Baby PIH: Parameterized Inapproximability of Min CSP

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Outline

- Background
  - Parameterized Complexity
  - Constraint Satisfaction Problem (CSP)
  - Parameterized Inapproximability Hypothesis (PIH)
- Our Result
  - Baby PIH
- Proof Overview

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Parameterized Complexity

- Each input instance x is associated with a parameter $k \in \mathbb{N}$
- Complexity is measured as a function of both $n = |x|$ and k.

- FPT (Fixed-Parameter Tractable):
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k-Vertex Cover

- Input:
  - $G = (V, E)$
- Output:
  - $\exists v_1, \ldots, v_k \in V$ covering all the edges?

(Multi-colored) k-Clique

- Input:
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FPT $\subseteq$ W[1]

Constraint Satisfaction Problem

2-CSP

- Input: $\Pi = (X, \Sigma, \Phi)$
  - X: a set of variables
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CSP Value

max. fraction of constraints satisfiable by some $\sigma: X \to \Sigma$

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  - PCP Theorem:
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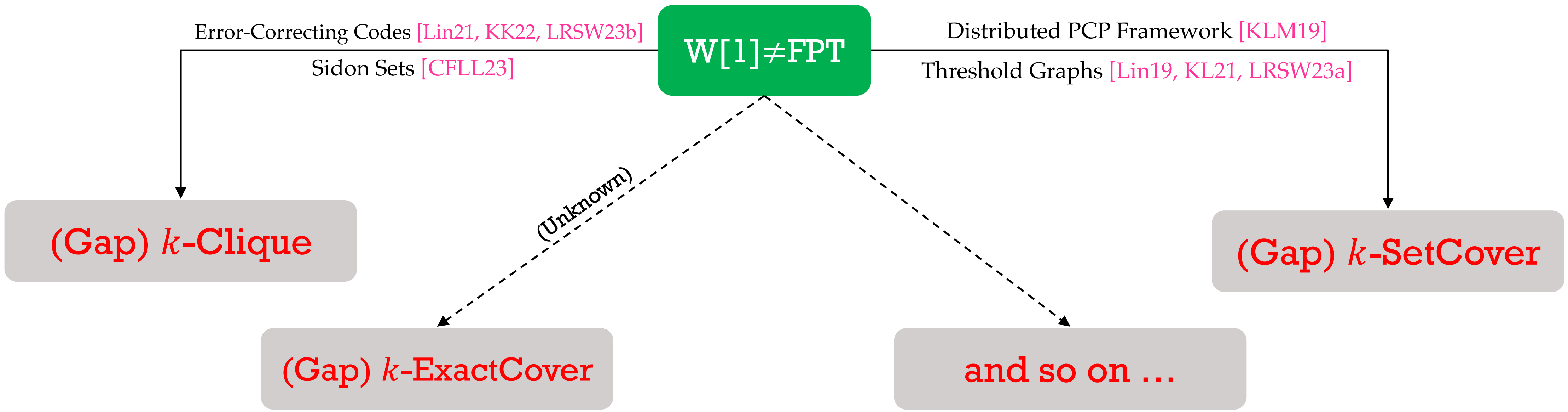
    - no $f(k) \cdot n^{O(1)}$ time algorithm assuming W[1]≠FPT
  - PIH (Parameterized Inapproximability Hypothesis) [LRSZ20]:
    - no $f(k) \cdot n^{O(1)}$ time algorithm for (1 vs 0.9) gap version assuming W[1]≠FPT
    - Parameterized analog of the PCP theorem!

Parameterized Inapproximability Hypothesis



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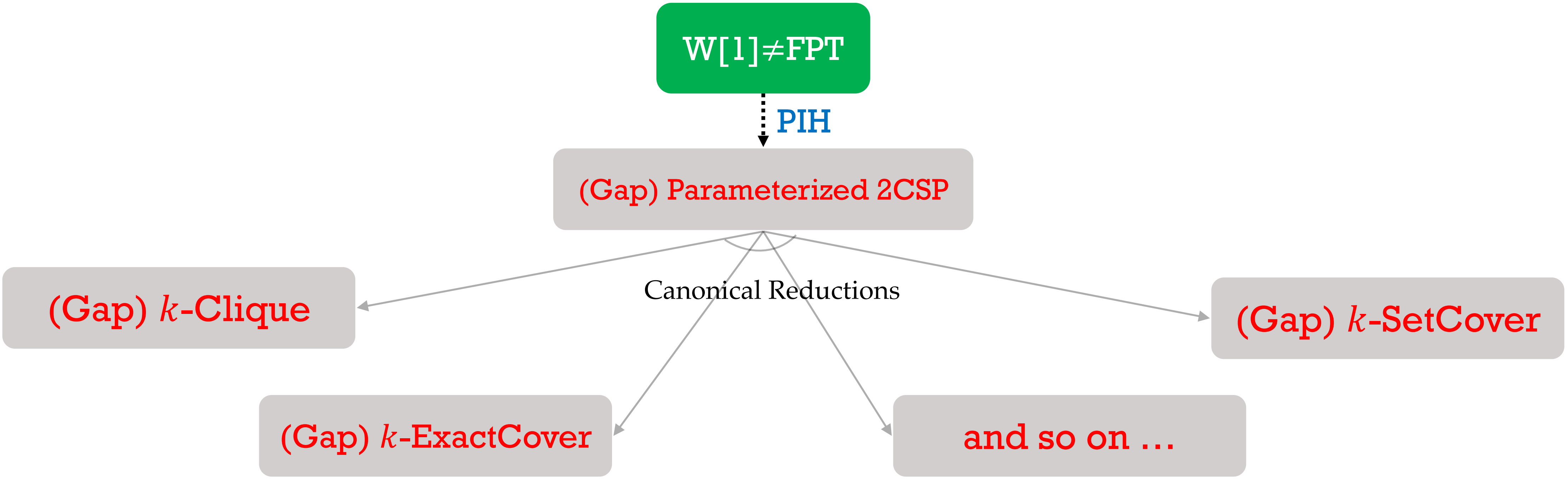

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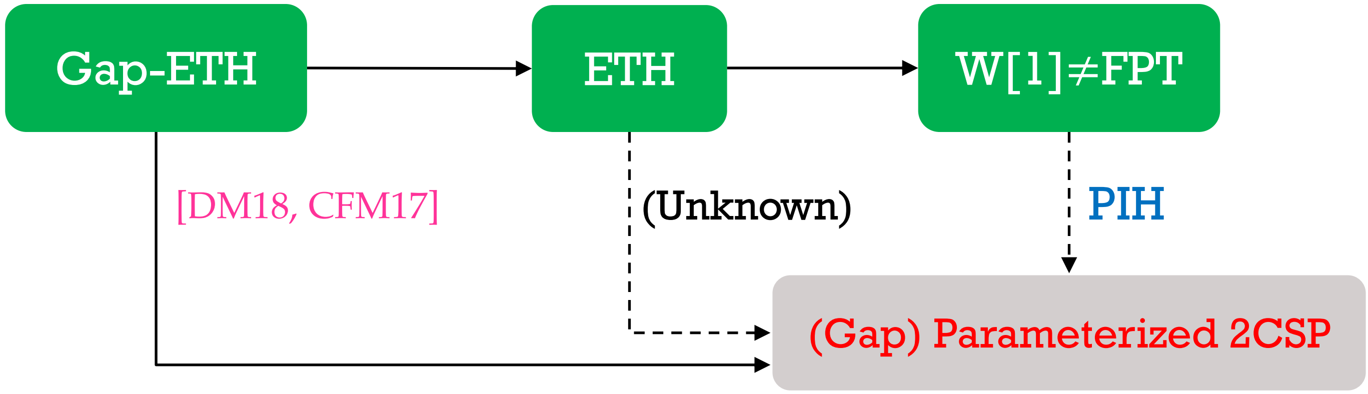

