Multivariable (φ, Γ)-modules and local-global compatibility

Florian Herzig
University of Toronto

(joint with C. Breuil, Y. Hu, S. Morra, and B. Schraen)

Banff, January 2023

Mod p Langlands

$K/\mathbb{Q}_p$ finite extension, $\mathbb{F}/\mathbb{F}_p$ finite (large)

Hope

There is a natural relationship:

$$\left\{ \begin{array}{c} \text{Galois representations} \\ \overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{F}) \end{array} \right\} \overset{?}{\longleftrightarrow} \left\{ \begin{array}{c} \text{(some) adm. smooth reps.} \\ \text{of } \mathrm{GL}_n(K) \text{ over } \mathbb{F} \end{array} \right\}.$$

$G = \mathrm{GL}_2(\mathbb{Q}_p)$: $\exists$ correspondence, realized by Colmez functor
(Breuil, Colmez, Kisin, Emerton, Paškūnas)

$\cong$ Colmez

$\{$étale (φ, Γ)-modules over $\mathbb{F}((T))\}$

$$\varphi(T) = (1+T)^p - 1, \quad \gamma(T) = (1+T)^\gamma - 1 \quad \forall \gamma \in \Gamma = \mathbb{Z}_p^\times.$$

Mod p Langlands

Hope

There is a natural relationship:

$$\left\{ \begin{array}{c} \text{Galois representations} \\ \overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{F}) \end{array} \right\} \overset{?}{\longleftrightarrow} \left\{ \begin{array}{c} \text{(some) adm. smooth reps.} \\ \text{of } \mathrm{GL}_n(K) \text{ over } \mathbb{F} \end{array} \right\}.$$

Our focus: $n = 2$, $K/\mathbb{Q}_p$ unramified.
Basic problems for $K \neq \mathbb{Q}_p$:

1. there are a lot more supersingular representations of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ (Breuil–Paškūnas)
2. supersingular reps. of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ are *not* of finite presentation (Schraen, Z. Wu)

Global setting

- $F/\mathbb{Q}$ totally real, p inert in F
- $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ irreducible, automorphic
- X_U a suitable Shimura curve over F

Let

$$\boxed{\pi = \pi(\overline{r}) := \varinjlim_{U_p} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\left(\overline{r}, H^1_{et}(X_{U_pU^p} \times_F \overline{F}, \mathbb{F})\right) \neq 0},$$

an admissible smooth representation of $\mathrm{GL}_2(F_p)$ over $\mathbb{F}$.

Assume level U^p is optimal ("multiplicity 1").

Question: does $\pi(\overline{r})$ only depend on $\overline{r}|_{\mathrm{Gal}(\overline{F}_p/F_p)}$?

Multivariable $(\varphi, \mathcal{O}_K^\times)$-modules: the ring A

- $k := \mathcal{O}_K/\mathfrak{m}_K \cong \mathbb{F}_{p^f}$, fix $\sigma_0 : k \hookrightarrow \mathbb{F}$.
- $Y_i := \sum_{\lambda \in k^\times} \lambda^{-p^i}[\widetilde{\lambda}] \in \mathbb{F}[[\mathcal{O}_K]]$, so $\mathbb{F}[[\mathcal{O}_K]] = \mathbb{F}[[Y_0, \dots, Y_{f-1}]]$.
- $\boxed{A := \widehat{\mathbb{F}[[\mathcal{O}_K]]_{Y_0 \cdots Y_{f-1}}}}$ (with natural valuation topology).

$$\rightsquigarrow \quad A \cong \mathbb{F}((Y_i))\left\langle \left(\frac{Y_j}{Y_i}\right)^{\pm 1} : j \neq i \right\rangle$$

$$\cong \left\{ \sum_{n=0}^{\infty} \lambda_n \underline{Y}^{\underline{i}(n)} : \lambda_n \in \mathbb{F},\ \underline{i}(n) \in \mathbb{Z}^f,\ \|\underline{i}(n)\| \to \infty \right\}$$

Also, $\operatorname{Spa}(A) = \{|Y_0| = \cdots = |Y_{f-1}| \neq 0\} \subset \operatorname{Spa} \mathbb{F}[[\mathcal{O}_K]]$ (open)

The actions of φ, $\mathcal{O}_K^\times$ extend to A.

