A universal algebra approach to free objects

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Background and introduction

Unanswered 'categorical' questions asked at workshop on ordered Banach algebras in Leiden in 2014

- Is there a unitisation of a vector lattice algebra?
- Is there a sensible notion of a free Banach lattice algebra over a set?
- Included in a list of problems published by Wickstead in 2017

Today

- Algebraic and analytic part
- Analytic part: joint with Walt van Amstel (Pretoria); partly also with Mitchell Taylor
- Today's message: many such problems are easy to answer
- For existence: *just need one single theorem from universal algebra* 'which the Positivity community seemed to have forgotten'
- Unitisation of vector lattice algebra: exists
- Free Banach lattice algebra over a non-empty set: exists only in a certain sense

Outline

- Categories and free objects
- Examples of free objects in algebraic categories
- Universal algebra: Part I
- Familiar (ordered) structures are abstract algebras with relations
- Universal algebra: Part II (the 'forgotten theorem')
- Existence of free objects in several (ordered) algebraic categories
- Examples of 'concrete models' of free objects in algebraic categories
- Free objects in analytic categories

Guiding examples in the algebraic part: free unital vector lattice algebra over a set and over a vector space where 'familiar constructions will not help you'

Some algebraic categories

Algebraic categories

- Set: the sets, with set maps
- Lat: the lattices, with lattice homomorphisms
- VS: the (real) vector spaces, with linear maps
- VL: the vector lattices, with vector lattice homomorphisms
- VLA: the vector lattice algebras, with vector lattice algebra homomorphisms (algebra homomorphism and vector lattice homomorphism)
- VLA^1: the unital vector lattice algebras, with the unital vector lattice algebra homomorphisms
- VLA^{1+}: the unital vector lattice algebras with a positive identity element, with the unital vector lattice algebra homomorphisms

Free objects

Definition

Suppose that Cat_1 and Cat_2 are categories, and that $G\colon \mathsf{Cat}_2 \mapsto \mathsf{Cat}_1$ is a faithful functor.[a] Take an object O_1 of Cat_1. Then a *free object over O_1 of Cat_2 with respect to G* is a pair $(j, \mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1])$, where $\mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]$ is an object of Cat_2 and $j : O_1 \to \mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]$ is a morphism of Cat_1, with the property that, for every object O_2 of Cat_2 and every morphism $\varphi : O_1 \to O_2$ of Cat_1, there exists a unique morphism $\overline{\varphi} : \mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1] \to O_2$ of Cat_2 such that the diagram

$$\begin{array}{ccc} O_1 & \xrightarrow{\;j\;} & G\left(\mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]\right) \\ & \underset{\varphi}{\searrow} & \Big\downarrow G(\overline{\varphi}) \\ & & G(O_2) \end{array}$$

in Cat_1 is commutative.

[a] Recall that G is *faithful* when the associated map $G\colon \mathrm{Hom}_{\mathsf{Cat}_2}(O_2, O_2') \to \mathrm{Hom}_{\mathsf{Cat}_1}(G(O_2), G(O_2'))$ is injective for all objects O_2, O_2' of Cat_2

Comments

- A pair $(j, \mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1])$ as in the definition need not exist.
- A free object over O_1 of Cat_2 with respect to G, if it exists, is determined up to an isomorphism of Cat_2
- So: speak of 'the' free object over O_1 (the morphism j being understood)

Simplification in our case

- The categories are always categories of sets (Set, VLA, BA, . . .).
- The functor G is always the obvious forgetful functor, or a variation on that: always faithful
- *Will omit G from the notation*

Example: unitisation is a free object

Take $\mathsf{Cat}_1 = \mathsf{VLA}$ and $\mathsf{Cat}_2 = \mathsf{VLA}^1$ with forgetful functor G

Let A be an object of VLA. A free object over A of VLA^1 is a pair $(j, \mathrm{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A])$, where $\mathrm{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A]$ is an object of VLA^1 and $j : A \to \mathrm{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A]$ is a vector lattice algebra homomorphism, with the property that, for every object A^1 of VLA^1 and every vector lattice algebra homomorphisms $\varphi : A \to A^1$, there exists a unique unital vector lattice algebra homomorphism $\overline{\varphi} : \mathrm{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A] \to A^1$ such that the following diagram commutes in VLA:

$$\begin{array}{ccc} A & \xrightarrow{\;j\;} & \mathrm{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A] \\ & \underset{\varphi}{\searrow} & \big\downarrow \overline{\varphi} \\ & & A^1 \end{array}$$

