Eisenstein series, p-adic deformations, Galois representations, and the group G_2

Sam Mundy

Princeton University

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The main theorem

Setup

Fix:

- p a prime,
- F a cuspidal holomorphic eigenform with
  - even weight $k \geq 4$,
  - level N with $p \nmid N$,
  - trivial nebentypus.

Then we get:

- π_F a cuspidal automorphic representation of $GL_2(\mathbb{A})$,
- $\rho_F : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ the Galois representation, by Deligne.

Main Theorem

Assume:

- $L(s, \pi_F, \mathrm{Sym}^3)$ vanishes to odd order at $s = \frac{1}{2}$,
- F is not CM,
- $4, 9 \nmid N$,
- The Hecke polynomial of F at p has simple roots.

Then, under Arthur's conjectures, we have that the Bloch–Kato Selmer group

$$H^1_f(\mathbb{Q}, (\mathrm{Sym}^3 \rho_F)^\vee(\tfrac{3k}{2} - 1))$$

is nontrivial.

This is a special case of the Bloch–Kato conjecture, which predicts in this case:

$$\mathrm{ord}_{s=1/2} L(s, \pi_F, \mathrm{Sym}^3) = \dim H^1_f(\mathbb{Q}, (\mathrm{Sym}^3 \rho_F)^\vee(\tfrac{3k}{2} - 1)).$$

Some remarks:

• In the special case of $N = 1$, the four bulleted hypotheses in the theorem are automatic. So the theorem reads:

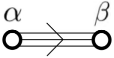
$$\dim H^1_f(\mathbb{Q}, (\mathrm{Sym}^3\, \rho_F)^\vee(\tfrac{3k}{2} - 1)) \neq 0$$

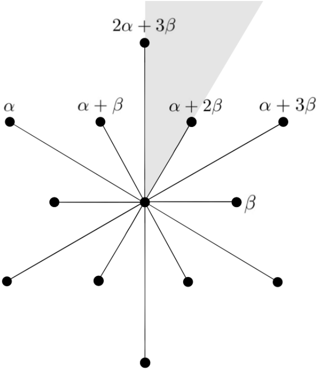
under Arthur.

• The proof of the main theorem makes use of a p-adic deformation of automorphic representations on the exceptional group G_2.

- Arthur, both local and global, is needed for:
  - Proving a multiplicity formula for certain CAP forms on G_2,
  - Studying those CAP forms at infinity,
  - Cases of functoriality from G_2 to GL_7.

Let G_2 now be the split group over $\mathbb{Q}$ with Dynkin diagram:





Roots and parabolics

Let $P = MN$ be the long root parabolic; M contains α.
Let $P^\vee = M^\vee N^\vee$ be the short root parabolic; $M^\vee$ contains β.

Then $M \cong GL_2$ and $M^\vee \cong GL_2$.

$G_2 = G_2^\vee$, and passing to dual switches long and short simple roots.

The three main steps

Step 1. Cohomology.

For this step, $k \geq 4$, N is arbitrary, and F is allowed to be CM. Assume $L(\frac{1}{2}, \pi_F, \mathrm{Sym}^3) = 0$.

Parabolically induce π_F along P to Π, an automorphic representation of $G_2(\mathbb{A})$. More precisely, Π is the Langlands quotient of

$$\mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F \otimes \delta_{P(\mathbb{A})}^{1/10+1/2}).$$

Locate the finite part Π_f in the cohomology of the locally symmetric spaces attached to G_2,

$$H^*(X_{G_2}, V_{\lambda_0}).$$

Here λ_0 is the weight $\frac{k-4}{2}(2\alpha + 3\beta)$, and V_{λ_0} denotes the representation of $G_2(\mathbb{C})$ of highest weight λ_0

Step 1. Cohomology.

We locate Π_f separately in both:

- Eisenstein cohomology. (Key input: Franke–Schwermer plus Langlands–Shahidi.)
- Cuspidal cohomology. (Key input: Arthur's multiplicity formula, and Adams–Johnson.)

In cuspidal cohomology, Π_f appears in different degrees depending on the sign $\epsilon(\frac{1}{2}, \pi_F, \mathrm{Sym}^3)$.

Step 2. p-adic deformation.

Now assume $p \nmid N$, and $\epsilon(\frac{1}{2}, \pi_F, \mathrm{Sym}^3) = -1$.

- Take a particular "critical p-stabilization" Π_f^{crit} of Π_f.
- Use Step 1 along with the machinery of Urban's eigenvariety to compute the "cuspidal overconvergent multiplicity" of Π_f^{crit}. It depends on the "classical" multiplicity of Π_f in $H^*(X_{G_2}, V_{\lambda_0})$, and is nonzero when $\epsilon(\frac{1}{2}, \pi_F, \mathrm{Sym}^3) = -1$.
- Urban's eigenvariety p-adically deforms Π_f^{crit} in a generically cuspidal family $\mathcal{E}$ over all p-adic weights.
- Use the theory of types (due to Fintzen) to show that the members of $\mathcal{E}$ have the same ramification properties at Π_f at primes $\ell | N$; here we use $4, 9 \nmid N$.

