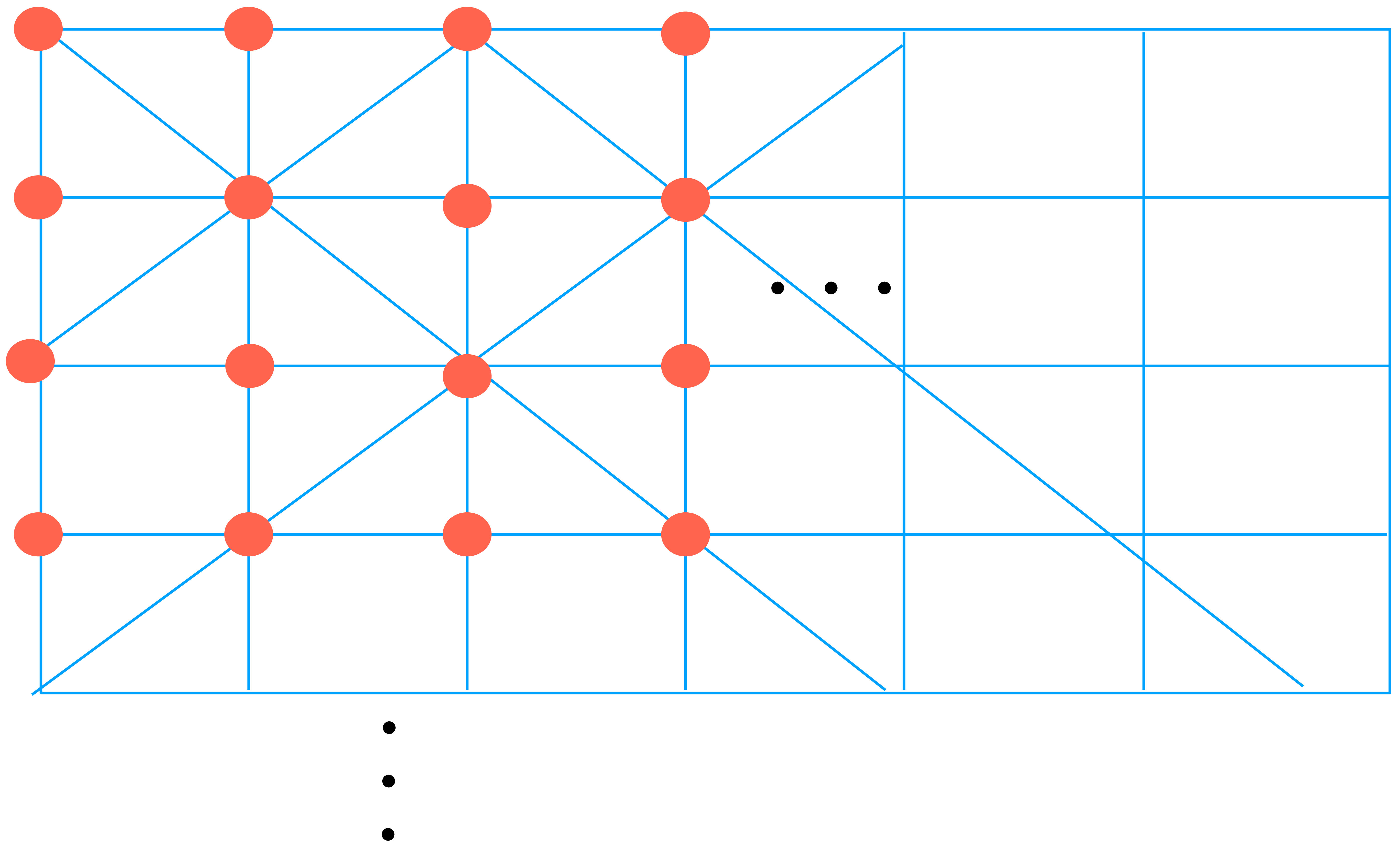
is a fixed point of the RG operator if T is basically the three point function between chiral operators belonging to the family a1,a2,a3. The I1, I2, I3 auxiliary indices would then correspond labels of the descendent of the given chiral family subjected to the condition :

We checked this for Ising model numerically and find up to 4 sigfig accuracy with a truncated bond dimensions



The Turaev -Viro TQFT would dictate that the conformal boundary condition is ``closeable” in exactly the same way as the proposed “entanglement brane boundary condition” . LYH, Wong 2020

