An Alternative Approach to Covariance Propagation (and a Generalized Gaspari-Cohn Correlation Function)

Shay Gilpin[1], Tomoko Matsuo[2,1], Stephen E. Cohn[3]
[1]Department of Applied Mathematics, University of Colorado, Boulder, [2]Smead Aerospace Engineering Sciences, University of Colorado, Boulder, [3]Global Modeling and Assimilation Office, NASA Goddard Space Flight Center, Greenbelt, Maryland

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An Alternative to Standard Covariance Propagation

Covariance propagation, e.g.,

$$\boldsymbol{P}_{k+1} = \boldsymbol{M}_{k+1,k}(\boldsymbol{M}_{k+1,k}\boldsymbol{P}_k)^T + \boldsymbol{Q}_k$$

Issues with Covariance Propagation:

- Inaccurate variance propagation
- Computational expense

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"Local covariance evolution" (Cohn, 1993) ⇒ Parametric Kalman Filter (e.g., Pannekoucke et al. 2016)

Local Covariance Evolution, 1D

Consider covariances $P = P(x_1, x_2, t)$ associated with states $q = q(x, t)$ on the unit circle (S_1^1),

$$q_t + vq_x + bq = 0, \qquad P_t + v_1 P_{x_1} + v_2 P_{x_2} + (b_1 + b_2)P = 0,$$
$$q(x, t_0) = q_0(x) \qquad P(x_1, x_2, t_0) = P_0(x_1, x_2)$$

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Variance Equation:

$$\sigma_t^2 + v\sigma_x^2 + 2b\sigma^2 = 0,$$
$$\sigma^2(x, t_0) = \sigma_0^2(x)$$

Correlation Length Equation:

$$L_t + vL_x - v_x L = 0,$$
$$L(x, t_0) = L_0(x)$$

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Variance Equation:	Correlation Length Equation:
$\sigma_t^2 + v\sigma_x^2 + 2b\sigma^2 = 0,$	$L_t + vL_x - v_x L = 0,$
$\sigma^2(x, t_0) = \sigma_0^2(x)$	$L(x, t_0) = L_0(x)$

1. Evolve σ^2 and L from initial condition $P_0(x_1, x_2) = \sigma_0(x_1)C_0(x_1, x_2)\sigma_0(x_2)$.
2. Approximate $P(x_1, x_2, t) = \sigma(x_1, t)C(x_1, x_2, t)\sigma(x, t)$ with evolved σ^2 and L using a *parametric correlation function*.

The Gaspari and Cohn (1999) Correlation Function

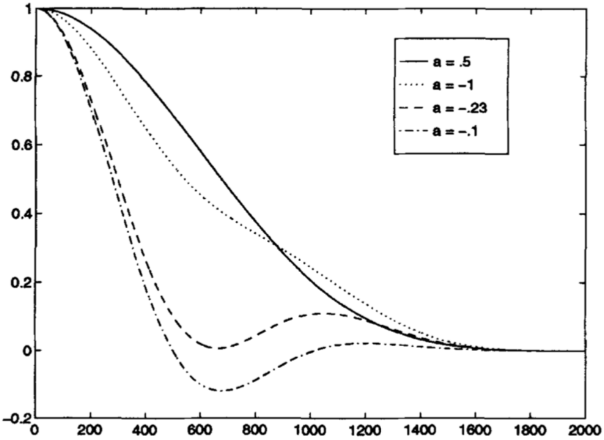

Figure 7. The function $C_0(z, a, c)$ of the example in section 4(c) for $c = 1000$ km and various values of a. See text for explanation.

Figure 1: Figure 7 from Gaspari and Cohn (1999). The function $C_0(z, a, c)$ is the general form of the compactly-supported, fifth-order, piecewise rational correlation function derived in their Sec. 4(c). **Typically, a $= 1/2$ (solid black)**.

The Generalized Gaspari-Cohn (GenGC) Correlation Function

Correlation length for the compactly-supported, piecewise rational:

$$L = c \left(\frac{3(22a^2 + 3a + 1)}{40(8a^2 - 2a + 1)} \right)^{1/2}, \quad a = 1/2 \Rightarrow L = \sqrt{0.3}c$$

Need to generalize this correlation function to allow for *variable L*.

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Generalized Gaspari-Cohn

Allow $a = a_k$ and $c = c_k$ to vary over the spatial index k.