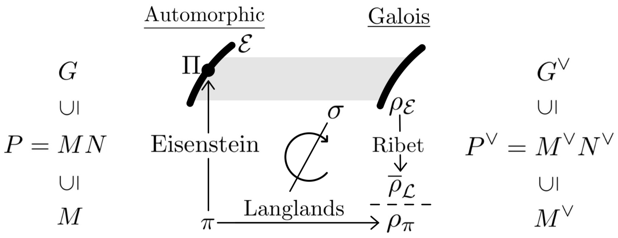
Step 3. Galois representations.

Now assume F is not CM. Then ρ_F has large image, and $\mathrm{Sym}^2\,\rho_F$ and $\mathrm{Sym}^3\,\rho_F$ are irreducible.

- Construct a Galois representation $\rho_{\mathcal{E}} : G_{\mathbb{Q}} \to G_2(\overline{\mathrm{Frac}(\mathcal{O}(\mathcal{E}))})$ interpolating those for the members of $\mathcal{E}$. (Key input: Lafforgue's pseudocharacters.)
- Construct a particular lattice $\mathcal{L}$ in $\rho_{\mathcal{E}}$.
- Show that the specialization of $\mathcal{L}$ at Π_f^{crit} gives an unramified (by the local properties of $\mathcal{E}$), crystalline (use here in particular that the Hecke polynomial of F at p has simple roots) extension

$$0 \to (\mathrm{Sym}^3\,\rho_F)^\vee(\tfrac{3k}{2} - 1) \to E \to \overline{\mathbb{Q}}_p \to 0.$$

Then $\sigma := [E] \in H^1_f(\mathbb{Q}, (\mathrm{Sym}^3\,\rho_F)^\vee(\tfrac{3k}{2} - 1))$.



The Skinner–Urban method; we use $G = G_2$, $\pi = \pi_F$.

More on the p-adic family $\mathcal{E}$

Overview

• To construct the family $\mathcal{E}$ deforming Π_f^{crit}, we need to use the machinery in the construction of Urban's eigenvariety.

• Urban's eigenvariety is constructed for groups with discrete series, which G_2 has.

• The family $\mathcal{E}$ will be a family of *representations*, not a family of forms.

Preliminaries: Groups

B is the Borel with maximal torus T and unipotent radical U.

Let $K_f^p \subset G_2(\mathbb{A}_f^p)$ be an open compact subgroup such that Π_f has fixed vectors by $K_f^p \cdot G_2(\mathbb{Z}_p)$.

Let $I \subset G_2(\mathbb{Z}_p)$ be the Iwahori subgroup;

$$I = \{g \in G_2(\mathbb{Z}_p) \mid (g \bmod p) \in B(\mathbb{F}_p)\}.$$

Let $T^+ \subset T(\mathbb{Q}_p)$ be the monoid defined by

$$T^+ = \left\{t \in T(\mathbb{Q}_p) \,\middle|\, tU(\mathbb{Z}_p)t^{-1} \subset U(\mathbb{Z}_p)\right\}.$$

Preliminaries: Hecke algebras

Let $\mathcal{U}_p$ be the $\mathbb{Z}_p$-subalgebra of $C_c^\infty(I\backslash G_2(\mathbb{Q}_p)/I, \mathbb{Z}_p)$ generated by

$$\frac{1}{\mathrm{Vol}(I)}\,\mathrm{char}(ItI), \qquad t \in T^+.$$

Let $\mathcal{H}_p(K_f^p)$ be the Hecke algebra defined by

$$\mathcal{H}_p(K_f^p) = \mathcal{U}_p \otimes_{\mathbb{Z}_p} C_c^\infty(K_f^p\backslash G_2(\mathbb{A}_f^p)/K_f^p, \mathbb{Q}_p).$$

It is a subalgebra of $C_c^\infty(G_2(\mathbb{A}_f), \mathbb{C})$, identifying $\overline{\mathbb{Q}}_p \cong \mathbb{C}$.

Distributions

For μ a dominant integral weight of T, define the distribution

$$I_0^{\mathrm{cl}}(f,\mu,K_f^p) = \mathrm{Tr}(f|H^*_{\mathrm{cusp}}(X_{G_2},V_\mu)), \qquad f\in\mathcal{H}_p(K_f^p).$$

Let $\mathfrak{X}$ be the rigid analytic *weight space*, defined by

$$\mathfrak{X}(L) = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p),L^\times),$$

for $L/\mathbb{Q}_p$ finite.