Factorising CFT partition function



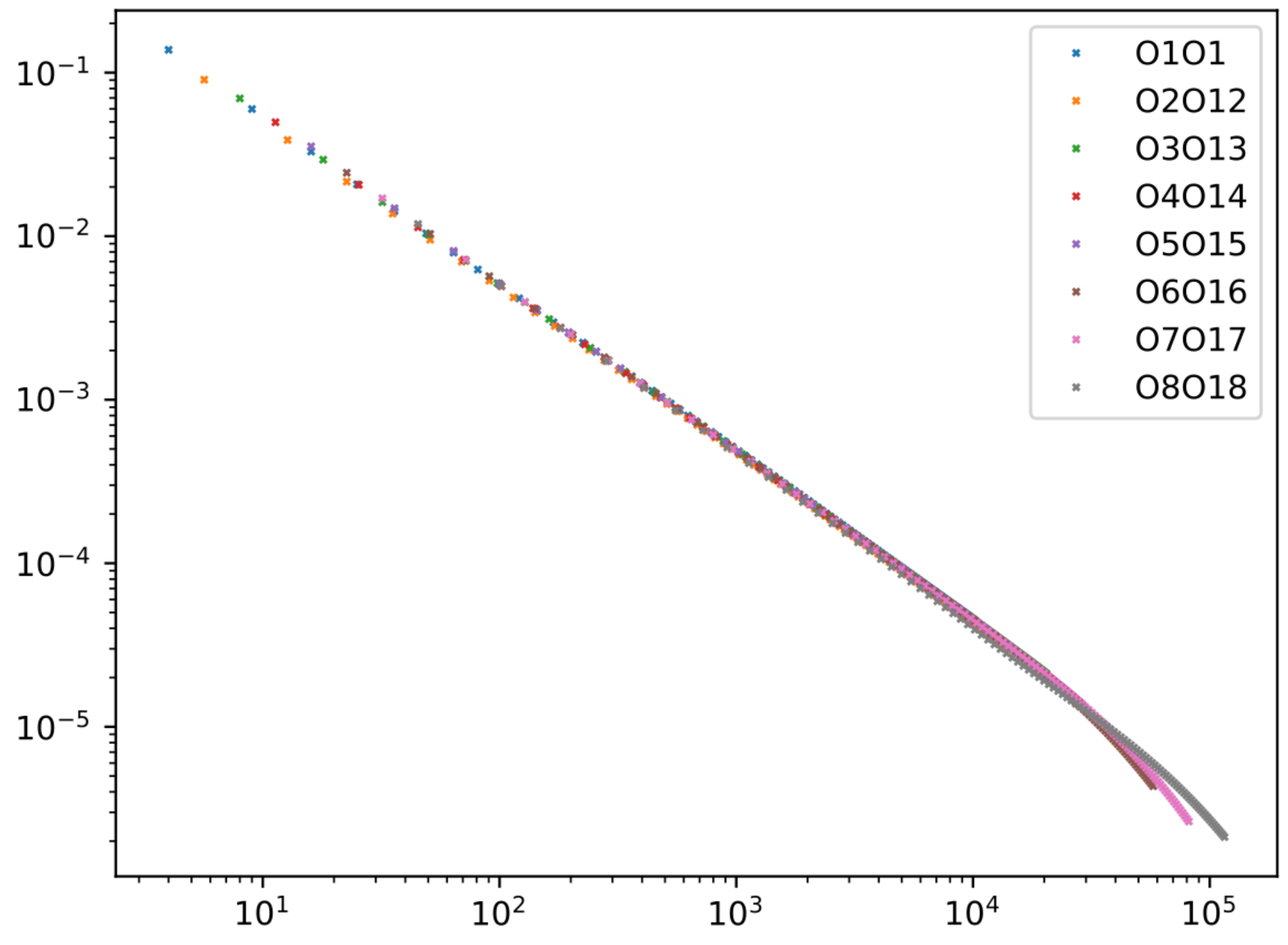
The Turaev-Viro TQFT construction can describe continuous field theory because the fixed point tensors can be sewed together and finally the holes can be contracted — the connection between lattice construction and continuous theory can be understood in this light. This picture can be generalised to non-diagonal RCFT.

Is this kind of holography the same kind as AdS/CFT?

Minimal models and Levin Wen models — and holography

Preliminary result for a bulk boundary propagator:

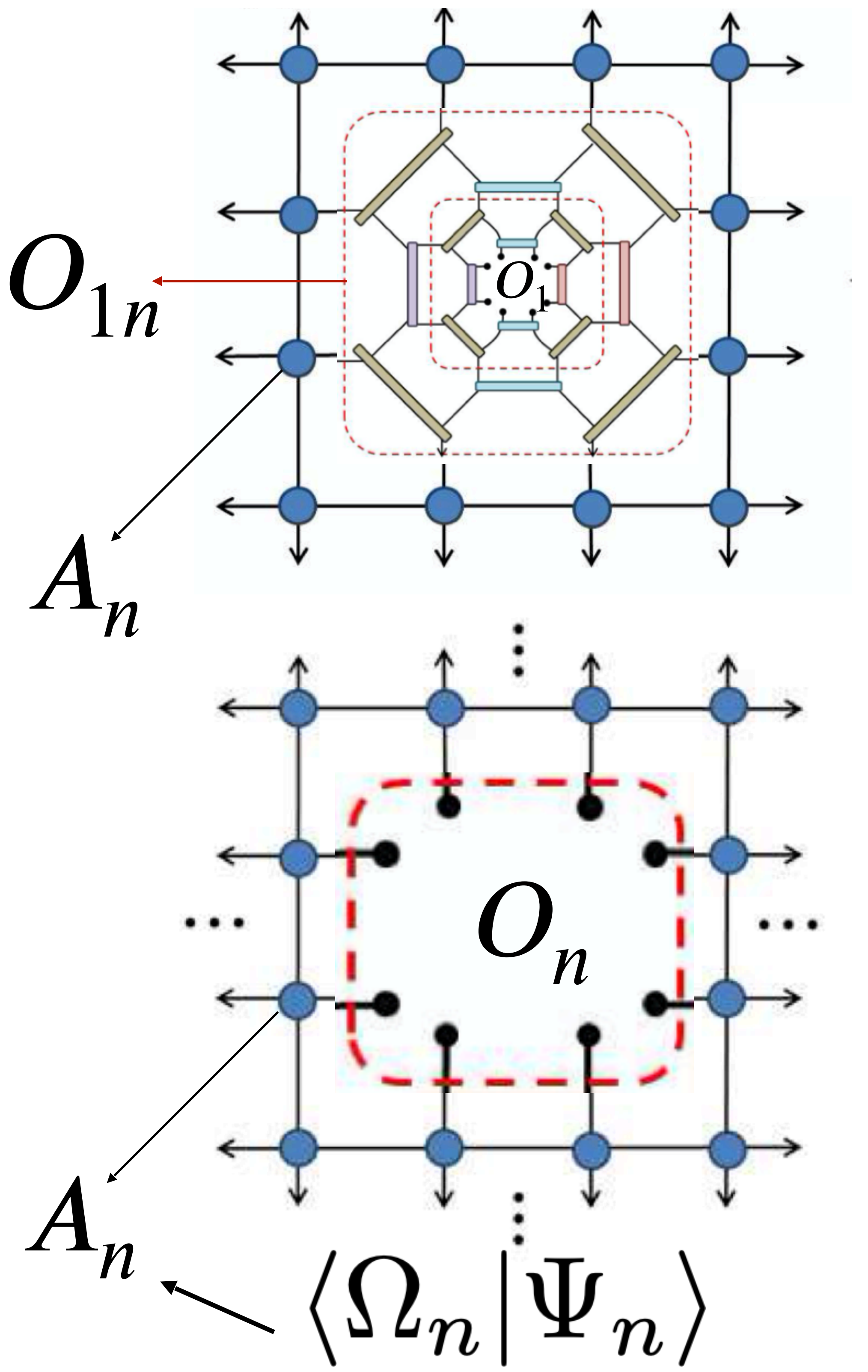
$< O_1 O_{1n} > vs . z_n x_n^2$



$$< O_n O_{1n} > \sim \left(\frac{z_n}{x_1^2 + z_n^2}\right)^{\Delta} = \left(\frac{1}{z_n(x_n^2+1)}\right)^{\Delta} \qquad x_1 = z_n x_n,\, z_n = \left(\sqrt{2}\right)^{n-1}$$

er.. looks like the bulk insertion didn't recover the right descendants, but only the primary! — this should be an issue of correctly dealing with sub-AdS locality in the network .