Remark: $\varphi(Y_i) = Y_{i-1}^p$, $\quad \widetilde{a}(Y_i) = a^{p^i} Y_i \ \ \forall a \in k^\times$

Multivariable $(\varphi, \mathcal{O}_K^\times)$-modules: $D_A(\pi)$

Suppose

- $\overline{r}|_{G_{F_p}}$ is tamely ramified and strongly generic (+TW assumptions).
- $\pi = \pi(\overline{r})$ as before.

BHHMS (2021)

Constructed (functorially) $D_A(\pi)$, an étale $(\varphi, \mathcal{O}_K^\times)$-module over A.

Also, $\mathrm{tr} : \mathbb{F}[[\mathcal{O}_K]] \twoheadrightarrow \mathbb{F}[[\mathbb{Z}_p]] = \mathbb{F}[[T]]$ induces $\mathrm{tr} : A \twoheadrightarrow \mathbb{F}((T))$ and

$$D_A(\pi)/\ker(\mathrm{tr}) \cong D_{\mathrm{Breuil}}(\pi)$$

is the étale $(\varphi, \mathbb{Z}_p^\times)$-module corresponding to $\mathrm{Ind}_{G_{F_p}}^{\otimes G_{\mathbb{Q}_p}}(\overline{r}|_{G_{F_p}})$.

Aside: genericity condition

A tamely ramified $\overline{\rho} : G_{F_p} \to \mathrm{GL}_2(\mathbb{F})$ is *strongly generic* if

- $\overline{\rho}$ reducible: $\overline{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \omega_f^{r_0+pr_1+\cdots+p^{f-1}r_{f-1}} & \\ & 1 \end{pmatrix}$ up to twist
- $\overline{\rho}$ irreducible: $\overline{\rho}|_{I_{F_p}} \cong \begin{pmatrix} \omega_{2f}^{r_0+pr_1+\cdots+p^{f-1}r_{f-1}} & \\ & \omega_{2f}^{p^f(\text{same})} \end{pmatrix}$ up to twist

with $\max\{14, 2f+1\} \le r_i \le p - \max\{14, 2f+1\}\ \forall i$.

Main results

Recall: $K/\mathbb{Q}_p$ unramified.
Construct functor

$$D_A^{\otimes} : \{\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{F})\} \to \left\{\text{étale } (\varphi, \mathcal{O}_K^{\times})\text{-modules over } A\right\}.$$

Theorem (BHHMS, 2022)

If $\overline{r}|_{G_{F_p}}$ is tamely ramified and strongly generic, then

$$D_A(\pi) \cong D_A^{\otimes}(\overline{r}|_{G_{F_p}}).$$

Remarks

1. Conjecturally can remove "tame" and "strongly generic".
2. In fact, $D_A^{\otimes}(\overline{\rho}) \cong \bigotimes_{\sigma : k \to \mathbb{F}} D_{A,\sigma}(\overline{\rho})$ for (exact) functors $D_{A,\sigma}$.
3. If $\overline{\rho}$ tame, then $D_A^{\otimes}(\overline{\rho})$ is explicit.

Reminder on $D_A(\pi)$

- $I_1 := \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix}$, $N_0 := \begin{pmatrix} 1 & \mathcal{O}_K \\ & 1 \end{pmatrix} \cong \mathcal{O}_K$.
- π adm. smooth rep. of $\mathrm{GL}_2(K)$ over $\mathbb{F}$, with central character
- $\pi^\vee := \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$, a f.g. $\mathbb{F}[[I_1]]$-module with $\mathfrak{m}_{I_1}$-adic topology.
- Define $\boxed{D_A(\pi) := A\widehat{\otimes}_{\mathbb{F}[[N_0]]}\pi^\vee}$. (Recall $A = \widehat{\mathbb{F}[[N_0]]_{Y_0 \cdots Y_{f-1}}}$.)