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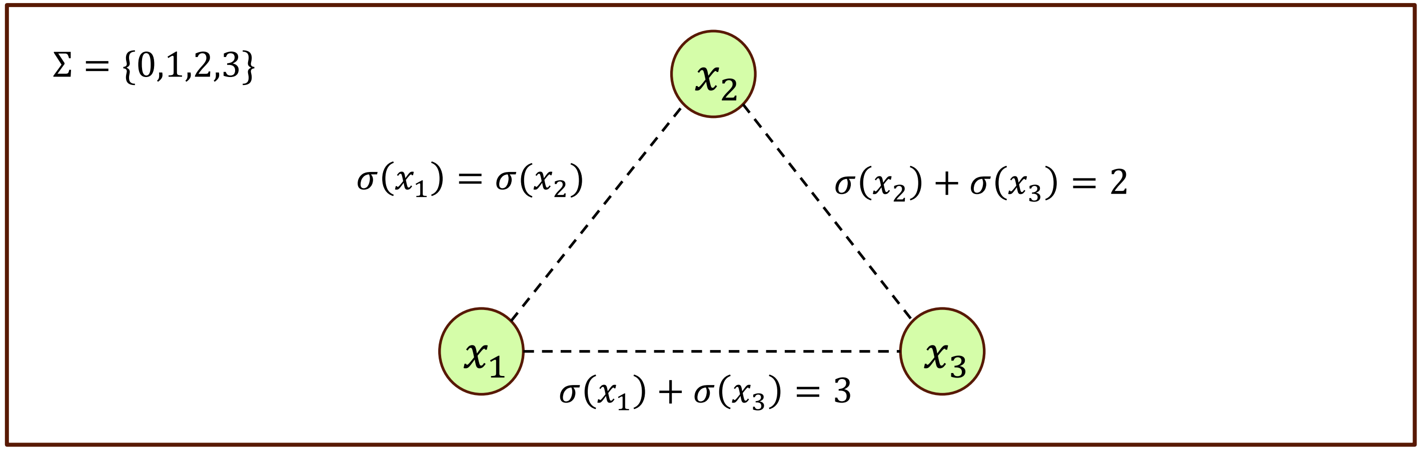

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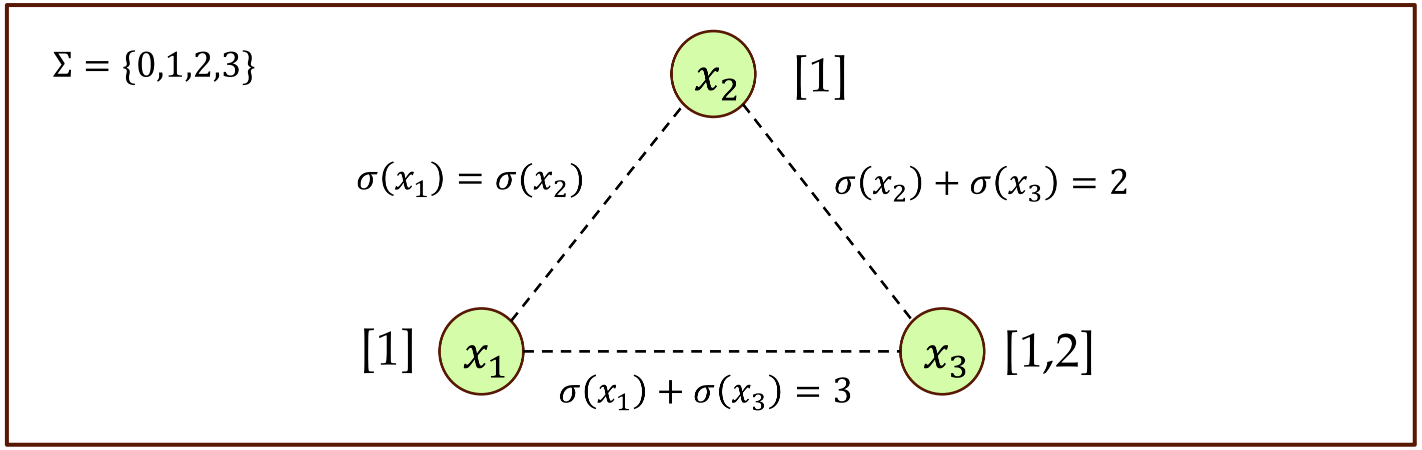

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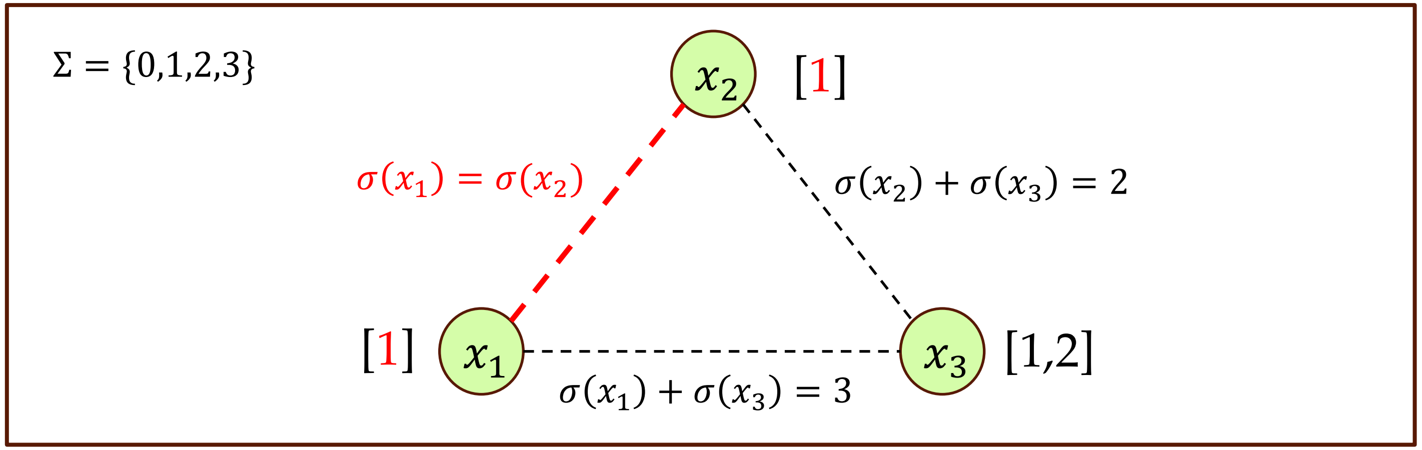

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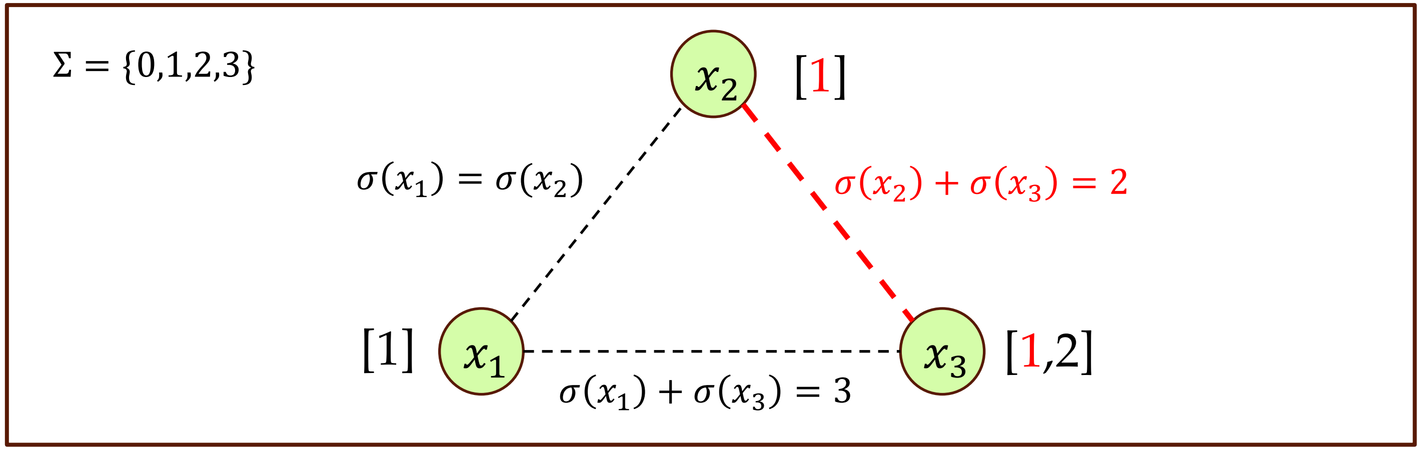