Picture courtesy Vidal et al 2014



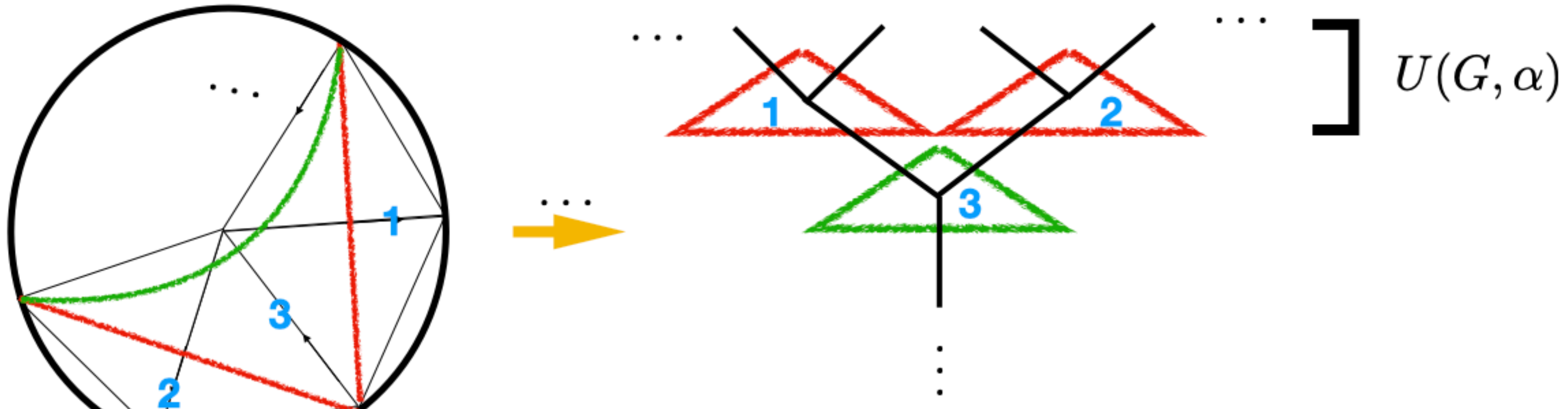
Bar, Can, Carroll, Chatwin-Davies, Hunter-Jones, Pollack, Remmen, 2015

Applying to large C 2d CFTs?

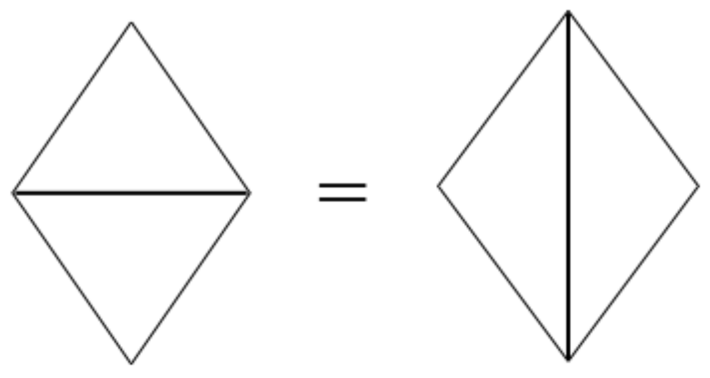
- There are universal forms of boundary operator OPE in the large C limit. They are closely related to those of Liouville theory, which are basically given by the 6j symbols of the quantum group U_q(sl2) Numasawa, Tsiares 2022; Teschner 2000; Zamolodchikov^2 2001 ……..
- Techmuller TQFT is a sub-sector of AdS3 gravity — and this is closely related to U_q(sl2) Ponsot, Teschner 2001; Verlinde 1990; More Recently Collier, Eberhardt, Zhang 2023; See also Mertens, Simon, Wong 2022;
- Our framework appears to suggest an alternative Turaev- Viro construction of the Techmuller TQFT that is related but not identical to the existing literature (work in progress)— but the main lesson is that the holographic relation between 2d CFT and the 3d TQFT here appears to coincide with the usual notion of AdS3/CFT2.

Generalizations to arbitrary dimensions,
including 3+1 d bulk 2+1 d boundary
or 1+1 d bulk and 0+1 d boundary

2D Dijkgraaf- Witten Theory as bulk and 1D G-symmetric TQFT



These boundary conditions produce 0+1 dimensional TQFT with global symmetry G.