BHHMS (2020)

If $\overline{r}|_{G_{F_p}}$ is tamely ramified and strongly generic, then $\mathrm{gr}(\pi^\vee)$ is annihilated by an explicit ideal $J \lhd \mathrm{gr}(\mathbb{F}[[I_1/Z_1]])$.

$\Longrightarrow$ $\mathrm{gr}\, D_A(\pi)$ is a f.g. $\mathrm{gr}(A)$-module
$\Longrightarrow$ $D_A(\pi)$ is a f.g. A-module (in fact, finite free)

Reminder on $D_A(\pi)$

$$\boxed{D_A(\pi) := A\widehat{\otimes}_{F[[N_0]]}\pi^\vee}$$

Action of $\begin{pmatrix} \mathcal{O}_K^\times & \\ & 1 \end{pmatrix} \implies D_A(\pi)$ has semilinear $\mathcal{O}_K^\times$-action
Action of $\begin{pmatrix} p & \\ & 1 \end{pmatrix} \implies D_A(\pi)$ has $\psi : D_A(\pi) \to D_A(\pi)$, $\psi(\varphi(a)x) = a\psi(x)$.

- From ψ get $\widetilde{\psi} : D_A(\pi) \to A \otimes_{\varphi,A} D_A(\pi)$.
- $D_A(\pi)$ has largest étale quotient $D_A(\pi)^{ét}$; invert $\widetilde{\psi}$ to make it into étale $(\varphi, \mathcal{O}_K^\times)$-module.

1. Introduction
2. Multivariable (φ, Γ)-modules
3. Construction of $D_A^{\otimes}(\overline{\rho})$
4. Proof of Theorem (outline)

Construction of $D_A^{\otimes}(\overline{\rho})$

Starting point: Lubin–Tate $(\varphi, \mathcal{O}_K^\times)$-modules.

$$G_{\mathrm{LT}} := \operatorname{Spf} \mathcal{O}_K[[T_K]], \text{ Lubin–Tate formal } \mathcal{O}_K\text{-module}$$
$$\Big(\log(T_K) = \sum_{n=0}^{\infty} \frac{T_K^{p^{fn}}}{p^n}\Big).$$

$\rightsquigarrow$ Lubin–Tate étale $(\varphi, \mathcal{O}_K^\times)$-module $D_{\mathrm{LT}}(\overline{\rho})$ over $\mathbb{F} \otimes k((T_K))$

- $D_{\mathrm{LT}}(\overline{\rho}) = \bigoplus_{\sigma : k \to \mathbb{F}} D_{\mathrm{LT},\sigma}(\overline{\rho})$ over $\mathbb{F} \otimes k((T_K)) \cong \prod_{\sigma : k \to \mathbb{F}} \mathbb{F}((T_K))$
- $D_{\mathrm{LT},\sigma}(\overline{\rho})$ is an étale $(\varphi^f, \mathcal{O}_K^\times)$-module over $\mathbb{F}((T_K))$.

Problem: comparing $\mathbb{F}((T_K))$ and A with $\mathcal{O}_K^\times$-actions is difficult!

Construction of $D_A^{\otimes}(\overline{\rho})$

Solution: universal cover $\widetilde{G}_{\mathrm{LT},\mathbb{F}} := \varprojlim_{[p]} G_{\mathrm{LT},\mathbb{F}} \cong \mathrm{Spa}\,\mathbb{F}[[T_K^{1/p^\infty}]]$.

$$\circlearrowleft$$
$$K$$

Fargues–Fontaine: for $R \in \mathrm{Perf}_{\mathbb{F}}$ define the topological ring $B^+(R)$

(a certain Fréchet completion of $W(R^\circ)[1/p]$).