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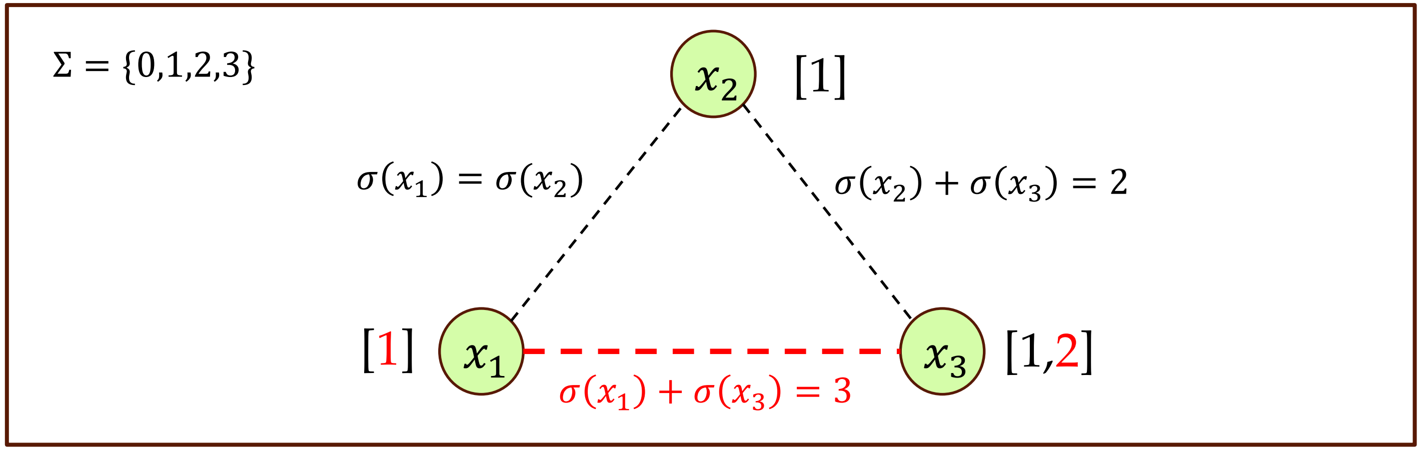

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  - For any $r > 1$, It's NP-hard to distinguish between [1-list satisfiable] and [not even r-list satisfiable].
- $\Leftarrow$ PCP $\quad\Uparrow\quad\quad \Uparrow$ (when $\varepsilon < 1/r^2$)
  - For any $\varepsilon > 0$, It's NP-hard to distinguish between [CSP Value $=1$] and [CSP Value $<\varepsilon$].

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  - Assuming NP≠P, for any $r > 1$, distinguishing between [1-list satisfiable] and [not even r-list satisfiable] cannot be done in $|\Pi|^{O(1)}$ time.
  - (A combinatorial proof)
  - (Enough to prove the NP-hardness of some PCSPs (e.g., $(2+\varepsilon)$-SAT))

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  - Assuming W[1]≠FPT, … cannot be done in $f(|X|) \cdot |\Sigma|^{O(1)}$ time.
  - (An itself interesting inapproximability result for list-satisfiability of CSP)
  - (A step towards PIH)
  - (Enough to get some applications of PIH?)

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  - (Enough to get some applications of PIH?)
    - not sure…, but something stronger is enough!
    - PIH ⇒ Average Baby PIH ⇒ Baby PIH

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- Follows from and extends [Barto-Kozik'22]'s proof of Baby PCP Theorem
- Direct Product Construction

2-CSP

$\Pi = (X, \Sigma, \Phi)$

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***t*-wise Direct Product 2-CSP**

$$\Pi^{\odot t} = \left(\binom{X}{t}, \Sigma^t, \Phi'\right)$$

partial satisfying assignments
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- (Want to show):
- For any $r > 1$, there exists t depending on r, such that for every Π,
  - (Completeness) If Π is satisfiable, then so is $\Pi^{\odot t}$.
  - (Soundness) If Π is not satisfiable, then $\Pi^{\odot t}$ is not r-list satisfiable.

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- Reduction time: $n^{O_r(1)}$ where $n = |\Pi|$
  - a unified proof for both Baby PCP and Baby PIH!

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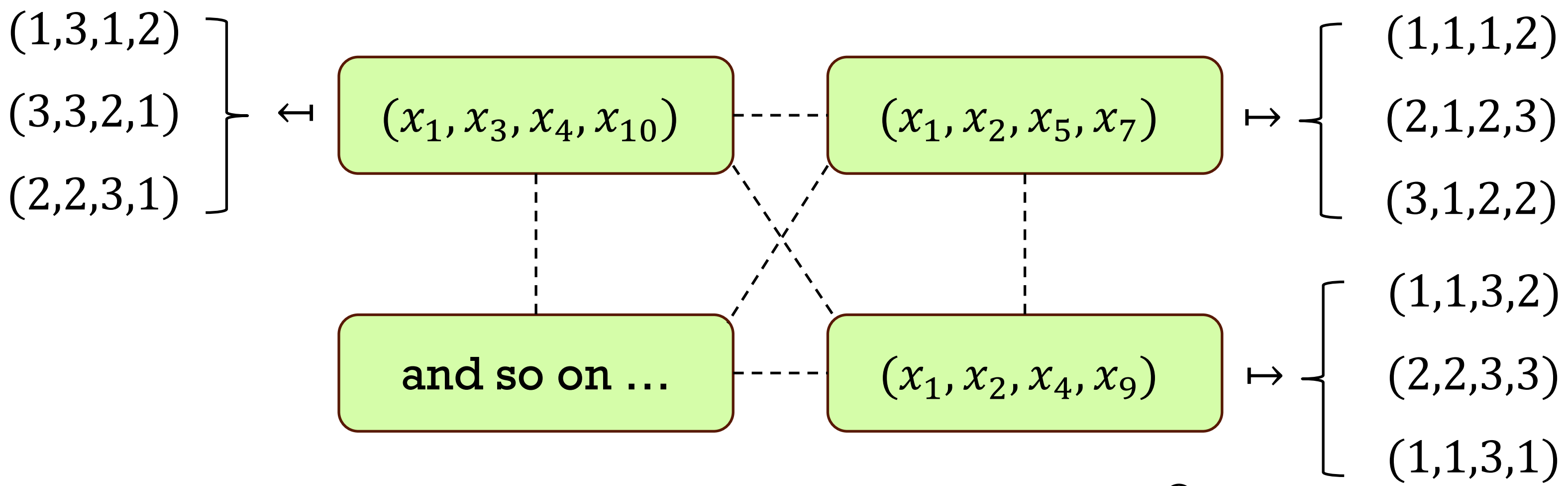