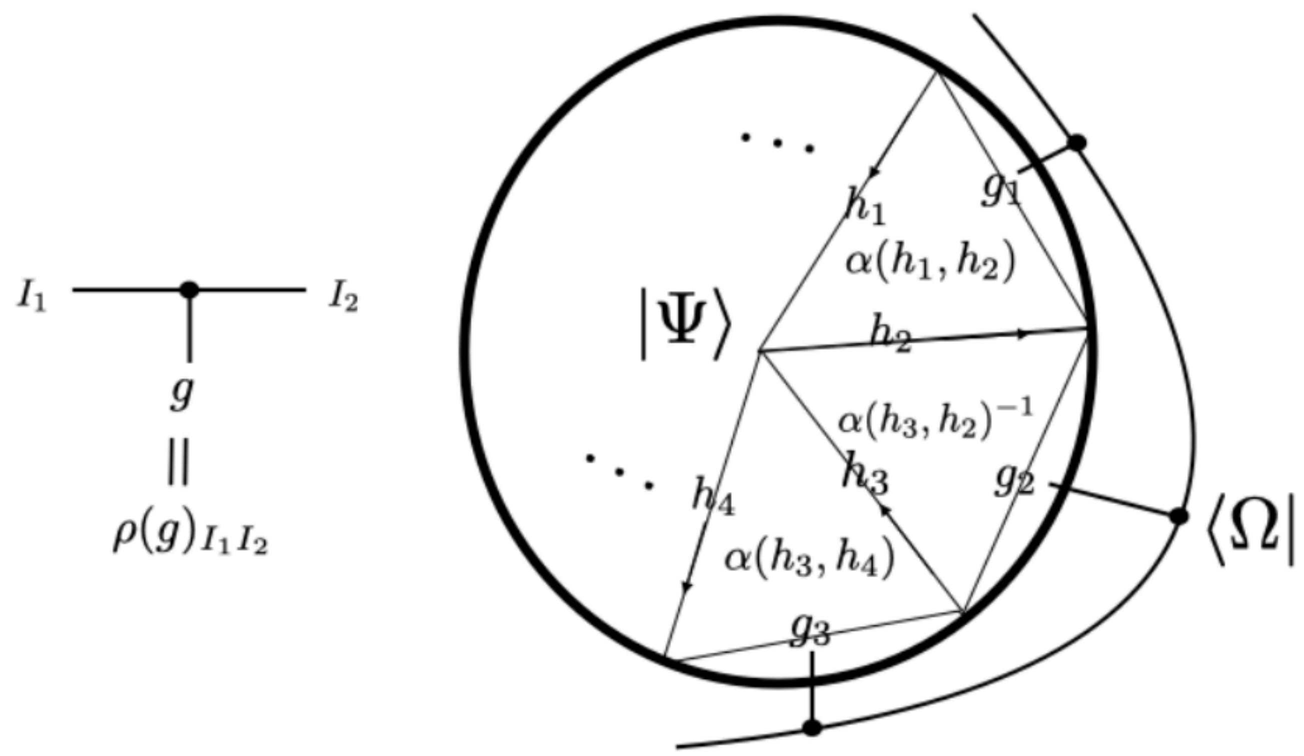


$$\frac{\alpha(g_1, g_2)\alpha(g_1 g_2, g_3)}{\alpha(g_1, g_2 g_3)\alpha(g_2, g_3)} = 1.$$

General Fixed point boundaries can be decompose as (projective) representations of the group

$$\langle \Omega | = \sum_{\{g\}} \mathrm{tr}(\cdots \rho(g_i)\rho(g_{i+1})\rho(g_{i+2})\cdots)\langle \cdots g_i, g_{i+1}, g_{i+2} \cdots |,$$

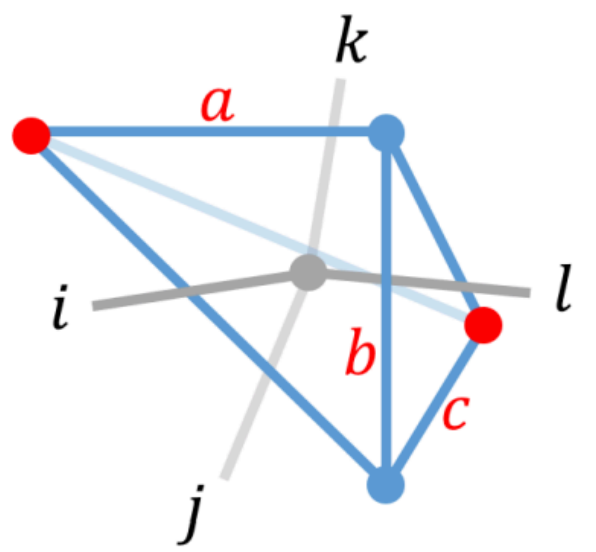
$$(\rho(g_1)\rho(g_2))_{ac} \equiv \sum_b \rho(g_1)_{ab}\rho(g_2)_{bc} = \rho(g_1 g_2)_{ac}, \qquad (8)$$