Comments

- $\mathbb{R} \oplus A$ is a unital vector lattice algebra, but this does not work!
- Still: a unitisation exists (see later)

Free objects in algebraic categories

Take $\mathsf{Cat}_1 = \mathsf{Set}$ and $\mathsf{Cat}_2 = \mathsf{VS}$: free vector space over a set

Let $S \neq \emptyset$. Is there a pair $(j, \mathrm{F}_{\mathrm{Set}}^{\mathrm{VS}}[S])$, where $\mathrm{F}_{\mathrm{Set}}^{\mathrm{VS}}[S]$ is a vector space and $j : S \to \mathrm{F}_{\mathrm{Set}}^{\mathrm{VS}}[S]$ is a map, with the property that, for every vector space V and map $\varphi : S \to V$, there exists a unique linear map $\overline{\varphi} : \mathrm{F}_{\mathrm{Set}}^{\mathrm{VS}}[S] \to V$ such that the following diagram is commutative?

$$\begin{array}{ccc} S & \xrightarrow{\;j\;} & \mathrm{F}_{\mathrm{Set}}^{\mathrm{VS}}[S] \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \overline{\varphi}} \\ & & V \end{array}$$

Two solutions

- Clever and informative: a vector space with basis indexed by S would solve the problem. There is one: the functions on S with finite support.
- 'Formal combinations $\sum_{s \in S} \lambda_s s$ with the obvious operations'
- The second solution is actually in the vein of the solution for a 'free abstract algebra in an equational class over a set': consists of equivalence classes of words

Abstract algebras of a given type

Suppose that $\mathcal{F} \neq \emptyset$ and that $\rho : \mathcal{F} \to \mathbb{N}_0$. Then the pair $(\mathcal{F}, \rho)$ is called a *type*. Let $A \neq \emptyset$ and suppose that, for each $f \in \mathcal{F}$, the following is given:

1. when $\rho(f) = 0$: an element f^A of A;
2. when $\rho(f) \geq 1$: a map $f^A : A^{\rho(f)} \to A$.

We set $\mathcal{F}^A := \{ f^A : f \in \mathcal{F} \}$. The pair $\langle A, \mathcal{F}^A \rangle$ is then called an *abstract algebra of type* $(\mathcal{F}, \rho)$. The elements of $\mathcal{F}$ are called *operation symbols*.
If $\rho(f) = 0$, then f^A is called a *constant of* A.
If $\rho(f) \geq 1$, then f^A is called an *operation on* A (taking $\rho(f)$ arguments).

Think of $(\mathcal{F}, \rho)$ as informing us how many distinguished elements (constants) there are in A, and what the numbers of variables are that the operations on A take.

Associating abstract algebras to familiar structures (and simplified notation)

- Example: vector space V
- Take $\mathcal{F} = \{0, \oplus, \ominus, \{ m_\lambda : \lambda \in \mathbb{R} \}\}$
- Take $\rho : \mathcal{F} \to \mathbb{N}_0$: $\rho(0) = 0$, $\rho(\oplus) = 2$, $\rho(\ominus) = 1$, $\rho(m_\lambda) = 1$
- Have naturally associated abstract algebra of type $(\mathcal{F}, \rho)$: $\langle V, \{0^V, \oplus^V, \ominus^V, \{ m_\lambda^V : \lambda \in \mathbb{R} \}\}\rangle$: zero element, addition, taking the additive inverse, multiplication by λ.

Not every abstract algebra $\langle V, \{0^V, \oplus^V, \ominus^V, \{ m_\lambda^V : \lambda \in \mathbb{R} \}\}\rangle$ of type $(\mathcal{F}, \rho)$ becomes a vector space when attempting to introduce vector space operations in the obvious way: $x + y := \oplus^V(x, y)$, $\lambda \cdot x := m_\lambda^V(x)$. One has to have $m_{\lambda_1\lambda_2}^V(x) = m_{\lambda_1}^V(m_{\lambda_2}^V(x))$, and more.

In examples: often omit mentioning the obvious type $(\mathcal{F}, \rho)$ and the superscript A in f^A. Above: simply speak of the abstract algebra $\langle V, \{0, \oplus, \ominus, \{ m_\lambda : \lambda \in \mathbb{R} \}\}\rangle$ associated to the vector space V.