A 3-list satisfying assignment for $\Pi^{\odot 4}$

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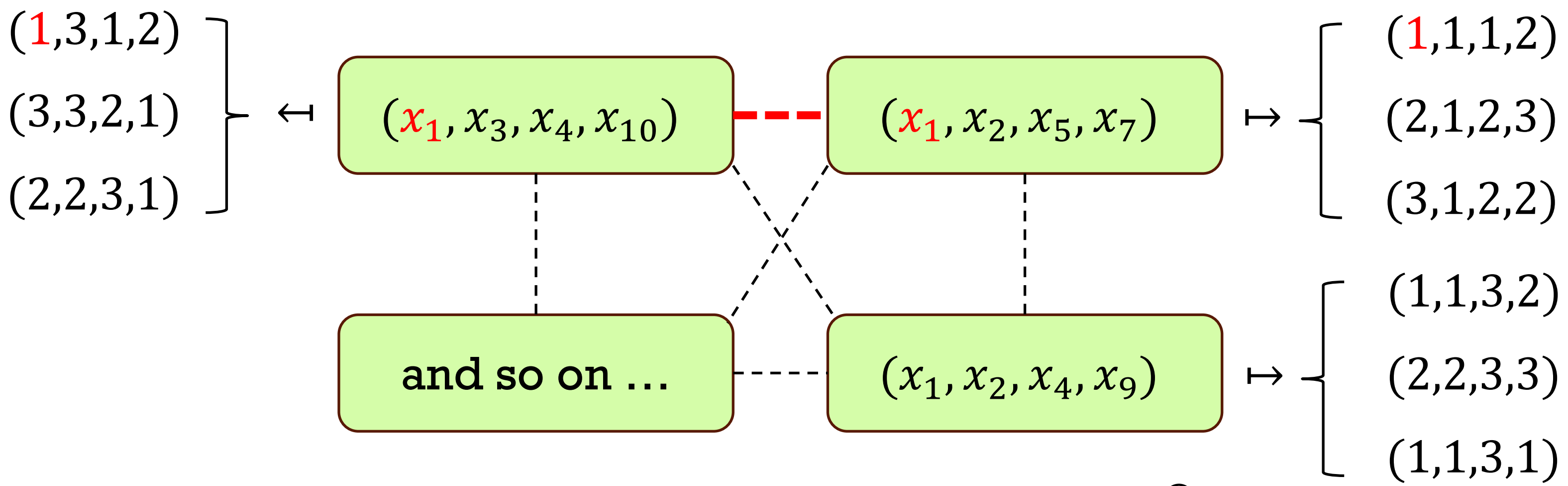

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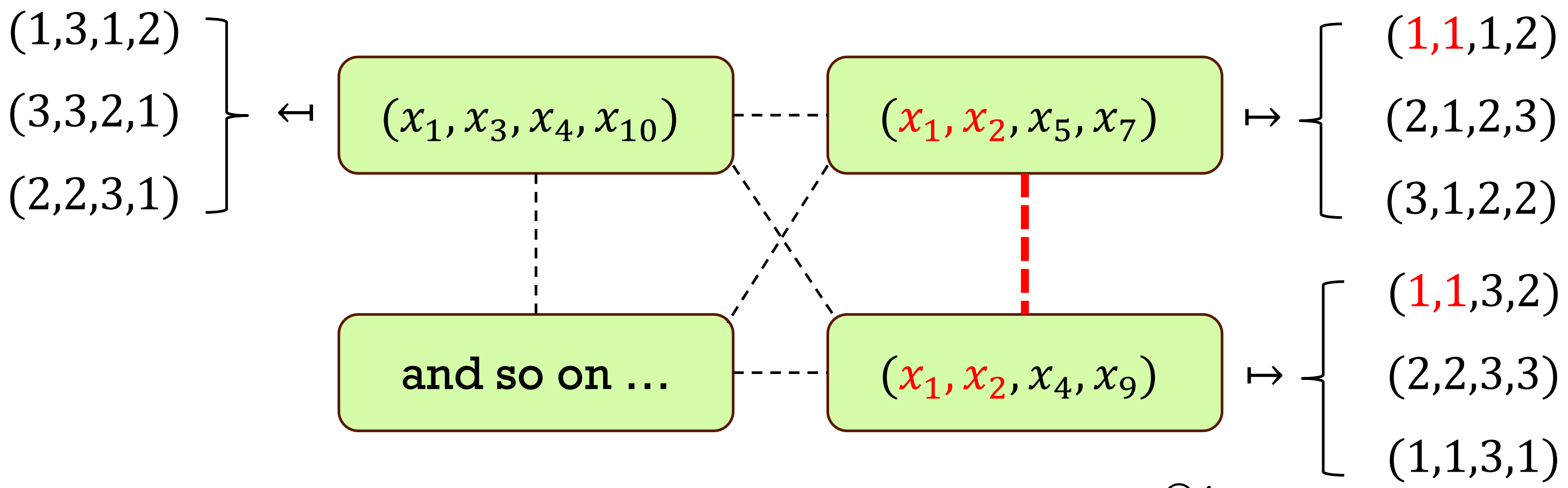

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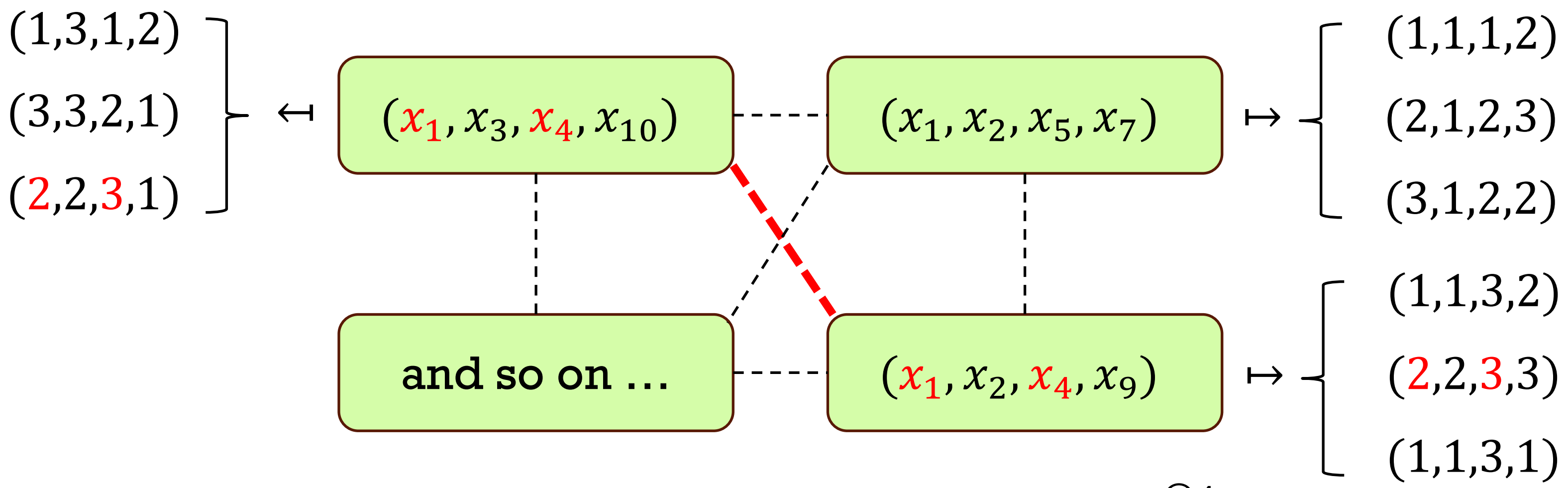

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    - for each set $S \in \binom{X}{t'}$, choose a set $T \in \binom{X}{t}$ with $S \subseteq T$
    - the list $\sigma'(S)$ is inherited from the list $\sigma(T)$ (at the hope of decreasing the list size by 1)