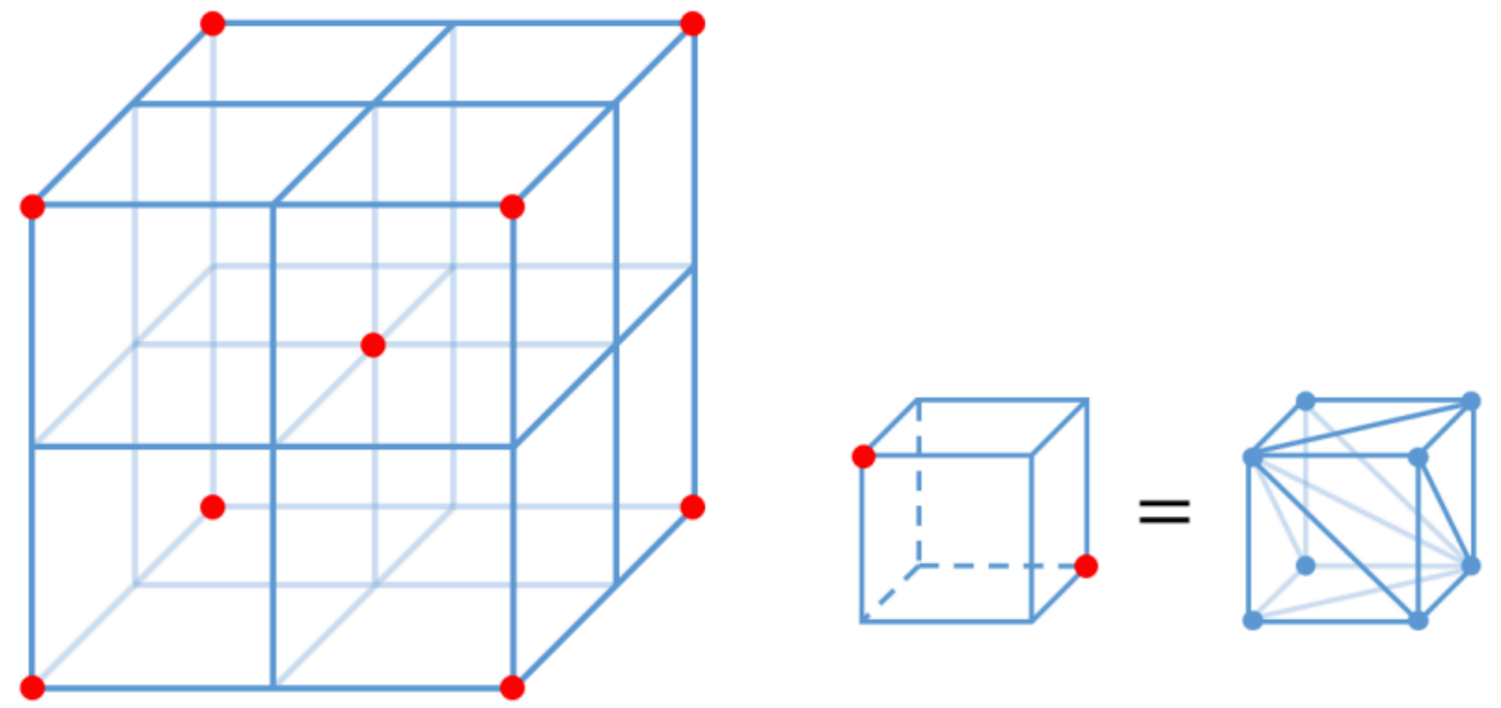


3+1 D TQFT and 2+1 D CFT

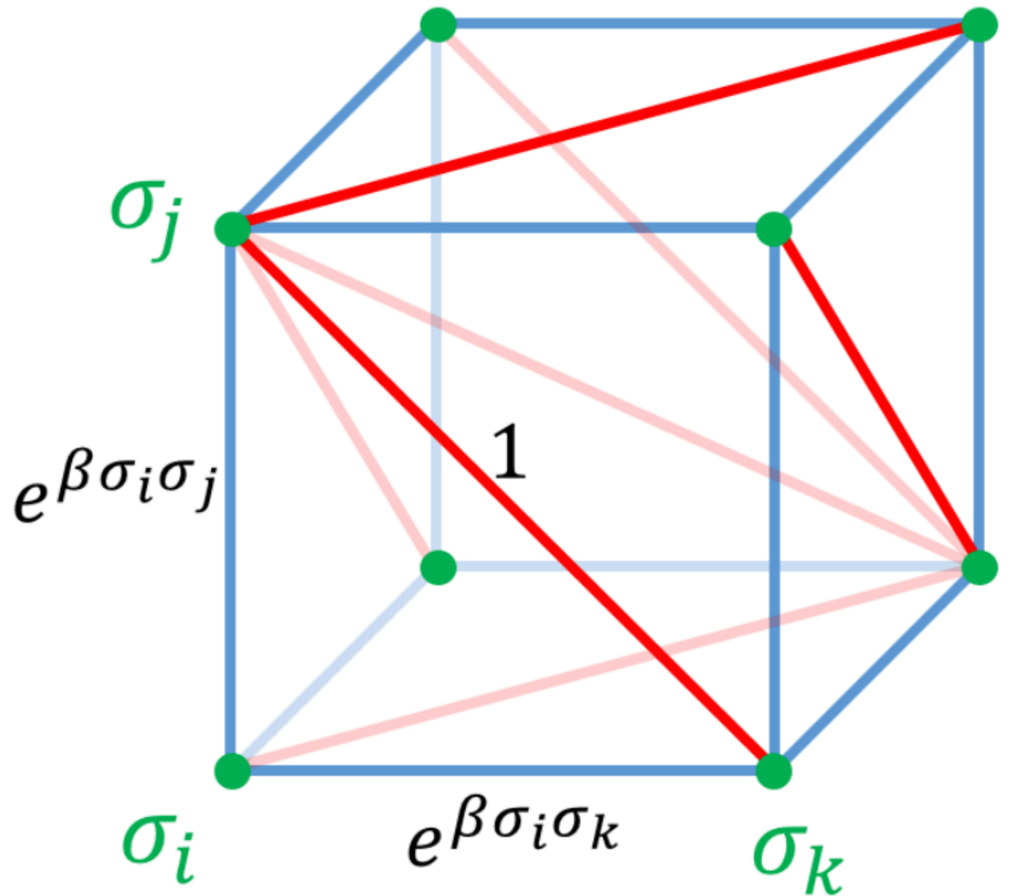
Example: 4D Z2 Dijkgraaf - Witten Theory and the Ising model

- Tensor Network Representation of the ground state wave-function of Dijkgraaf-Witten theory:





Boundary conditions for the Ising model:



3+1D DW Theories and RG operator

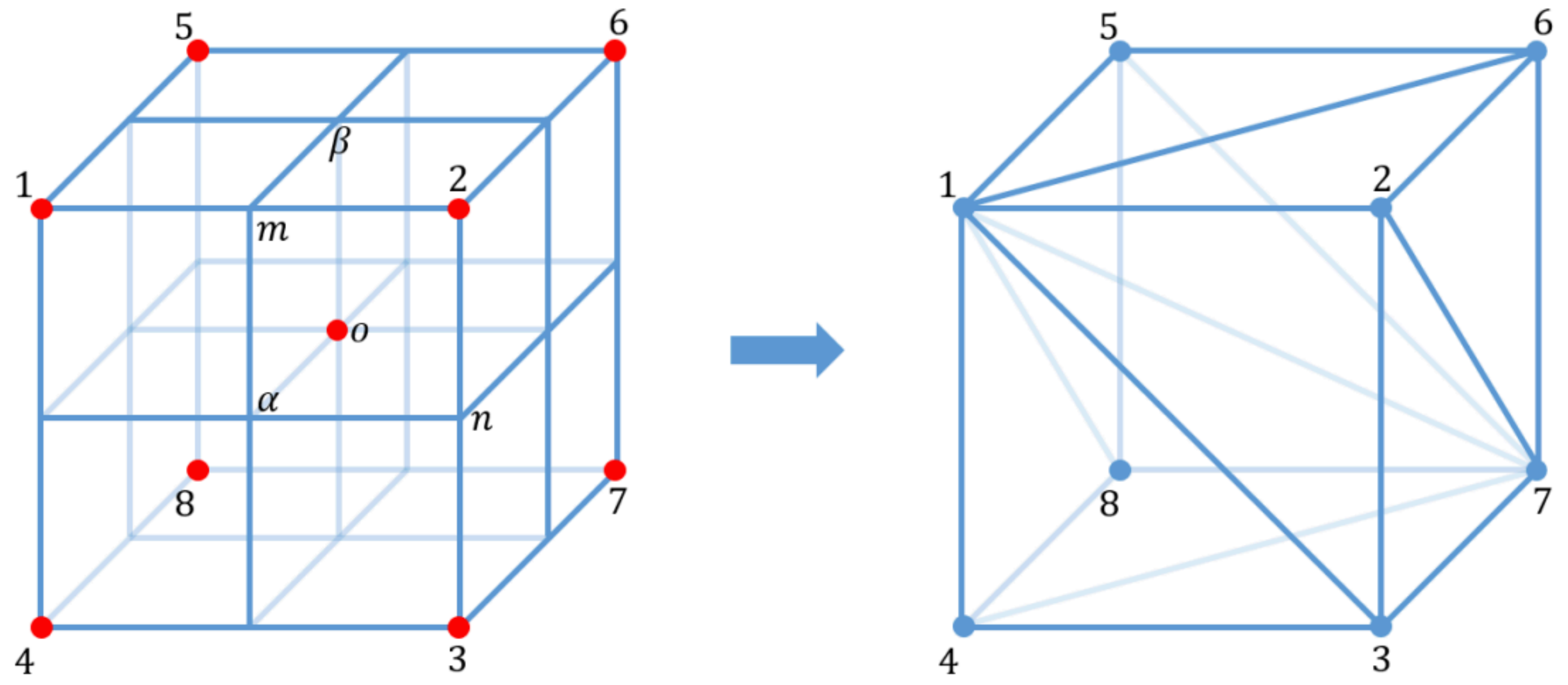


FIG. 17. Coarse grain the $2 \times 2 \times 2$ cube into $1 \times 1 \times 1$ cube.