Urban defines, this time for any $\mu\in\mathfrak{X}(\overline{\mathbb{Q}}_p)$, a distribution

$$I_0^\dagger(\cdot,\mu,K_p^f):\mathcal{H}_p(K_f^p)\to\overline{\mathbb{Q}}_p.$$

The distributions $I_0^\dagger(\cdot,\mu,K_p^f)$ are analytic in μ, and for μ dominant integral and regular, they contain $I_0^{\mathrm{cl}}(\cdot,\mu,K_f^p)$ as a summand.

p-stabilizations

A *p-stabilization* of an irreducible smooth $G_2(\mathbb{A}_f)$-representation σ with $I \cdot K_f^p$-fixed vectors is, by definition, an irreducible constituent of $\sigma^{I \cdot K_f^p}$ when it is viewed as an $\mathcal{H}_p(K_f^p)$-module.

The p-stabilizations of σ are determined by the I-fixed vectors in the local constituent σ_p of σ at p.

Each p-stabilization of σ has a *slope*, which is a $\mathbb{Q}$-valued weight of T determined by the p-adic valuation of eigenvalues of operators in $\mathcal{U}_p$.

Theorem (Urban)

If μ is dominant, integral and regular and τ is an irreducible constituent of $I_0^{\mathrm{cl}}(\cdot, \mu, K_f^p)$ whose slope is not too large with respect to μ (technically, noncritical), then there is a generically cuspidal p-adic family of automorphic representations passing through τ. This means there are

- *A rigid analytic space $\mathfrak{Y}$, generically finite over a neighborhood $\mathfrak{U}$ of μ in $\mathfrak{X}$,*
- *A distribution $I : \mathcal{H}_p(K_f^p) \to \mathcal{O}(\mathfrak{Y})$,*
- *A point $y_0 \in \mathfrak{Y}(\overline{\mathbb{Q}}_p)$ over μ,*

such that the specialization of I at y_0 contains τ as a summand, and for generic y in $\mathfrak{Y}$ over a dominant integral and regular weight μ_y, the specialization of I at y is an irreducible constituent of $I_0^{\mathrm{cl}}(\cdot, \mu_y, K_f^p)$.

Problem

Unfortunately, Urban's theorem does not apply here:

- The p-stabilization Π_f^{crit} of Π_f we must use has critical slope;
- The weight λ_0 of Π is also irregular.

Solution

Urban's theorem above is actually an immediate corollary of:

Theorem (Urban)

If μ is any p-adic weight and τ is an irreducible constituent of $I_0^{\dagger}(\cdot, \mu, K_f^p)$, then there is a generically cuspidal p-adic family of automorphic representations passing through τ.

So it suffices to show that Π_f^{crit} is a constituent of $I_0^{\dagger}(\cdot, \lambda_0, K_f^p)$.

One defines the distribution $I_0^\dagger(\cdot, \mu, K_f^p)$ in terms of another one, $I^\dagger(\cdot, \mu, K_f^p)$, and analogous distributions for smaller Levis of G_2.

Definition

We let

$$I_0^\dagger(f, \mu, K_f^p) = I^\dagger(f, \mu, K_f^p) - \sum_{Q \in \{P, P^\vee, B\}} \sum_{w \in W_{\mathrm{Eis}}^{M_Q}} (-1)^{\dim(N_Q) - \ell(w)} I_{M_Q,0}^\dagger(f_{M_Q,w}, w * \mu + 2\rho_Q, K_{f,M_Q}^p).$$

The sum over proper parabolics can be made completely explicit. What about the term $I^\dagger(\cdot, \mu, K_f^p)$?

Theorem (Urban)

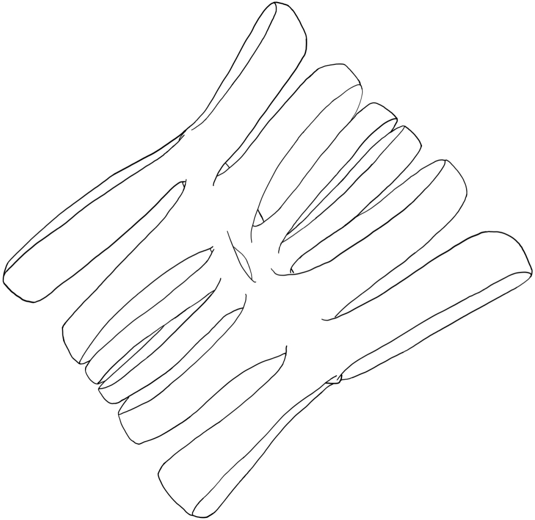
$$I^{\mathrm{cl}}(f,\mu,K_f^p)=\sum_{w\in W_{G_2}}(-1)^{\ell(w)}I^{\dagger}(f^{w,\mu},w*\mu,K_f^p).$$

Slope considerations show that Π_f^{crit} can only be a constituent of $I^{\dagger}(f^{w,\lambda_0},w*\lambda_0,K_f^p)$ for $w=1,w_\beta$.

Combined with the definition of $I_0^{\dagger}$, one computes everything explicitly and obtains:

Theorem

Π_f^{crit} *appears with multiplicity at least* 3 *in* $I_0^{\dagger}(\cdot,\lambda_0,K_f^p)$ *if* $\epsilon(\frac{1}{2},\pi_F,\mathrm{Sym}^3)=-1$ *(under Arthur).*



Thank you!