Abstract algebra homomorphisms

Let $\langle A, \mathcal{F}^A \rangle$ and $\langle B, \mathcal{F}^B \rangle$ be abstract algebras of the *same* type $(\mathcal{F}, \rho)$. Then $h : A \to B$ is an *abstract algebra homomorphism* when:

1. $h(f^A) = f^B$ for all $f \in \mathcal{F}$ such that $\rho(f) = 0$;
2. $h\big(f^A(a_1, \ldots, a_{\rho(f)})\big) = f^B\big(h(a_1), \ldots, h(a_{\rho(f)})\big)$ for all $f \in \mathcal{F}$ such that $\rho(f) \geq 1$.

Categorical aspects

- Clear: abstract algebras of a given type form and their abstract algebra homomorphisms form a category $\mathsf{AbsAlg}_{(\mathcal{F},\rho)}$
- Will show: if $S \neq \emptyset$, then $\mathsf{F}_{\mathsf{Set}}^{\mathsf{AbsAlg}_{(\mathcal{F},\rho)}}[S]$ exists
- $\mathsf{F}_{\mathsf{Set}}^{\mathsf{AbsAlg}_{(\mathcal{F},\rho)}}[S]$ consists of strings
- It captures the concept of repeatedly applying operations, labelled by $\mathcal{F}$, to their appropriate number of variables, labelled by S, thus building operations of ever increasing complexity

Definition of the abstract term algebra over a set: consists of words

Let $(\mathcal{F},\rho)$ be a type. Let $S \neq \emptyset$ be disjoint from $\mathcal{F}$. Set

$$T_0(S) := \{s : s \in S\} \cup \{f \in \mathcal{F} : \rho(f) = 0\}.$$

For $n \geq 1$, set

$$T_{n+1}(S) := T_n(S) \cup \left\{ f[t_1,\ldots,t_{\rho(f)}] : f \in \mathcal{F},\ \rho(f) \geq 1,\ t_1,\ldots,t_{\rho(f)} \in T_n(S) \right\}$$

Define $T_{(\mathcal{F},\rho)}(S) := \bigcup_{n\geq 0} T_n(S)$. The elements of $T_{(\mathcal{F},\rho)}(S)$ are called *terms of type* $(\mathcal{F},\rho)$ *over* S.

If $\rho(f) = 0$, set

$$f^{T_{(\mathcal{F},\rho)}(S)} := f.$$

If $\rho(f) \geq 1$, set

$$f^{T_{(\mathcal{F},\rho)}(S)}(t_1,\ldots,t_{\rho(f)}) := f[t_1,\ldots,t_{\rho(f)}]$$

for $t_1,\ldots,t_{\rho(f)} \in T_{(\mathcal{F},\rho)}(S)$.

This makes the *term algebra* $T_{(\mathcal{F},\rho)}(S)$ into an abstract algebra of type $(\mathcal{F},\rho)$.

Universal algebra: Part I

Theorem (from the textbooks)

Let $(\mathcal{F}, \rho)$ be a type, and let $S \neq \emptyset$ be disjoint from $\mathcal{F}$. For every abstract algebra A of type $(\mathcal{F}, \rho)$ and every map $h : S \to A$, there is a unique abstract algebra homomorphism $\overline{h} : T_{(\mathcal{F},\rho)}(S) \to A$ making the following diagram commutative:

$$\begin{array}{ccc} S & \subset & T_{(\mathcal{F},\rho)}(S) \\ & \searrow^{h} & \downarrow \overline{h} \\ & & A \end{array}$$

That is, $\mathrm{F}_{\mathrm{Set}}^{\mathrm{AbsAlg}(\mathcal{F},\rho)}[S]$ exists and is equal to $T_{(\mathcal{F},\rho)}(S)$.

Proof

Every element of $T_{(\mathcal{F},\rho)}(S)$ (such as $\oplus[m_\lambda[\oplus[s_1, s_2]], \ominus[s_2]]$ in the example) has an interpretation as 'operations applied to the elements of S'. Replace the operation symbols in a term by the corresponding actual operations in A, brackets with parentheses, and every $s \in S$ with $h(s) \in A$. This gives $\overline{h}$.

Vector lattice algebras as abstract algebras *with relations*

Vector spaces as abstract algebras 'with relations'

- Take an abstract algebra $\langle V, \{0, \oplus, \ominus, \{ m_\lambda : \lambda \in \mathbb{R} \}\}\rangle$ of the same type as you get from a vector space V
- If the operations happen to be such that $m_{\lambda_1}(m_{\lambda_2}(x)) = m_{\lambda_1\lambda_2}(x)$, $\oplus(x, \oplus(y, z)) = \oplus(\oplus(x, y), z)$, $\oplus(x, \ominus(x)) = 0$, etc., then it is clear how to make V into a vector space.
- The vector spaces are the abstract algebras of a fixed type in which certain relations between the operations hold that are expressed by equalities.