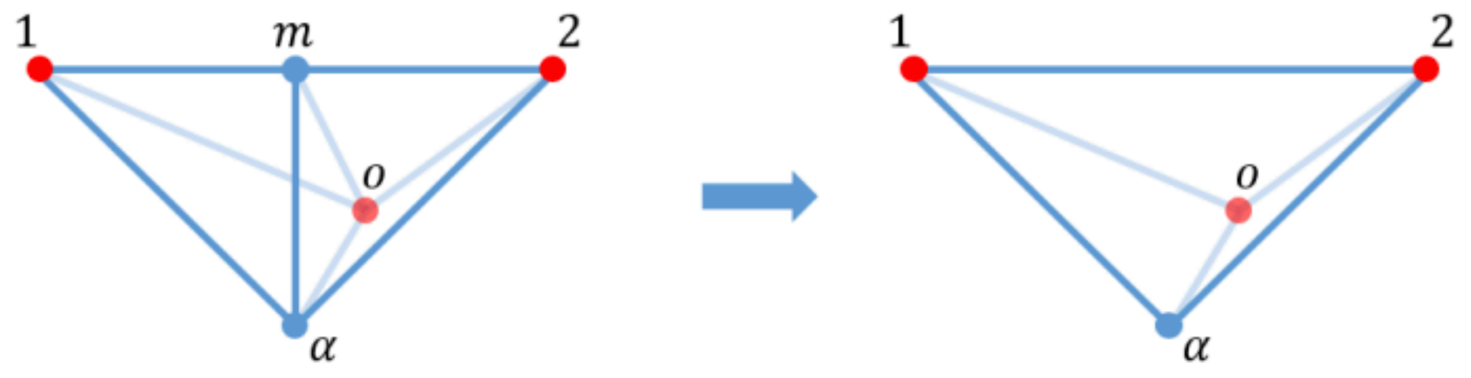


FIG. 18. Combine $1m\alpha o$ and $2m\alpha o$ to get a bigger tetrahedron $12\alpha o$.

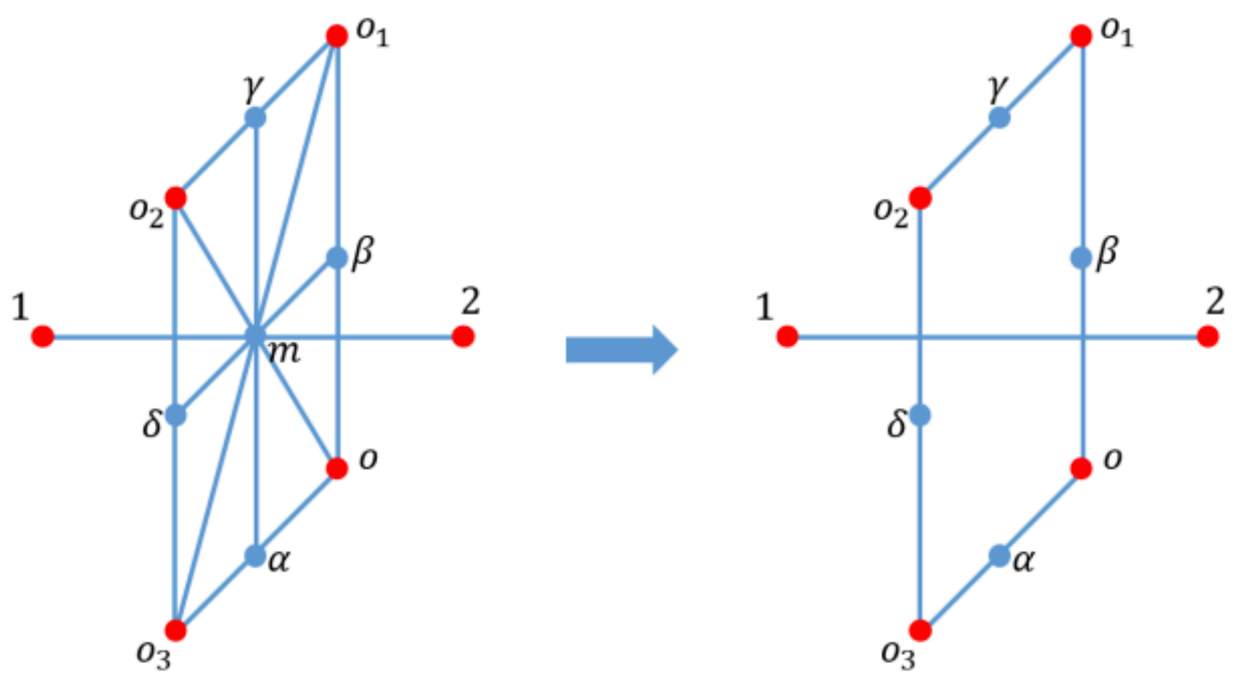


FIG. 19. The first step is to eliminate the vertices like m and to obtain edges like 12. To avoid clutter, we omit the edges connecting $1, 2$ with $\alpha, \beta, \gamma, \delta, o, o_1, o_2, o_3$. On the left hand side, there is a vertex m, and there are 16 small tetrahedron $1m\alpha o, 1m\beta o, 1m\beta o_1, \ldots, 2m\alpha o, 2m\beta o, 2m\beta o_1 \ldots$. On the right hand side, there is no m, and there are 8 bigger tetrahedron $12\alpha o, 12\beta o, 12\beta o_1, \ldots$. They are on the two boundaries of a 4D body which consists of eight 4-simplices $12m\alpha o$, $12m\beta o$, $12m\beta o_1$, $12m\gamma o_1$, $12m\gamma o_2$, $12m\delta o_2$, $12m\delta o_3$, $12m\alpha o_3$.