Now $L = L_k$ can vary!

Construction of the GenGC Correlation Function

For fixed $c_k > 0$ and $a_k \in \mathbb{R}$, define the following compactly-supported, radially symmetric functions $\mathbb{R}^3$:

$$h_k(\boldsymbol{r}; a_k, c_k) = \begin{cases} (2(a_k - 1)||\boldsymbol{r}||/c_k + 1)n_k, & 0 \leq ||\boldsymbol{r}|| \leq c_k/2, \\ 2a_k n_k(1 - ||\boldsymbol{r}||/c_k), & c_k/2 \leq ||\boldsymbol{r}|| \leq c_k, \\ 0, & c_k \leq ||\boldsymbol{r}||, \end{cases}$$

with $n_k = (44a_k^2 + 6a_k + 2)^{-1/2}$.

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For each fixed $k, \ell = 1, 2, ..., m$, GenGC is defined by the convolution,

$$B_{k\ell}(\boldsymbol{r}, \boldsymbol{s}) = (h_k * h_\ell)(\boldsymbol{r} - \boldsymbol{s}) = \int_{\mathbb{R}^3} h_k(\boldsymbol{v}) h_\ell(\boldsymbol{r} - \boldsymbol{s} - \boldsymbol{v}) d\boldsymbol{v}.$$

Generalized Gaspari-Cohn (GenGC), 1D Example

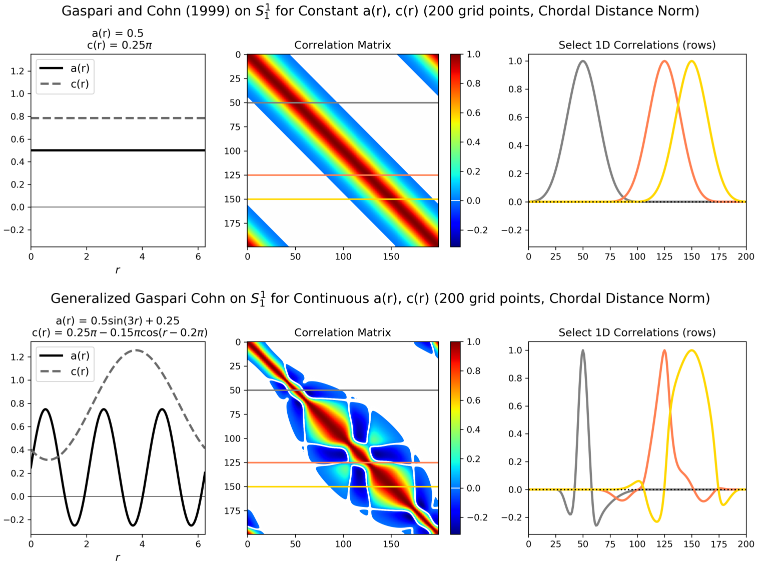

Figure 2: Correlations constructed on the unit circle (S_1^1) for constant a and c (top row) and spatially-varying, continuous a and c (bottom row). White regions in the correlation matrix (middle column) correspond to correlations between -0.003 and 0.003.

Demonstration

Correlations from Direct, Full Rank Covariance Propagation, $t_f = T$

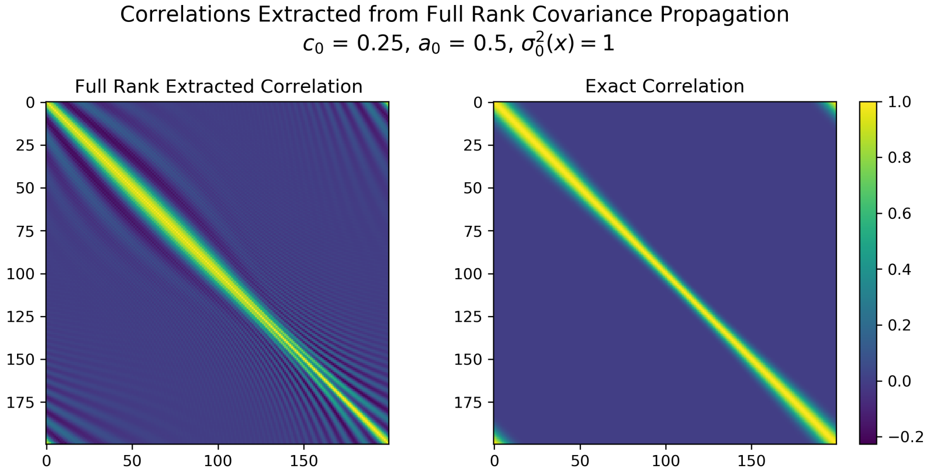

Figure 3: *Adapted from Gilpin et al. (2022):* Correlation matrices extracted from covariances evolved from the Gaspari-Cohn correlation function for $a_0 = 1/2$, $c_0 = 0.25$, and $\sigma_0^2 = 1$, evolved in time up to slightly after a full time period. Errors in full rank propagation are between -0.713 and 0.450