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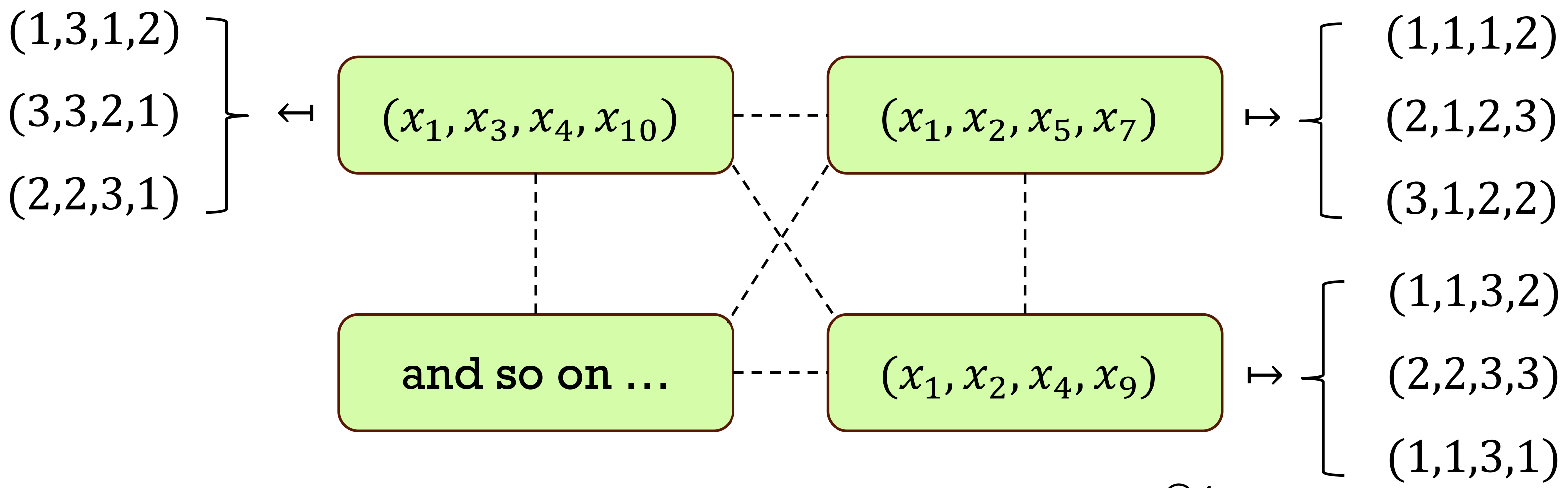

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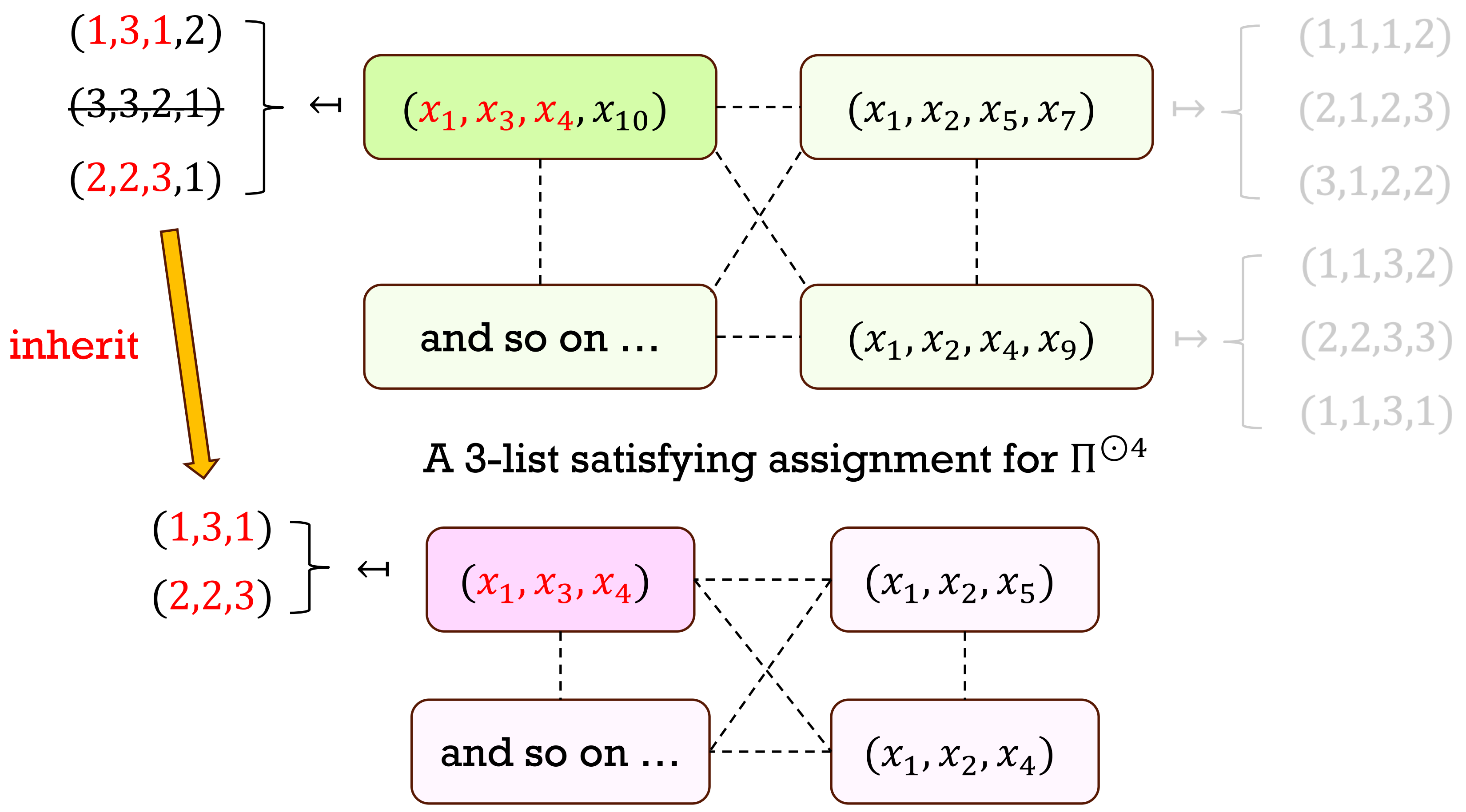