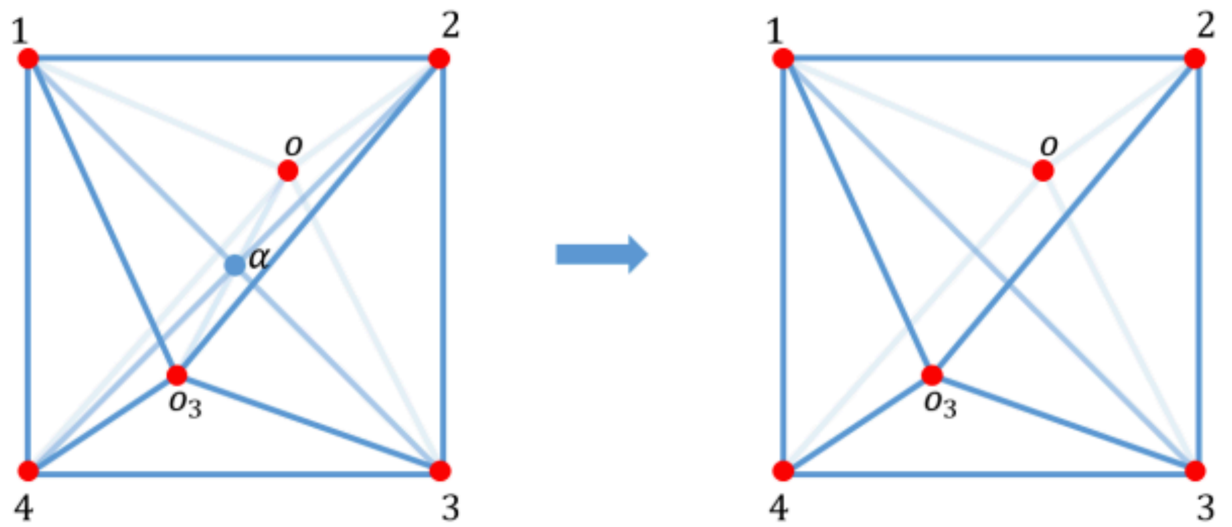


FIG. 21. The second step is to eliminate the vertices like α. Here we choose to connect vertices $1,3$ since in the target coarse grained cubic there is a 13 edge as shown in figure 17. On the left hand side, there is a vertex α, and there are 8 tetrahedra $12\alpha o, 23\alpha o, 34\alpha o, 41\alpha o, 12\alpha o_3, 23\alpha o_3, 34\alpha o_3, 41\alpha o_3$. On the right hand side, there is no α, and there are 4 tetrahedra $123o, 341o, 123o_3, 341o_3$. They are on the two boundaries of a 4D body which consists of four 4-simplices $123\alpha o, 123\alpha o_3, 341\alpha o_3, 341\alpha o$.

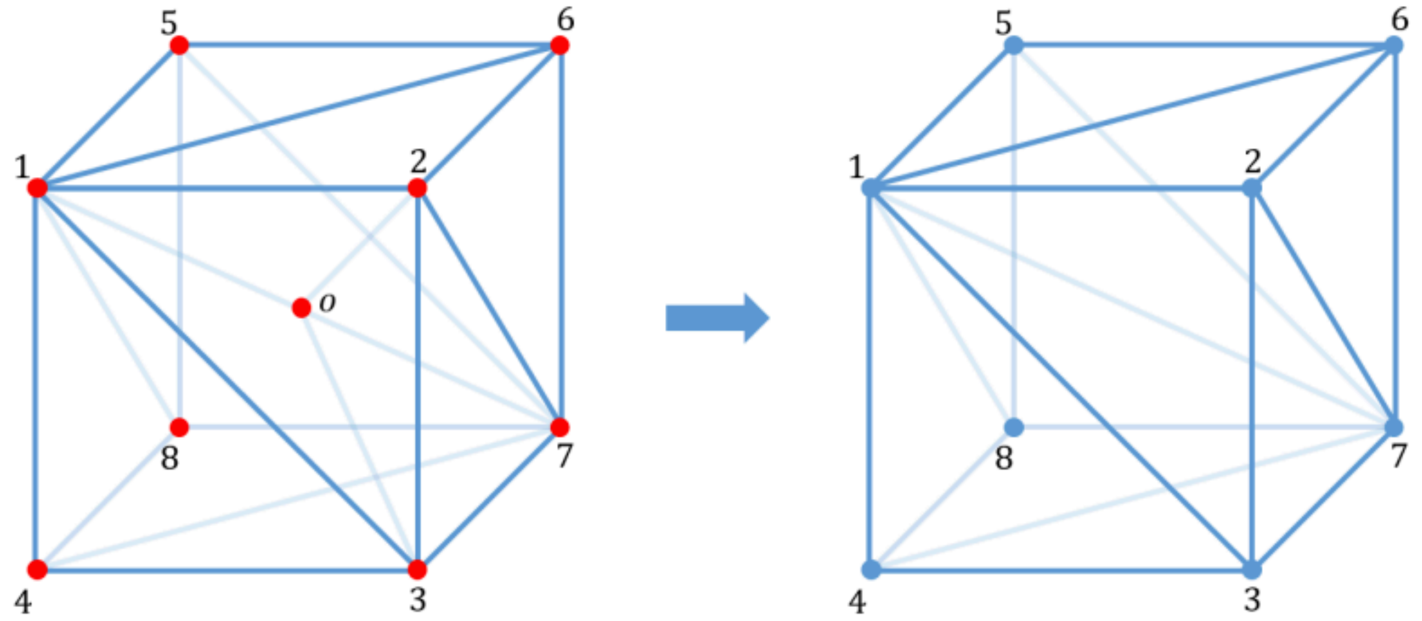


FIG. 22. The third step is to eliminate the vertex o and to obtain the edge 17. To avoid clutter, we only show some of the edges connecting o with $1,2,3,4,5,6,7,8$. On the left hand side, there is a vertex o, and there are 12 tetrahedra $123o, 143o$, $237o, 267o$, $126o, 156o$, $148o, 158o$, $487o, 437o$, $567o, 587o$. On the right hand side, there is no o, and there are 6 tetrahedra 1237, 1267, 1567, 1587, 1487, 1437. They are on the two boundaries of a 4D body which consists of six 4-simplices $1237o, 1267o, 1567o, 1587o, 1487o, 1437o$. Combining $123o$ and $237o$ to get the tetrahedron 1237 can be read off from this figure.