Then $\boxed{\widetilde{G}_{\mathrm{LT},\mathbb{F}}(\cdot) \cong B^+(\cdot)^{\varphi^f = p}}$ as pro-ét. sheaves of K-v.sp. on $\mathrm{Perf}_{\mathbb{F}}$.

Also, $\boxed{\widetilde{G}_{\mathcal{O}_K,\mathbb{F}}(\cdot) \cong B^+(\cdot)^{\varphi^f = p^f}}$ as pro-ét. sheaves of K-v.sp. on $\mathrm{Perf}_{\mathbb{F}}$, where

$$G_{\mathcal{O}_K} := \widehat{\mathbb{G}}_{m,\mathbb{F}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \implies \widetilde{G}_{\mathcal{O}_K,\mathbb{F}} \cong \mathrm{Spa}(\mathbb{F}[[\widehat{\mathcal{O}_K]]^{1/p^\infty}}).$$

Construction of $D_A^{\otimes}(\overline{\rho})$

Recall: $\widetilde{G}_{\mathrm{LT},\mathbb{F}}(\cdot) \cong B^+(\cdot)^{\varphi^f = p}$, $\widetilde{G}_{\mathcal{O}_K,\mathbb{F}}(\cdot) \cong B^+(\cdot)^{\varphi^f = p^f}$

Let $\boxed{Z_{\mathrm{LT}} := (\widetilde{G}_{\mathrm{LT},\mathbb{F}} \setminus \{0\})^f}$, $\boxed{Z_{\mathcal{O}_K} := \widetilde{G}_{\mathcal{O}_K,\mathbb{F}} \setminus \{0\}}$ (perfectoids)

Multiplication $(B^+(\cdot)^{\varphi^f = p})^f \to B^+(\cdot)^{\varphi^f = p^f}$ induces a morphism

$$\begin{array}{ccc} Z_{\mathrm{LT}} & \xrightarrow{m} & Z_{\mathcal{O}_K} \\ \circlearrowleft & & \circlearrowleft \\ (K^\times)^f \rtimes S_f & \longrightarrow & K^\times \end{array}$$

Let $\Delta := \ker((K^\times)^f \twoheadrightarrow K^\times)$.

Theorem (Fargues)

$(\Delta \rtimes S_f)\backslash Z_{LT} \xrightarrow{\sim} Z_{\mathcal{O}_K}$ *as pro-étale sheaves on* $\mathrm{Perf}_{\mathbb{F}}$.

Define $Z_{\mathcal{O}_K}^{gen} := \{|Y_0| = \cdots = |Y_{f-1}| \neq 0\} \subset Z_{\mathcal{O}_K}$ (open).
Then $Z_{\mathcal{O}_K}^{gen} \cong \mathrm{Spa}(A_\infty)$, where $A_\infty := \widehat{A^{1/p^\infty}}$ (perfectoid, φ bijective!).

$$\begin{array}{ccc} Z_{\mathrm{LT}}^{gen} := m^{-1}(Z_{\mathcal{O}_K}^{gen}) & \longrightarrow & Z_{\mathcal{O}_K}^{gen} \\ \cap & & \cap \\ Z_{\mathrm{LT}} & \xrightarrow{\quad m \quad} & Z_{\mathcal{O}_K} \end{array}$$

Proposition

$m : Z_{LT}^{gen} \to Z_{\mathcal{O}_K}^{gen}$ is a pro-étale $\Delta \rtimes S_f$-torsor.