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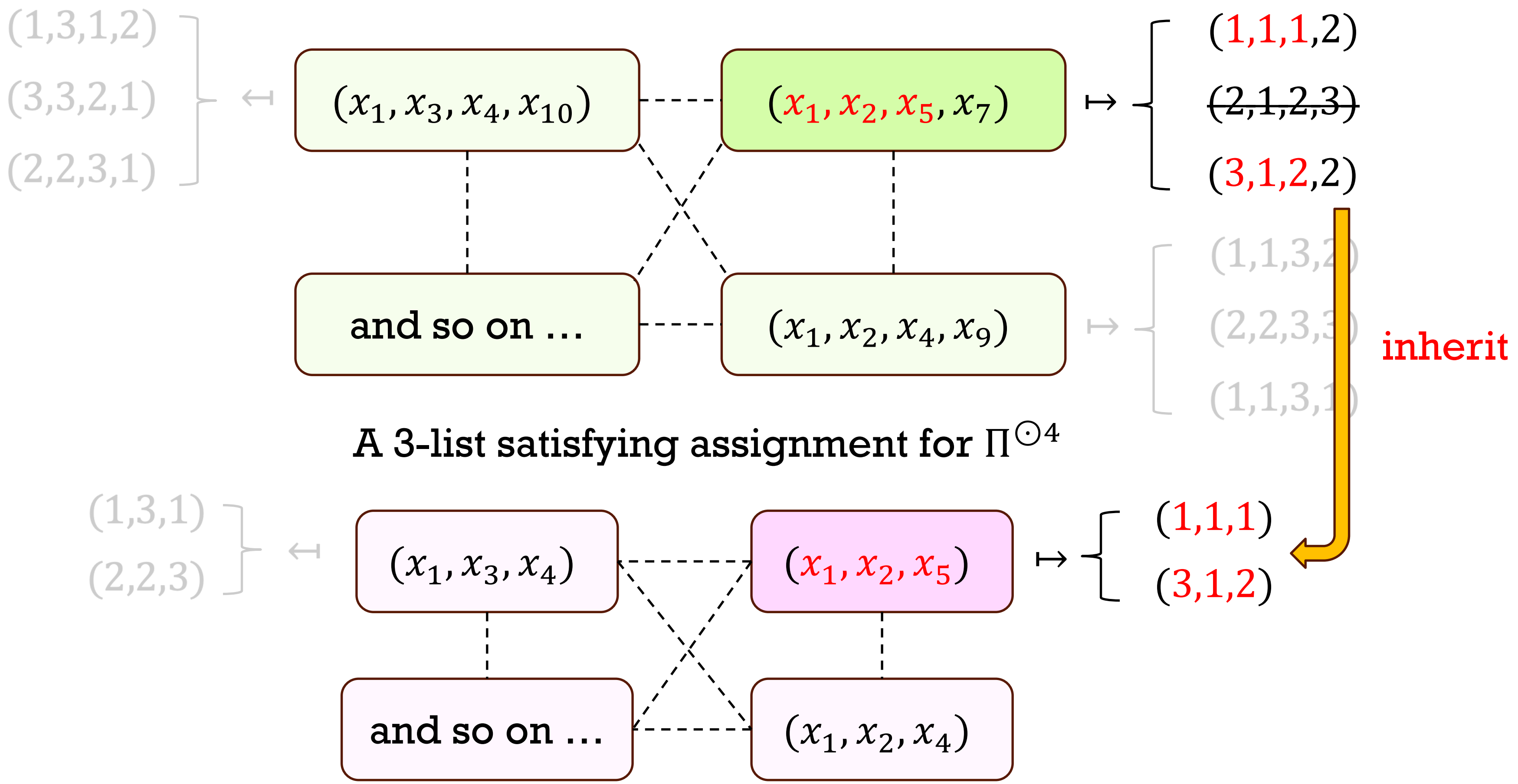

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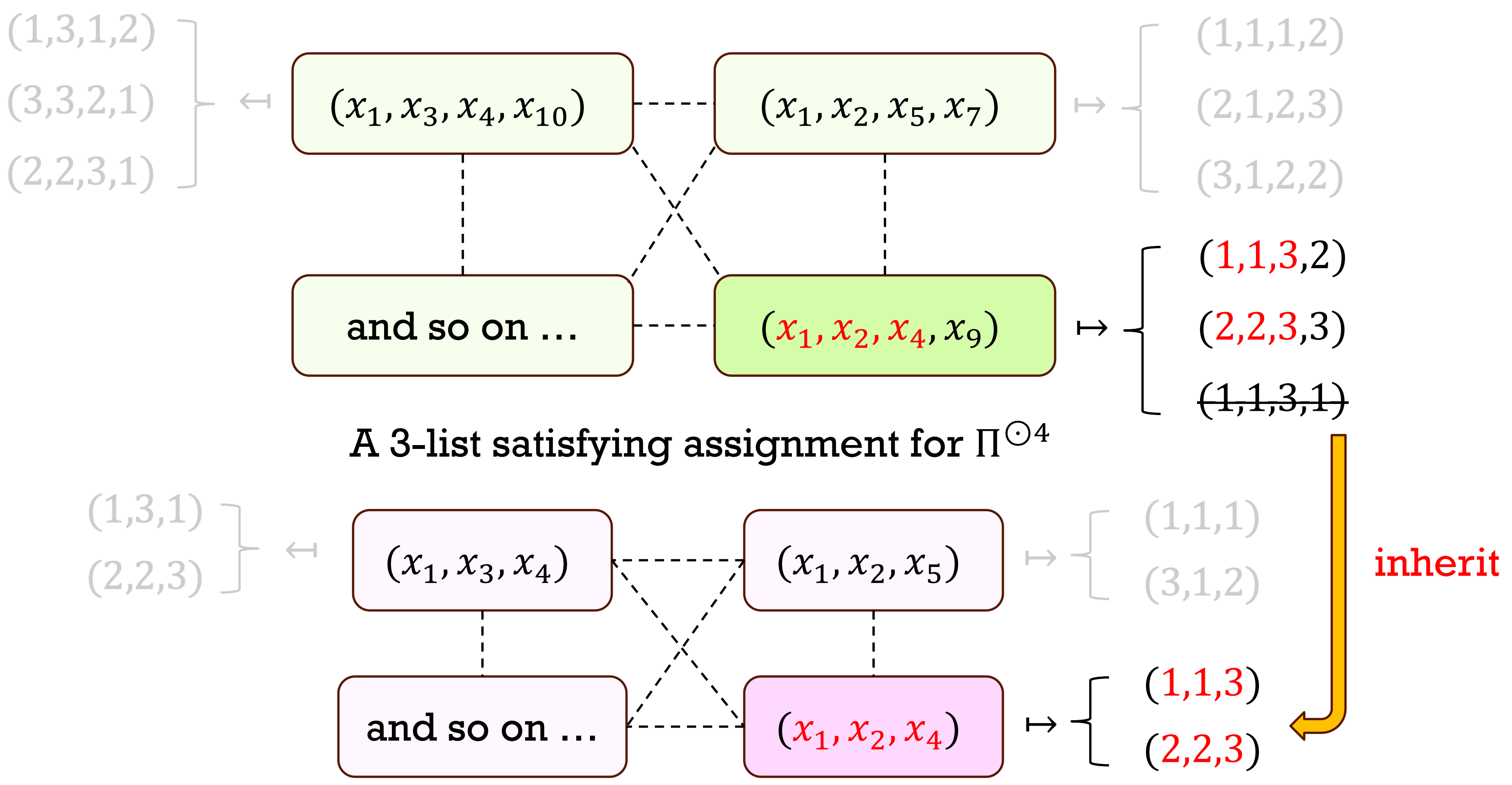

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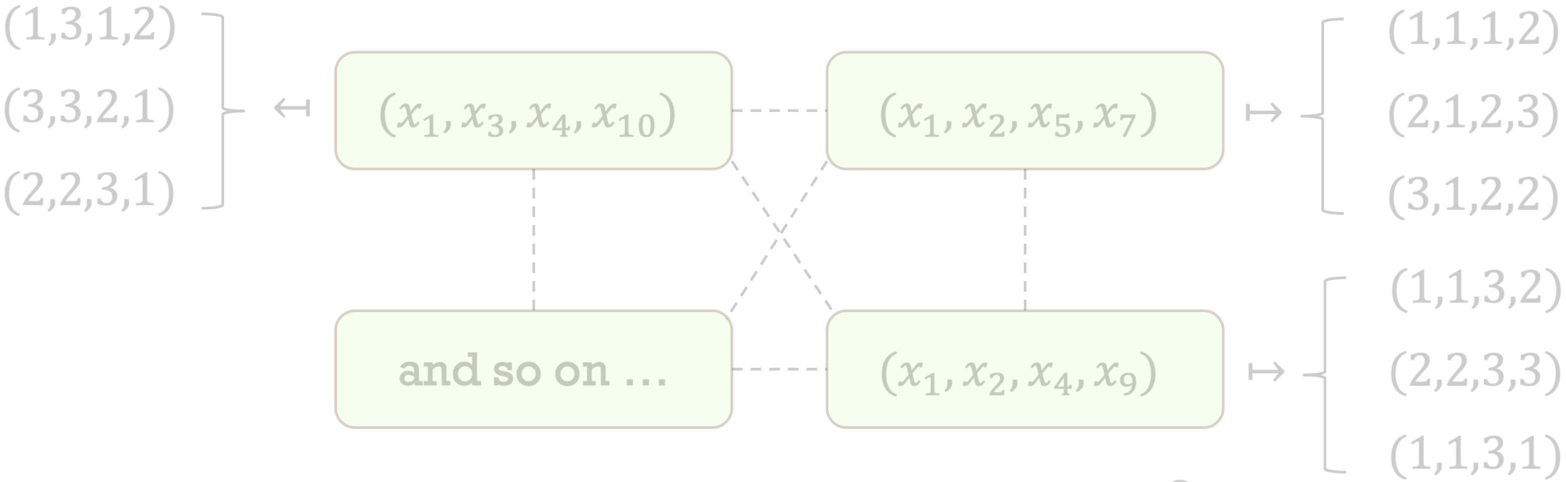