Current Work, Correlation Reconstruction with GenGC, $t_f = T$

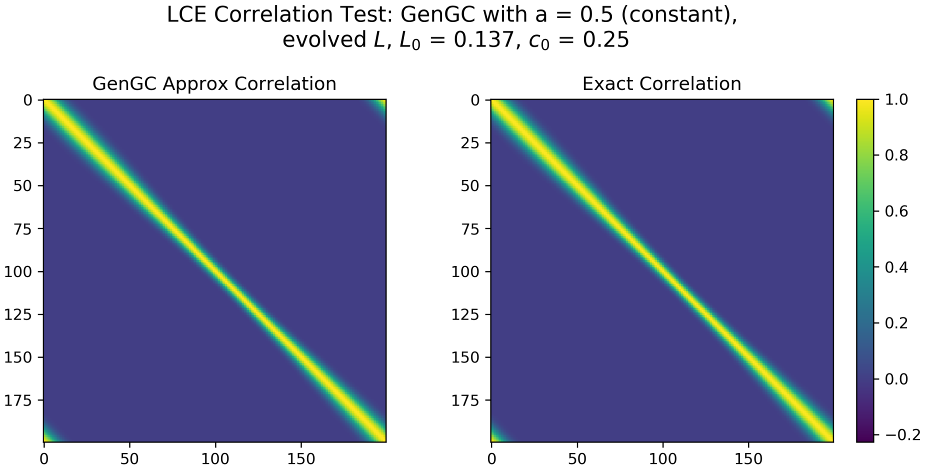

Figure 4: Left: correlation matrix approximated with GenGC using evolved correlation lengths L, $a = 0.5$ constant, $c_0 = 0.25$. Right: the exact correlation matrix. Errors in the GenGC approximation are between -0.0009 and 0.0008.

Correlations from Direct, Full Rank Covariance Propagation, $t_f = T/2$

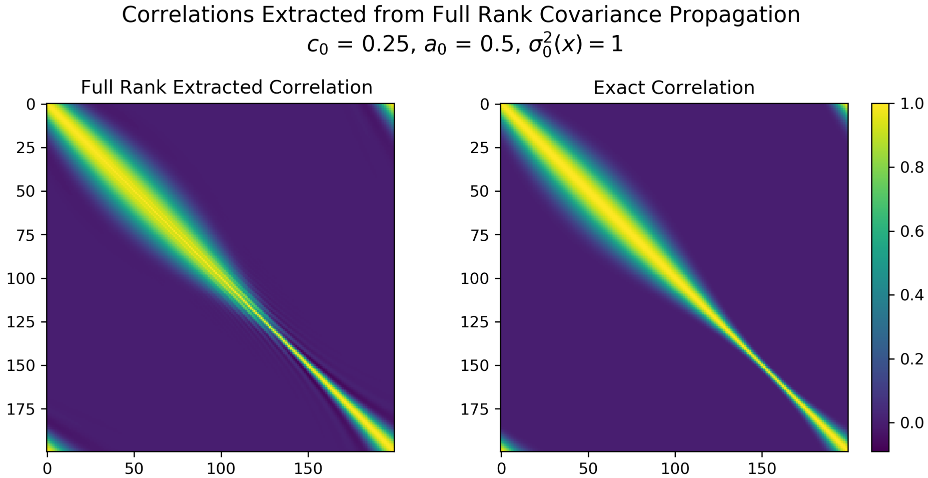

Figure 5: *Adapted from Gilpin et al. (2022):* Correlation matrices extracted from covariances evolved from the Gaspari-Cohn correlation function for $a_0 = 1/2$, $c_0 = 0.25$, and $\sigma_0^2 = 1$, evolved in time up to slightly after half a time period. Errors in full rank propagation are between -0.557 and 0.339