How about structures that are *lattices*?

- Can we do something similar for vector lattice algebras (and for other structures that are lattices)?
- Not so clear: there are *inequalities* in the definition of a vector lattice, and *existence* of suprema 'is a property, not a relation'.

Partially ordered lattices and algebraic lattices

Two kinds of lattices (in ad hoc terminology and human notation) for a set $S \neq \emptyset$

1. Suppose that $\leq$ is a partial ordering on S. Then the partially ordered set $(S, \leq)$ is a *partially ordered lattice* if, for all $x, y \in S$, the supremum $x \vee y$ and the infimum $x \wedge y$ exist in S.
2. Suppose that the abstract algebra $(S, \{\owedge, \ovee\})$ has two binary operations. Then $(S, \{\owedge, \ovee\})$ is an *algebraic lattice* if

$$
\begin{aligned}
x \owedge (y \owedge z) &= (x \owedge y) \owedge z, & x \ovee (y \ovee z) &= (x \ovee y) \ovee z, \\
x \owedge x &= x, & x \ovee x &= x, \\
x \owedge y &= y \owedge x, & x \ovee y &= y \ovee x, \\
x \owedge (x \ovee y) &= x, \quad \text{and} & x \ovee (x \owedge y) &= x.
\end{aligned}
$$

Key observation

- There is a natural correspondence between these two types of structures

Lemma (from the textbooks)

Let $S \neq \emptyset$.

1. *Suppose that $(S, \leq)$ is a partially ordered lattice. For $x, y \in S$, set $x \owedge y := x \wedge y$ and $x \ovee y := x \vee y$. Then the abstract algebra $(S, \{\ovee, \owedge\})$ is an algebraic lattice.*
2. *Suppose that the abstract algebra $(S, \{\owedge, \ovee\})$ is an algebraic lattice. Say that $x \leq y$ if $x \owedge y = x$. Then $\leq$ is a partially ordering on S, and $(S, \leq)$ is a partially ordered lattice. Moreover, for $x, y \in S$, we have $x \wedge y = x \owedge y$ and $x \vee y = x \ovee y$, where $x \wedge y$ and $x \wedge y$ refer to the infimum and the supremum in the partial ordering $\leq$.*
3. *The above transitions from $(S, \leq)$ to $(S, \{\owedge, \ovee\})$ and vice versa are mutually inverse, with lattice homomorphisms corresponding to abstract algebra homomorphisms*

Now: use this result to see that, for example, the unital vector lattice algebras can also be identified with the abstract algebras (of an obvious type) where the operations *satisfy certain relations*.

Unital VLAs as abstract algebras satisfying relations

Proposition

Let A be an abstract algebra with (not necessarily different) constants 0 and 1, a binary map $\oplus$, a unary map $\ominus$, a unary map m_λ for every $\lambda \in \mathbb{R}$, a binary map $\odot$, and binary maps $\owedge$ and $\ovee$. Suppose that all of the following are satisfied:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ *for all* $x, y, z \in A$
2. $x \oplus 0 = x$ *for all* $x \in A$
3. $x \oplus (\ominus x) = 0$ *for all* $x \in A$
4. $x \oplus y = y \oplus x$ *for all* $x, y \in A$
5. $m_\lambda(x \oplus y) = m_\lambda(x) \oplus m_\lambda(y)$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y \in A$
6. $m_{\lambda+\mu}(x) = m_\lambda(x) \oplus m_\mu(x)$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $x \in A$
7. $m_{\lambda\mu}(x)) = m_\lambda(m_\mu(x))$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $x \in A$
8. $m_1(x) = x$ *for all* $x \in A$

Unital VLA's as abstract algebras satisfying relations

Proposition (continued)

9. $(x \odot y) \odot z = x \odot (y \odot z)$ *for all* $x, y, z \in A$;
10. $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ *for all* $x, y, z \in A$
11. $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ *for all* $x, y, z \in A$
12. $m_\lambda(x \odot y) = m_\lambda(x) \odot y = x \odot m_\lambda(y)$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y \in A$
13. $1 \odot x = x \odot 1 = x$ *for all* $x \in A$
14. $x \owedge (y \owedge z) = (x \owedge y) \owedge z$ *and* $x \ovee (y \ovee z) = (x \ovee y) \ovee z$ *for all* $x, y, z \in A$
15. $x \owedge x = x$ *and* $x \ovee x = x$ *for all* $x \in A$
16. $x \owedge y = y \owedge x$ *and* $x \ovee y = y \ovee x$ *for all* $x, y \in A$
17. $x \owedge (x \ovee y) = x$ *and* $x \ovee (x \owedge y) = x$ *for all* $x, y \in A$
18. $x \oplus (y \owedge z) = (x \oplus y) \owedge (x \oplus z)$ *for all* $x, y, z \in A$
19. $m_\lambda(0 \owedge x) = 0 \owedge (m_\lambda(x))$ *for all* $\lambda \in \mathbb{R}_{\geq 0}$ *and* $x \in A$
20. $0 \owedge ((x \ovee (\ominus x)) \odot (y \ovee (\ominus y))) = 0$ *for all* $x, y \in A$.