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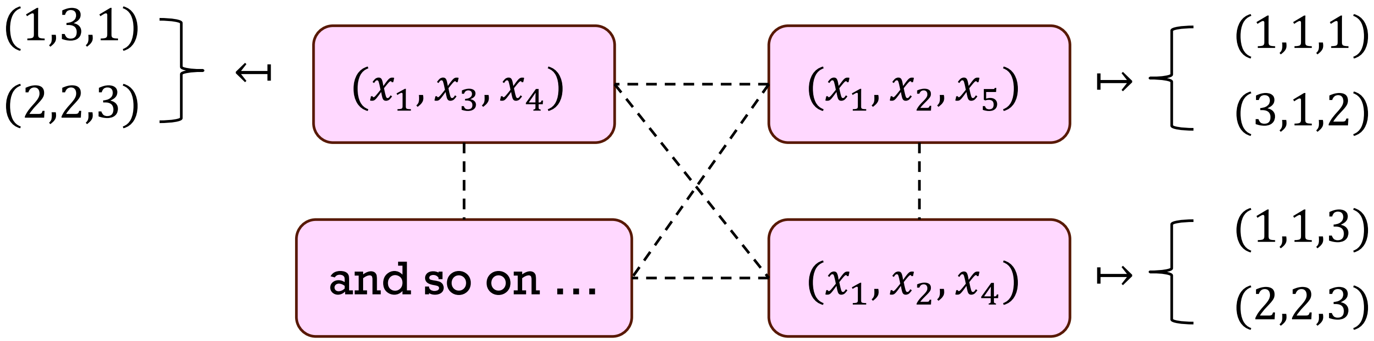

A 2-list satisfying assignment for $\Pi^{\odot 3}$?

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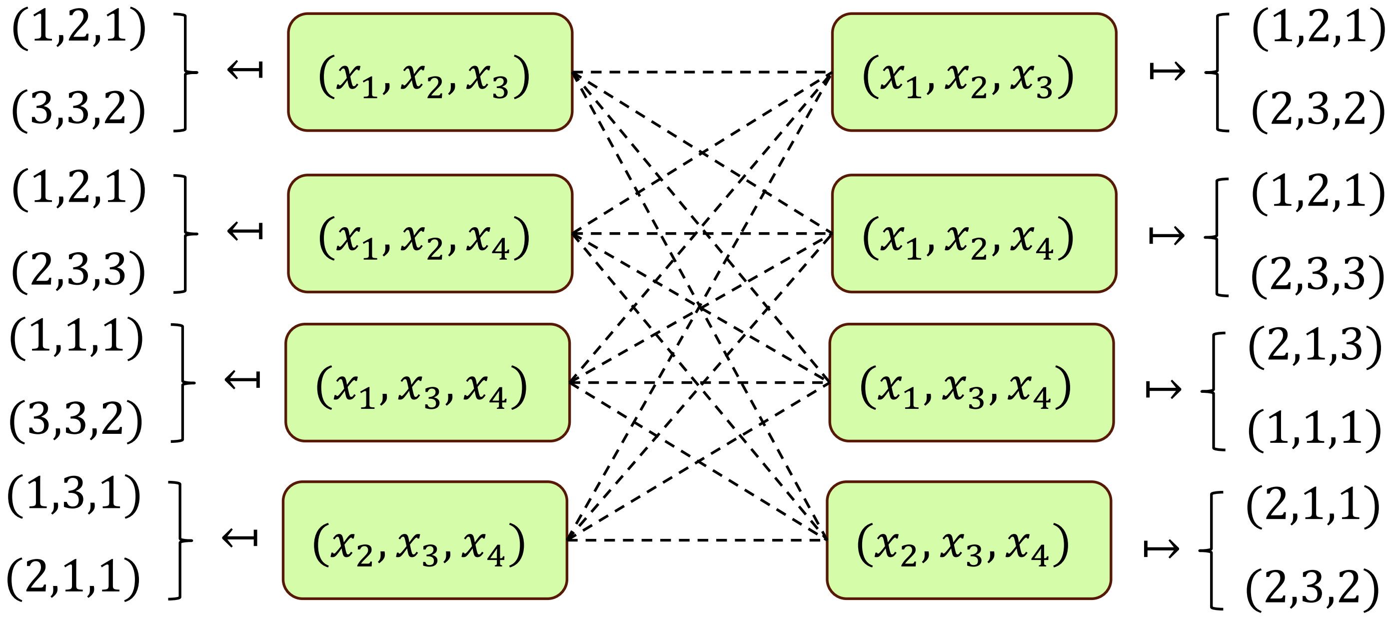

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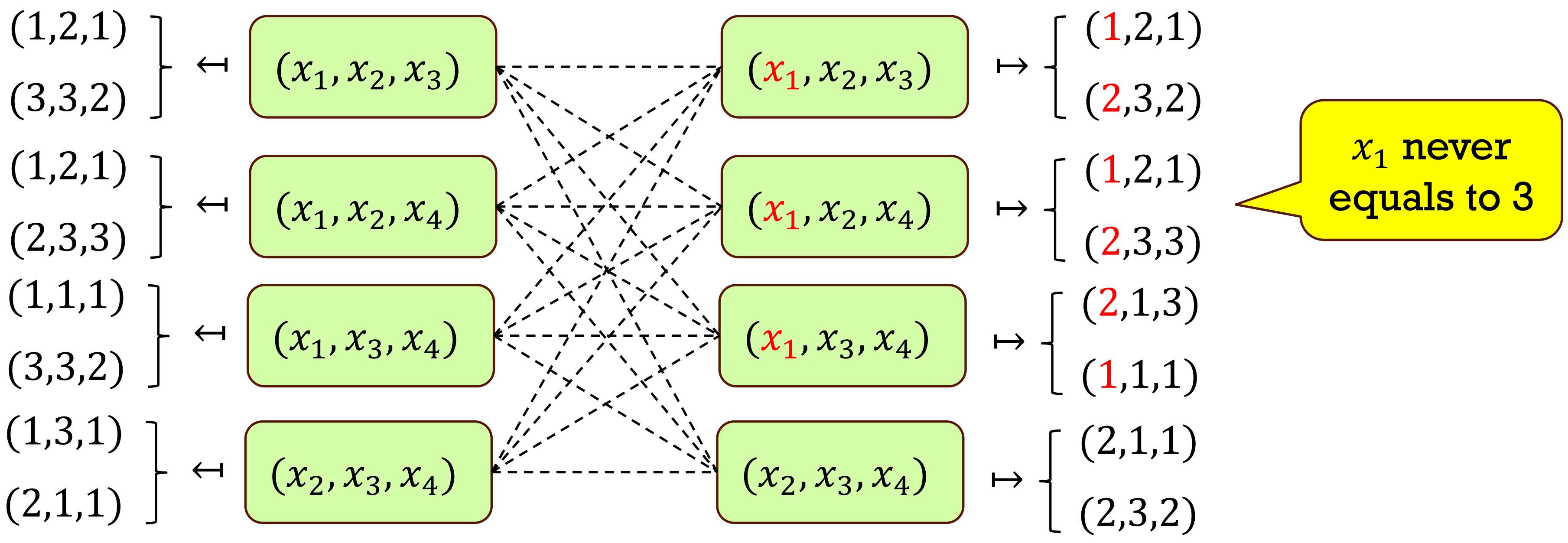