Current Work, Correlation Reconstruction with GenGC, $t_f = T/2$

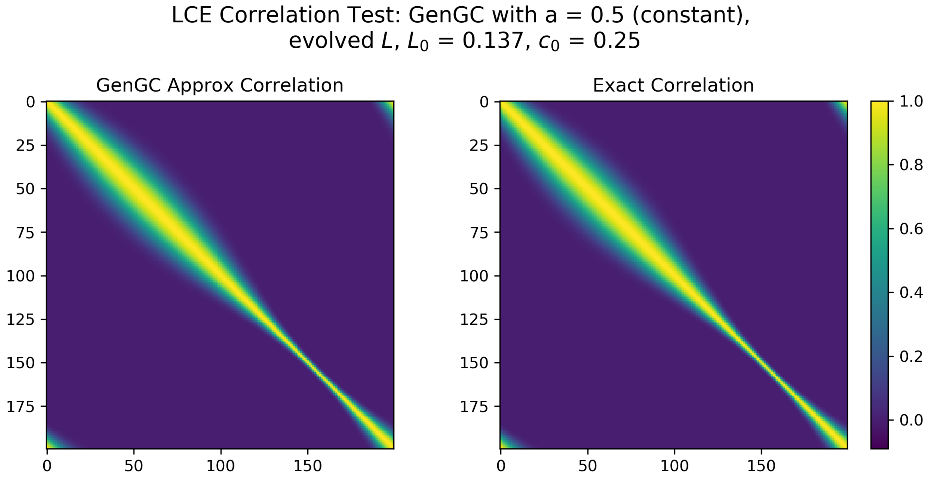

Figure 6: Left: correlation matrix approximated with GenGC using evolved correlation lengths L, $a = 0.5$ constant, $c_0 = 0.25$. Right: the exact correlation matrix. Errors in the GenGC approximation are between -0.0124 and 0.0107.

Concluding Remarks and Further Investigation

- Local covariance evolution is an alternative means of mitigating problems associated with covariance propagation.

- The Generalized Gaspari-Cohn correlation function has several additional applications (e.g., covariance modeling, localization, coupled data assimilation).

Acknowledgements and Questions

For more information or further discussion, contact Shay at **shay.gilpin@colorado.edu**

Relevant work:

Gilpin, Matsuo, and Cohn, (2023): *A generalized, compactly-supported correlation function for data assimilation applications*, submitted to QJRMS.

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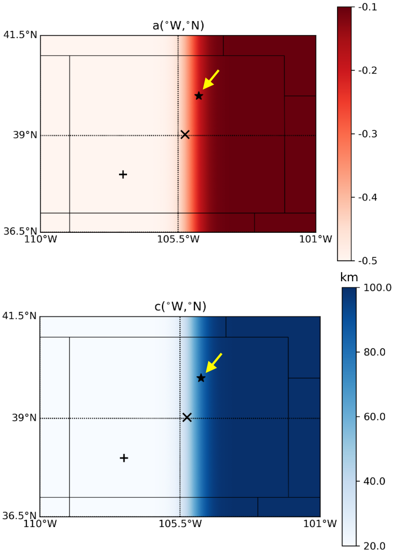
Extra Slides

Generalized Gaspari-Cohn (GenGC), 2D Example



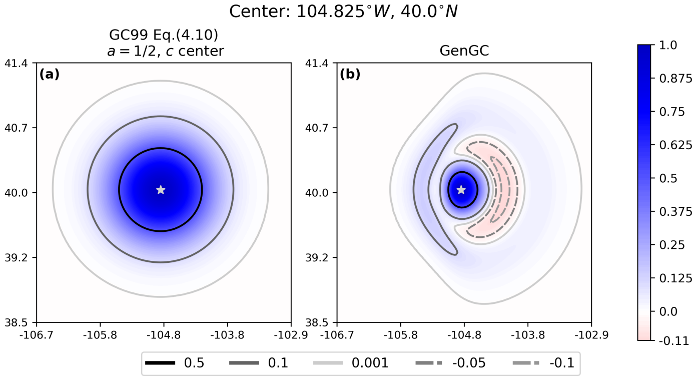


Figure 7: Left: Background a and c fields as functions of longitude and latitude over Colorado. Right: 2D correlations with respect to $\star$ for the given a and c fields, plotted using a Mercator projection.