Unital VLA's as abstract algebras satisfying relations

Proposition (continued)

Define

(a) $x + y := x \oplus y$ *for* $x, y \in A$

(b) $\lambda \cdot x := m_\lambda(x)$ *for* $\lambda \in \mathbb{R}$ *and* $x \in A$

(c) $xy := x \odot y$ *for* $x, y \in A$

(d) $x \leq y$ *when* $x \owedge y = x$

Then A *is a unital vector lattice algebra with zero element 0 and identity element 1.*

Conversely, every unital vector lattice algebra gives rise, in the obvious way, to an abstract algebra with operations as above satisfying these 20 relations. These two passages are mutually inverse, with unital vector lattice algebra homomorphisms corresponding to abstract algebra homomorphisms.

Where do we stand?

- Have free abstract algebra of a given type $(\mathcal{F}, \rho)$ over a set S: $T_{(\mathcal{F},\rho)}(S)$
- For example: the free abstract algebra of the *type* of unital vector lattice algebras over a set S exists
- This is not yet what we want: we need a free algebra over a set in *'the category of abstract algebras of that type where these 20 relations between the operations are satisfied'*
- How can one, in fact, formalise the concept of *the 'same' relations between operations (on different sets!) being satisfied?*

Expressing that relations between operations are satisfied

- Suppose that $(\mathcal{F}, \rho)$ is a type, and that $\oplus \in \mathcal{F}$ is such that $\rho(\oplus) = 2$
- Let V be an abstract algebra of type $(\mathcal{F}, \rho)$
- Take a set $S_{\aleph_0}$ with at least three elements s_1, s_2, s_3
- Take $t_1 = \oplus[s_1, \oplus[s_2, s_3]]$ and $t_2 = \oplus[\oplus[s_1, s_2], s_3] \in T_{(\mathcal{F},\rho)}(S_{\aleph_0})$
- Take $x_1, x_2, x_3 \in V$ and choose any map $\mu_{x_1,x_2,x_3} : S_{\aleph_0} \to V$ such that $\mu_{x_1,x_2,x_3}(s_i) = x_i$ $(i = 1, 2, 3)$. The unique extension to an abstract algebra homomorphisms $\overline{\mu}_{x_1,x_2,x_3} : T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \to V$ then satisfies

$$\overline{\mu}_{x_1,x_2,x_3}(t_1) = \oplus^V(x_1, \oplus^V(x_2, x_3))$$
$$\overline{\mu}_{x_1,x_2,x_3}(t_2) = \oplus^V(\oplus^V(x_1, x_2), x_3)$$

- So: if $h(t_1) = h(t_2)$ for *all* abstract algebra homomorphisms $T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \to V$, then $\oplus^V(x_1, \oplus^V(x_2, x_3)) = \oplus^V(\oplus^V(x_1, x_2), x_3)$ for *all* $x_1, x_2, x_3 \in V$ (and conversely)

This leads to the following definition.

Definition

Let $(\mathcal{F}, \rho)$ be a type. Let $S_{\aleph_0}$ be a countably infinite set. Take two terms $t_1, t_2 \in T_{(\mathcal{F},\rho)}(S_{\aleph_0})$. Let A be an abstract algebra of type $(\mathcal{F}, \rho)$. Then A *satisfies* $t_1 \approx t_2$ when $h(t_1) = h(t_2)$ for every abstract algebra homomorphism $h : T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \to A$. For a subset Σ of $T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \times T_{(\mathcal{F},\rho)}(S_{\aleph_0})$, A *satisfies* Σ when A satisfies $t_1 \approx t_2$ for every pair $(t_1, t_2) \in \Sigma$.

For $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \times T_{(\mathcal{F},\rho)}(S_{\aleph_0})$, the class of all abstract algebras of type $(\mathcal{F}, \rho)$ satisfying Σ is called the *equational class defined by* Σ. Together with the abstract algebra homomorphism between them, it forms the subcategory $\mathsf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$ of $\mathsf{AbsAlg}_{(\mathcal{F},\rho)}$.

