SMOOTH AND TOPOLOGICAL PSEUDOCONVEXITY IN COMPLEX SURFACES

ROBERT E. GOMPF

Question. *a) When can an open subset of a complex surface be deformed (isotoped) into a Stein open subset (domain of holomorphy)?*
b) When can a compact 4-manifold embedded in a complex surface be isotoped to be (strictly) pseudoconvex (PC) (Stein domain)?

Theorem 1. [2] *Both work iff the inherited almost-complex structure J on the domain is homotopic to a Stein structure on it.*

Basic idea: Eliashberg's technique already works inside a complex manifold.
Now follows in all dimensions from a theorem in the Cieliebak–Eliashberg book.

Examples. a) Many compact, contractible Stein surfaces embed in $\mathbb{C}^2$. These can all be assumed PC. (J is unique.) Gives lots of PC embedded homology 3-spheres.
(What about Brieskorn spheres?)
b) Stein embedded exotic $\mathbb{R}^4$'s in $\mathbb{C}^2$. (Uncountably many diffeomorphism types.)
c) Every Stein open subset of $\mathbb{C}^2$ is homeomorphic, not diffeomorphic to uncountably many others.

For domains with homology, the condition on J is nontrivial. We can eliminate it by passing to the topological (C^0) category:

Theorem 2. [3] *An open subset U of a complex surface is* **topologically** *isotopic to a Stein open set iff it is homeomorphic to a 2-handlebody interior.*

This typically changes the smooth structure on U, resulting in infinite smooth topology.
Basic idea: Apply work of Casson–Freedman. The 2-handle cores may not be totally real or even smoothable. But can make them totally real *immersed*. Iterate.

Examples. a) Every embedding $S^2 \hookrightarrow \mathbb{C}^2$ has neighborhood $S^2 \times \mathbb{R}^2$ that can be made Stein (but with no smooth S^2 inside).
b) For K topologically but not smoothly slice, its trace X_K embeds unsmoothably in $\mathbb{C}^2$ and its interior can be made Stein.
c) Theorem 1 gives smooth, compact Stein domains in $\mathbb{C}^2$ with the homotopy type of S^2 (not knot traces).

Now suppose X is a compact 2-handlebody topologically embedded in a complex surface, with a topological collar $I \times \partial X$ outside it (so not wild).
Call X *topologically pseudoconvex (TPC)* if it is also a Stein compact, i.e., has a Stein neighborhood system. This implies $\operatorname{int} X$ is Stein.
Theorem 2 extends to make X TPC (by *ambient* isotopy).
Now examples (a,b) give TPC embeddings of $S^2 \times D^2$ and X_K.
Digging deeper into Freedman's proof extends to an uncountable nest of TPC embeddings:

Definition 1. [4] A *Stein onion* consists of a closed 3-manifold M, a Stein surface U and a continuous surjection $\psi\colon [0,1)\times M\to U$ restricting to a homeomorphic embedding on $(0,1)\times M$ such that the open subset $\psi([0,\sigma)\times M)$ is Stein whenever σ is in the Cantor set.

Its *core* $\psi(0)$ is then a Stein compact with an uncountable nested system of Stein neighborhoods (all homeomorphic) with TPC closures.

Theorem 3. [3] *A collared, topologically embedded 2-handlebody in a complex surface is topologically ambiently isotopic to a layer of a Stein onion.*

Examples. a) Every compact, tame, topological 2-complex in a complex surface is ambiently isotopic to the core of a Stein onion. Its cells become smooth and totally real, except for one singularity on each 2-cell.
b) For every complex surface S, every $\alpha\in H_2(S)$ can be realized by a smooth surface. Any such becomes the core of a Stein onion. This can be chosen so that the Stein neighborhoods realize all sufficiently large minimal genera, or stabilize at a preassigned upper bound.
If $\alpha\cdot\alpha\le 0$, each such minimal genus is realized by uncountably many diffeomorphism types within the given Stein onion.

Call a topologically embedded 3-manifold M *TPC* if it has a neighborhood biholomorphic to a neighborhood of the boundary of some TPC 4-manifold (in a possibly different complex surface).
Like smooth PC embeddings, a TPC embedding in a Stein manifold cuts out a Stein compact (so an open Stein surface).
PC 3-manifolds inherit contact structures. On any 3-manifold M (up to homotopy)

$$\text{plane fields on } M \iff \text{almost complex structures on } \mathbb{R}\times M.$$

A TPC 3-manifold M inherits a complex structure on a neighborhood *homeomorphic* to $\mathbb{R}\times M$. These can be classified homeomorphism-invariantly [4].
TPC embeddings are much more common than smooth PC embeddings:

Theorem 4. *a)* [3] *Every closed, oriented M^3 admits a TPC embedding in every simply connected, compact complex surface S with $b_\pm(S)$ sufficiently large.*
b) [4] *Every J on M can be realized this way when* $\operatorname{div} c_1(S)$ *divides* $\operatorname{div} c_1(J)$ *(so whenever S is nonminimal, since* $\operatorname{div} c_1(S)=1$*).*

Basic idea: Explicitly realize these by Stein onions.

Smoothly pseudocon**cave** examples [2]:
a) Knot traces whenever $tb(\overline{K})\ge -1$ and the framing is sufficiently large
b) $I\times M$ where M is a circle bundle over F with $|e(M)|\le -\chi(F)$
or M the Brieskorn sphere $(p,q,npq-1)\ne(2,3,5)$
c) the corresponding Milnor fiber.
d) Every simply connected, compact complex surface contains a pseudoconcave contractible manifold.
It also contains uncountably many **topologically** pseudoconcave exotic $\mathbb{R}^4$'s [3].

References

[1] *Stein surfaces as open subsets of* $\mathbb{C}^2$, J. Symplectic Geom. **3** (2005), 565-587. (Expository warm-up)

[2] *Smooth embeddings with Stein surface images*, J. of Topology **6** (2013), 915–944.

[3] *Creating Stein surfaces by topological isotopy*, J. Differ. Geom., to appear, arXiv:2002.02042 .

[4] *Topological convexity in complex surfaces*, Asian J. Math., to appear, arXiv:2010.05114 .