Topological solutions = Higher Frobenius Algebra

Wang, Li, Hu, Wan, *JHEP* 10 (2018) 114,   Zhao, Lou, Zhang, Hung, Kong, Tian, 2208.07865

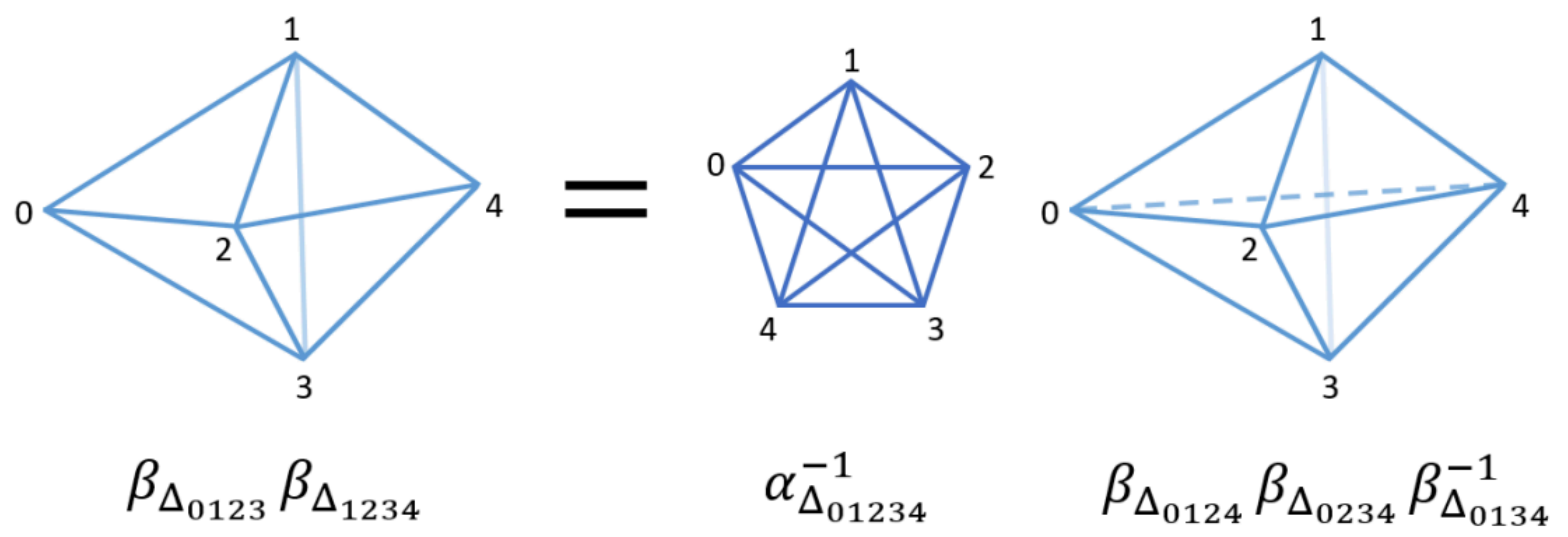


FIG. 23. There are 2 tetrahedra on the left and 3 tetrahedra on the right corresponding to two different triangulations of the boundary. We have $\beta_{\Delta_{0123}}\beta_{\Delta_{1234}} = \alpha^{-1}_{\Delta_{01234}}\beta_{\Delta_{0124}}\beta_{\Delta_{0234}}\beta^{-1}_{\Delta_{0134}}$. The powers of -1 are related to the orientations.

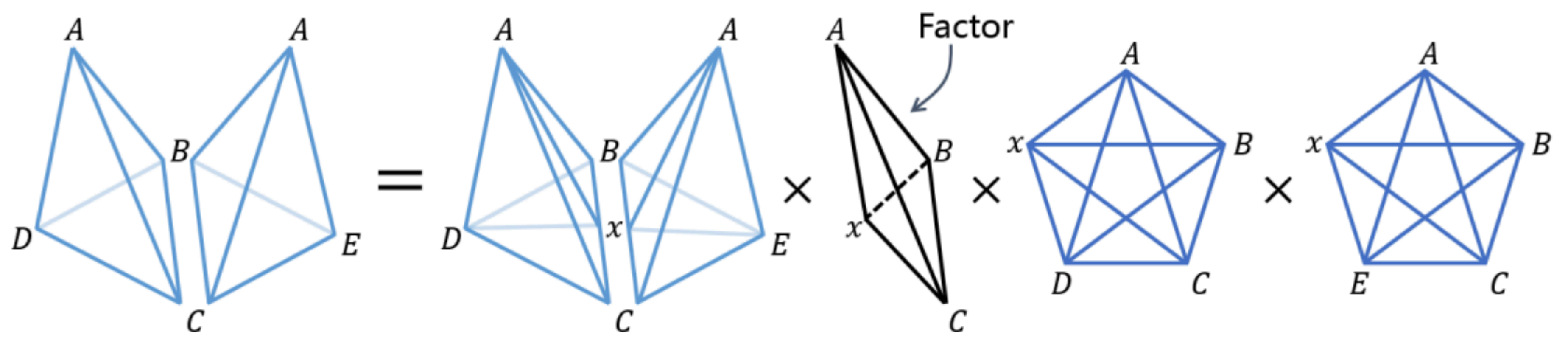


FIG. 24. The blue tetraheron corresponds to the boundary factors β. The pair of 4-simplices on the right hand side corresponds to the 4-cocycles of the DW theory. The black tetrahedron referred to as a "factor" is the analogue of a bubble that is contracted. The equality is based on absorbing this black tetrahedron and is thus the analogue of separability in 2+1 dimensional topological order. In the current model however the factor is equal to unity.

Search for critical point between electric and magnetic boundaries:

Using the same method — we can find the critical temperature of 2+1 D ising model as a phase transition between two of the three Higher Frobenius algebra of the 4D toric code.
— to appear soon

3D Ising: bond=1, transition temperature: 0.27-0.28

various values of D. For a discussion see the tex

D	β_c
Ising	0.22165463(8)
0.641	0.38567122(5)
0.655	0.387721735(25)
$\ln 2 = 0.69314718...$	0.39342239(8)
1.15	0.4756110(2)
1.5	0.5575303(10)

Outlook

- Exact eigenstates of RG operator that describe 2d CFT — they are open conformal blocks…. (Work in Progress)

- Generalization to higher dimensions? What is the counter part of these conformal blocks say in 3+1 d?

- Techmuller TQFT — it admits a Turaev-Viro construction — the construction in this paper would appear to work (?) This gives the usual AdS3-CFT2 now in a tensor network construction (?)

- Does that work in higher dimensions? How to construct gravitational operators in holographic bulk?

Thank you very much!