Universal algebra: Part II

Theorem ('what we should have remembered')

Let $(\mathcal{F},\rho)$ be a type. Take a countably infinite set $S_{\aleph_0}$ and $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \times T_{(\mathcal{F},\rho)}(S_{\aleph_0})$. Let $S \neq \emptyset$. There exists an equivalence relation θ on $T_{(\mathcal{F},\rho)}(S)$ such that $T_{(\mathcal{F},\rho)}(S)/\theta$ is an abstract algebra of type $(\mathcal{F},\rho)$ satisfying Σ, and with the following property: for every abstract algebra of type $(\mathcal{F},\rho)$ satisfying Σ and every map $h : S \to A$, there exists a unique abstract algebra homomorphism $\overline{h} : T_{(\mathcal{F},\rho)}(S)/\theta \to A$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{q_\theta|_S} & T_{(\mathcal{F},\rho)}(S)/\theta \\ & \searrow^{h} & \downarrow \overline{h} \\ & & A \end{array}$$

is commutative; here $q_\theta : T_{(\mathcal{F},\rho)}(S) \to T_{(\mathcal{F},\rho)}(S)/\theta$ denotes the quotient abstract algebra homomorphism.

That is, $\mathrm{F}_{\mathrm{Set}}^{\mathrm{AbsAlg}(\mathcal{F},\rho);\Sigma}[S]$ exists and is equal to $T_{(\mathcal{F},\rho)}(S)/\theta$.

Comments

- The equivalence relation θ on $T_{(\mathcal{F},\rho)}(S)$ is given explicitly: it is the smallest so-called congruence relation on $T_{(\mathcal{F},\rho)}(S)$ containing the pairs $(h'(t_1), h'(t_2))$ for all $(t_1, t_2) \in \Sigma$ and all abstract algebra homomorphisms $h' : T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \to T_{(\mathcal{F},\rho)}(S)$.
- These free objects consist of equivalence classes of words—recall the 'existence proof' for the free vector space over a set by introducing 'formal linear combinations with the 'obvious' operations'.

Free unital vector lattice algebras

Consequence

- Free unital vector lattice algebras over a set exist: the unital vector lattice algebras form an equational class
- Free objects in many other categories over sets also exist for the same reason

Not just over sets

- Construct free unital vector lattice algebra over a vector space V:
- Take $(j, \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)])$ where $j : V \to \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]$ is a set map
- Let I be the order-and-algebra ideal of $\mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]$ generated by all $j(x+y) - j(x) - j(y)$ and all $j(\lambda x) - \lambda j(x)$
- Then $\mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]\,/I$ is a unital vector lattice algebra
- With $q : \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)] \to \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]\,/I$, the pair $(q \circ j, \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]\,/I)$ is the free unital vector lattice algebra over V

Theorem (MdJ)

For a non-empty set S*, a vector space* V*, a vector lattice* E*, a vector lattice algebra* A*, and a unital vector lattice algebra* A^1*, the 15 free objects on the following slide all exist, with inclusions as indicated. The surjective unital vector lattice algebra homomorphisms in the rightmost column are the quotient maps corresponding to dividing out the order-and-algebra-ideal that is generated by* $(|1|-1)$*.*

Existence of free objects in some algebraic categories

$$S \subset \mathrm{F}_{\mathrm{Set}}^{\mathrm{VS}}[S] \subset \mathrm{F}_{\mathrm{Set}}^{\mathrm{VL}}[S] \subset \mathrm{F}_{\mathrm{Set}}^{\mathrm{VLA}}[S] \begin{array}{l} \subset \;\mathrm{F}_{\mathrm{Set}}^{\mathrm{VLA}^1}[S] \\ \qquad\quad \twoheaddownarrow \\ \subset \;\mathrm{F}_{\mathrm{Set}}^{\mathrm{VLA}^{1+}}[S] \end{array}$$

$$V \subset \mathrm{F}_{\mathrm{VS}}^{\mathrm{VL}}[V] \subset \mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[V] \begin{array}{l} \subset \;\mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}^1}[V] \\ \qquad\quad \twoheaddownarrow \\ \subset \;\mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}^{1+}}[V] \end{array}$$

$$E \subset \mathrm{F}_{\mathrm{VL}}^{\mathrm{VLA}}[E] \begin{array}{l} \subset \;\mathrm{F}_{\mathrm{VL}}^{\mathrm{VLA}^1}[E] \\ \qquad\quad \twoheaddownarrow \\ \subset \;\mathrm{F}_{\mathrm{VL}}^{\mathrm{VLA}^{1+}}[E] \end{array}$$

$$A \begin{array}{l} \subset \;\mathrm{F}_{\mathrm{VLA}}^{\mathrm{VLA}^1}[A] \\ \qquad\quad \twoheaddownarrow \\ \subset \;\mathrm{F}_{\mathrm{VLA}}^{\mathrm{VLA}^{1+}}[A] \end{array}$$

$$\mathrm{F}_{\mathrm{VLA}^1}^{\mathrm{VLA}^{1+}}\left[A^1\right]$$

Theorem (continued)