Construction

$\overline{\rho} \Longrightarrow$ étale $(\varphi^f, \mathcal{O}_K^\times)$-module $\mathbb{F}((T_K^{1/p^\infty})) \otimes_{\mathbb{F}((T_K))} D_{\mathrm{LT},\sigma_0}$

$\Longrightarrow K^\times$-equivariant vector bundle $\mathcal{V}_{\overline{\rho}}$ on $\widetilde{G}_{\mathrm{LT},\mathbb{F}} \setminus \{0\}$

$\Longrightarrow (K^\times)^f \rtimes S_f$-equivariant vector bundle $\mathcal{V}_{\overline{\rho}}^{\otimes f}$ on $(\widetilde{G}_{\mathrm{LT},\mathbb{F}} \setminus \{0\})^f = Z_{\mathrm{LT}}$
$\rightsquigarrow$ restrict to Z_{LT}^{gen}

$\overset{\text{descent}}{\Longrightarrow} K^\times$-equivariant v.b. $(m_* \mathcal{V}_{\overline{\rho}}^{\otimes f}|_{Z_{\mathrm{LT}}^{gen}})^{\Delta \rtimes S_f}$ on $Z_{\mathcal{O}_K}^{gen} = \mathrm{Spa}(A_\infty)$

$=$ étale $(\varphi, \mathcal{O}_K^\times)$-module $D_{A_\infty}^{\otimes}(\overline{\rho})$ over A_∞ of rank $(\dim \overline{\rho})^f$

$\Longleftrightarrow$ étale $(\varphi, \mathcal{O}_K^\times)$-module $D_A^{\otimes}(\overline{\rho})$ over A of rank $(\dim \overline{\rho})^f$

Theorem (BHHMS, 2022)

If $\overline{r}|_{G_{F_p}}$ is tamely ramified and strongly generic, then

$$D_A(\pi) \cong D_A^{\otimes}(\overline{r}|_{G_{F_p}}).$$

Compute LHS: Use crucially that diagram is known (Dotto–Le):

$$\begin{array}{ccc} \pi^{I_1} & \hookrightarrow & \pi^{1+pM_2(\mathcal{O}_K)} \\ \circlearrowleft & & \circlearrowleft \\ \begin{pmatrix} & 1 \\ p & \end{pmatrix} & & \mathrm{GL}_2(k) \end{array}$$

- $\exists\ 0 \neq \mu : A \to \mathbb{F}$ cts., $\mu \circ \psi_A = (-1)^{f-1}\mu$ (unique up to $\mathbb{F}^{\times}$)
- μ induces $\mathrm{Hom}_A(D_A(\pi), A) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cts}}(D_A(\pi), \mathbb{F})$.
- Use definition of $D_A(\pi)$ and weight cycling to define 2^f elements $x_\sigma \in \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cts}}(D_A(\pi), \mathbb{F})$ ($\sigma \in W(\overline{r}) =$ Serre weights of $\overline{r}$).
- (Most subtle) Prove that x_σ descends to $\mathrm{Hom}_A(D_A(\pi), A)$.
- Show the (x_σ) form basis + compute $(\varphi, \mathcal{O}_K^{\times})$-actions.

Theorem (BHHMS, 2022)

If $\overline{r}|_{G_{F_p}}$ is tamely ramified and strongly generic, then

$$D_A(\pi) \cong D_A^{\otimes}(\overline{r}|_{G_{F_p}}).$$

Compute RHS: Compute $D_A^{\otimes}(\overline{\rho})$ for any absolutely irreducible $\overline{\rho}$.
Key:

$$\begin{array}{ccc} Z_{\mathrm{LT}}^{gen} & \xrightarrow{\ \Delta \rtimes S_f\text{-tors.}\ } & Z_{\mathcal{O}_K}^{gen} \\ \text{open } \bigcup & \nearrow_{\Delta_1\text{-tors.}} & \\ U & & \end{array}$$

Here, $\Delta_1 := \ker((\mathcal{O}_K^{\times})^f \twoheadrightarrow \mathcal{O}_K^{\times}) \subset \Delta = \ker((K^{\times})^f \twoheadrightarrow K^{\times})$.
The affinoid $U = \mathrm{Spa}(A'_{\infty})$ is given by

$$|T_{K,0}| = |T_{K,1}|^{p^{-1}} = \cdots = |T_{K,f-1}|^{p^{-(f-1)}} \neq 0.$$