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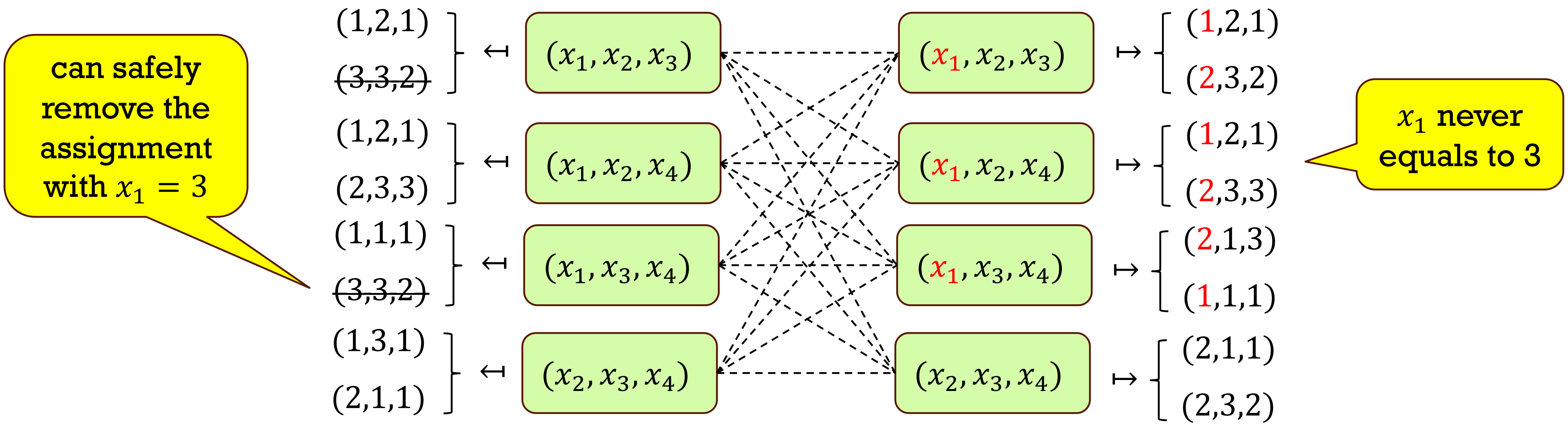

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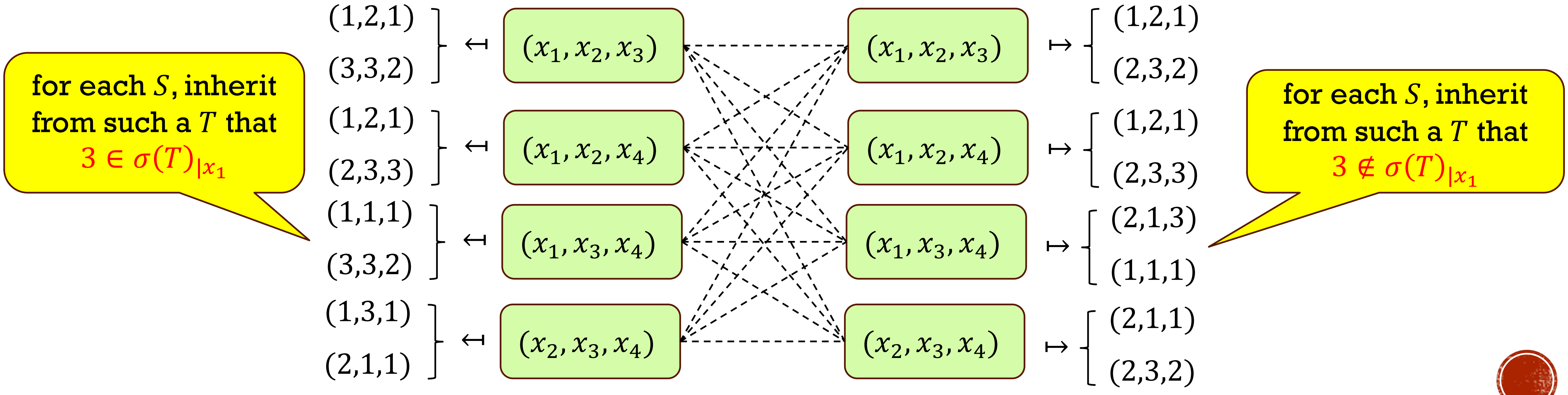

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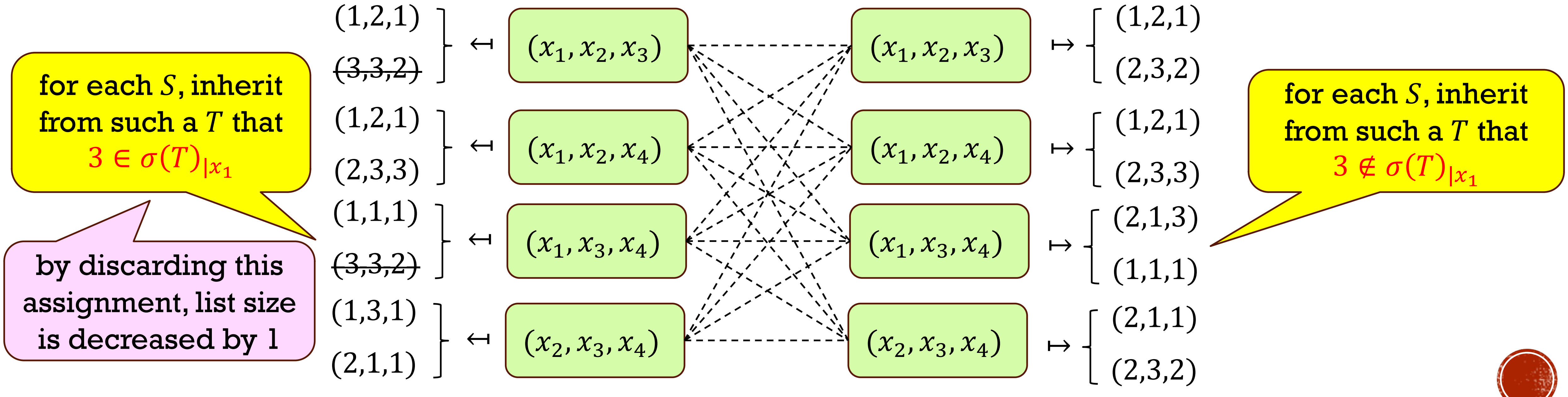

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- Bipartite $(r, 1)$-case

$\uparrow$

- Bipartite (r, q)-case

$\uparrow$

- Non-bipartite r-case

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Takeaway

- Parameterized Inapproximability Hypothesis – parameterized analog of PCP
- Baby PIH – inapproximability of the list-satisfiability of (parameterized) 2CSP
  - W[1]-hard to distinguish between [1-list satisfiable] and [not even r-list satisfiable]
  - Proof Idea: induction on the list size

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  - $\Rightarrow$ constant inapproximability of k-ExactCover

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- Thanks!