Let L be a (partially ordered or algebraic) lattice. Then the 4 free objects in

$$L \xrightarrow{j} \mathrm{F}_{\mathrm{Lat}}^{\mathrm{VL}}[L] \subset \mathrm{F}_{\mathrm{Lat}}^{\mathrm{VLA}}[V] \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} \mathrm{F}_{\mathrm{Lat}}^{\mathrm{VLA}^1}[L] \\ \twoheaddownarrow \\ \mathrm{F}_{\mathrm{Lat}}^{\mathrm{VLA}^{1+}}[L] \end{array}$$

all exist. The map j is injective if and only if L is distributive.

'Concrete models' of free objects in algebraic categories

Beyond existence: what do the abstractly constructed free objects 'look like'?

- Free vector space over a set: space of functions with set as basis
- Archimedean tensor product of Archimedean E and F is a double quotient of $\mathrm{F}^{\mathrm{VL}}_{\mathrm{Set}}[\mathrm{Set}(E \times F)]$ (de Pagter); Fremlin gives non-trivial properties
- Free vector lattice over a set: is a lattice of functions (Bleier)

Theorem (MdJ)

Let V be a vector space. Take a linear subspace $L^\sharp$ of the algebraic dual $V^\sharp$ that separates the points of V. Define the linear map $\Psi : V \to \mathrm{Fun}(L^\sharp, \mathbb{R})$ by setting

$$[\Psi(v)](l^\sharp) := l^\sharp(v) \quad (v \in V,\, l^\sharp \in L^\sharp).$$

Let $\mathcal{F}$ be the vector sublattice of $\mathrm{Fun}(L^\sharp, \mathbb{R})$ that is generated by its linear subspace $\Psi(V)$. Then $(\Psi, \mathcal{F})$ is a free vector lattice over V.

Free objects in analytic categories

Recipe (C^*-algebra theorist, Troitsky, Tradacete, Taylor/dJ,...)

- Remove analysis from the problem and start with free algebraic object
- Introduce seminorm on free algebraic object using its algebraic universal property
- Quotient out its kernel and, if necessary, complete

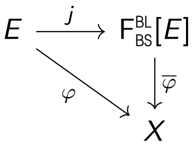
Example: construct $\mathsf{F}_{\mathsf{BS}}^{\mathsf{BL}}[E]$ *for a Banach space* E (BS: Banach space with bounded linear maps; BL: Banach lattices with (bounded) vector lattice homomorphisms).
Step 1: Start with $\mathsf{F}_{\mathsf{VS}}^{\mathsf{VL}}[E]$: for every vector lattice X and every linear $\varphi : E \to X$ there is a unique vector lattice homomorphism $\overline{\varphi}$ such that we have a commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\ j\ } & \mathsf{F}_{\mathsf{VS}}^{\mathsf{VL}}[E] \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\overline{\varphi}} \\ & & X \end{array}$$

Step 2: Introduce lattice seminorm on $\mathrm{F}_{\mathrm{VS}}^{\mathrm{VL}}[E]$: for $v \in \mathrm{F}_{\mathrm{VS}}^{\mathrm{VL}}[E]$, set

$$\rho(v) := \sup \left\{ \|\overline{\varphi}(v)\| : X \text{ is a Banach lattice}, \, \varphi : E \to X, \, \|\varphi\| \leq 1 \right\}$$

Since $j(E)$ generates $\mathrm{F}_{\mathrm{VS}}^{\mathrm{VL}}[E]$ as a vector lattice and the lattice seminorm ρ is finite on $j(E)$, ρ is finite on the whole $\mathrm{F}_{\mathrm{VS}}^{\mathrm{VL}}[E]$.
Step 3: Let $\mathrm{F}_{\mathrm{BS}}^{\mathrm{BL}}[E]$ be the completion of $\mathrm{F}_{\mathrm{VS}}^{\mathrm{VL}}[E] \, / \ker \rho$ and replace j with the composition of j, the quotient map, and the inclusion. Result: for every Banach lattice X and every bounded linear $\varphi : E \to X$, there is a unique vector lattice homomorphism $\overline{\varphi} : \mathrm{F}_{\mathrm{BS}}^{\mathrm{BL}}[E] \to X$ such that



Moreover, j is contractive, and $\|\varphi\| = \|\overline{\varphi}\|$.
Non-generic: j is even isometric (Hahn-Banach)

Free objects in analytic categories

Example Let L be a lattice. Analogously, one constructs a Banach lattice $\mathrm{F}^{\mathrm{BL}}_{\mathrm{Lat}}[L]$ and a lattice homomorphism $j : L \to \mathrm{F}^{\mathrm{BL}}_{\mathrm{Lat}}[L]$ with the following property: for every Banach lattice X and every bounded map $\varphi : E \to X$, there is a unique vector lattice homomorphism $\overline{\varphi} : \mathrm{F}^{\mathrm{BL}}_{\mathrm{BS}}[E] \to X$ such that

$$\begin{array}{ccc} E & \xrightarrow{\ j\ } & \mathrm{F}^{\mathrm{BL}}_{\mathrm{Lat}}[L] \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\overline{\varphi}} \\ & & X \end{array}$$

Moreover, j is contractive, and $\|\varphi\| = \|\overline{\varphi}\|$.
Non-generic: j is injective if and only if L is distributive (embedding theorem for distributive lattices).

Example: From the Archimedean vector tensor lattice product of two Banach lattices, one constructs a Banach lattice that has the same universal property with respect to lattice bimorphisms and lattice homomorphisms as their Fremlin tensor product. Hence they are isometrically isomorphic.

Comments

- Essential to the above constructions: family of initial morphisms φ can be scaled to be uniformly bounded in every point of the image of j
- For algebras this is not possible
- In fact: free Banach (lattice) algebras over non-empty sets or over non-zero Banach spaces do not exist
- For infinite sets this is also not possible
- In fact: free Banach spaces or free Banach lattices over infinite sets don't exist.

Still: an adaptation of the recipe produces something. Apply the usual construction, but now use M rather than 1 as an upper bound for the definition of ρ to see the following. (BLA: Banach lattice algebras with (bounded) vector lattice algebra homomorphisms).

Proposition

Let X be a Banach space. Take $M > 0$. There exist a Banach lattice algebra $\mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M$ and a linear $j_M : X \to \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M$ with $\|j_M\| \leq M$ with the following property: for every Banach lattice algebra A and every linear $\varphi : X \to A$ with $\|\varphi\| \leq M$, there exists a unique vector lattice algebra homomorphism $\overline{\varphi} : \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M \to X$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{j_M} & \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M \\ & \searrow^{\varphi} & \downarrow \overline{\varphi} \\ & & A \end{array}$$

is commutative, and where this unique $\overline{\varphi}$ is then contractive.

$F_{BS}^{BLA}[X]_M$ can still be interpreted as a free object

- replace BS with the category of Banach spaces with weights of the form $x \mapsto \alpha\|x\|$ for arbitrary $\alpha > 0$, and with weighted linear maps
- Modify the forgetful functor from BLA to BS by including the weight for $\alpha = 1$.
- Then a weighted Banach space X with weight $x \mapsto M\|x\|$ has $F_{BS}^{BLA}[X]_M$ as a free object in BLA over it with respect to this modified forgetful functor.

Free objects in analytic categories

Free objects via inverse limits

- Continue with $\mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M$ as above.
- Take $M_2 \geq M_1$. Then $\|j_{M_1}\| \leq M_1 \leq M_2$, so there is a contractive vector lattice algebra homomorphism $\overline{j_{M_1}} : \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_{M_2} \to \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_{M_1}$ with a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{j_{M_2}} & \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_{M_2} \\ & {\scriptstyle j_{M_1}} \searrow & \downarrow {\scriptstyle \overline{j_{M_1}}} \\ & & \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_{M_1} \end{array}$$

- Follows from this: the $\mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M$ for $M > 0$ form an inverse system.

More work shows: the inverse limit of this system of Banach lattice algebras in the category of complete locally convex-solid topological algebras is a free object in that category over the Banach space X, where X is seen as an object in the category of locally convex spaces.

Comments

- Inverse limit construction works also in other cases
- Example: free complete locally convex topological algebras over arbitrary sets exist
- For a one point set (and over $\mathbb{C}$) this is the algebra of entire functions in the topology of uniform convergence on compact subsets

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