Memory AMP and Recent Results on Its Implementation

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Outline

1. Preliminaries
   - Problem Formulation
   - AMP, OAMP/VAMP, and CAMP
   - MAMP and GD-MAMP
2. Overflow-Avoiding GD-MAMP
   - Overflow Problem of GD-MAMP
   - OA-GD-MAMP with eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$
   - OA-GD-MAMP without eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$
3. Complexity-Reduced GD-MAMP
   - Complexity analysis of GD-MAMP
   - Complexity-Reduced GD-MAMP
4. Conclusion

Problem Formulation

- System model:

$$\begin{aligned}\Gamma: &\quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{n},\\ \Phi: &\quad x_i \sim P_X(x),\ \forall i.\end{aligned}$$

  where $\boldsymbol{A} \in \mathbb{C}^{M\times N}$, $\boldsymbol{n} \sim \mathcal{CN}(0, \sigma^2 \boldsymbol{I}_M)$, and $\boldsymbol{A}, \boldsymbol{y}, \sigma^2, P_X(\cdot)$ are known.
- Assumptions:
  (1) $M, N \to \infty$ with fixed $\delta = M/N$.
  (2) $\boldsymbol{A}$ is right-unitarily-invariant.
  (3) $\boldsymbol{x}$ is IID. For convenience, $\mathrm{E}\{\boldsymbol{x}\} = \boldsymbol{0}$ and $\frac{1}{N}\mathrm{E}\{\|\boldsymbol{x}\|^2\} = 1$.
- Goal: Given $\{\boldsymbol{y}, \boldsymbol{A}, \Gamma, \Phi\}$, find an MMSE estimate of $\boldsymbol{x}$:

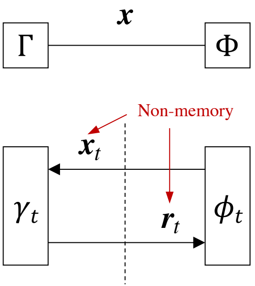
$$MSE \to \mathrm{mmse}\{\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{A}, \Gamma, \Phi\}$$

  For non-Gaussian $\boldsymbol{x}$, without the assumptions of $M, N \to \infty$ and $\boldsymbol{A}$, finding the optimal solution is generally NP-hard.

Approximate Message Passing (AMP)

- AMP-type algorithms:

$$\text{linear estimator (LE)}: \quad \boldsymbol{r}_t = \gamma_t\left(\boldsymbol{x}_t\right),$$
$$\text{non}-\text{linear esitmator (NLE)}: \quad \boldsymbol{x}_{t+1} = \phi_t\left(\boldsymbol{r}_t\right).$$



- AMP:

$$\text{LE}: \qquad \boldsymbol{r}_t = \boldsymbol{x}_t + \boldsymbol{A}^{\mathrm{H}}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t) + \boldsymbol{r}_t^{\text{Onsager}},$$
$$\text{NLE}: \quad \boldsymbol{x}_{t+1} = \phi(\boldsymbol{r}_t) = \mathrm{E}\{\boldsymbol{x}|\boldsymbol{r}_t\},$$

where $\boldsymbol{r}_t^{\text{Onsager}} = \beta\langle\phi'(\boldsymbol{r}_{t-1})\rangle(\boldsymbol{r}_{t-1} - \boldsymbol{x}_{t-1})$.

✓ Bayes optimal

✓ Low-complexity

× IID $\boldsymbol{A}$ is required

□ D. L. Donoho, A. Maleki, and A. Montanari, "Message-passing algorithms for compressed sensing," in *Proc. Nat. Acad. Sci.*, 2009.

Orthogonal/Vector AMP (OAMP/VAMP)

- OAMP/VAMP:

$$\text{LE}: \; \boldsymbol{r}_t = \boldsymbol{x}_t + \tfrac{1}{\epsilon_t^{\gamma}} \boldsymbol{A}^{\mathrm{H}} \big(\rho_t \boldsymbol{I} + \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}\big)^{-1} (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t),$$
$$\text{NLE}: \; \boldsymbol{x}_{t+1} = \tfrac{1}{\epsilon_{t+1}^{\phi}} \big[\hat{\phi}_t(\boldsymbol{r}_t) + (1 - \epsilon_{t+1}^{\phi}) \boldsymbol{r}_t\big],$$

where ϵ_t^{γ} and ϵ_t^{ϕ} are orthogonal parameters.

✓ Bayes optimal (replica)

✓ Unitarily-invariant $\boldsymbol{A}$

× High-complexity

□ J. Ma and L. Ping, "Orthogonal AMP," *IEEE Access*, 2017.

□ S. Rangan, P. Schniter, and A. Fletcher, "Vector approximate message passing," *IEEE Trans. Inf. Theory*, 2019.

Convolutional AMP (CAMP)

- CAMP:

$$\begin{aligned} \text{LE}: \quad & \boldsymbol{r}_t = \boldsymbol{x}_t + \boldsymbol{A}^H(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t) + \boldsymbol{r}_t^{\text{Onsager}}, \\ \text{NLE}: \quad & \boldsymbol{x}_{t+1} = \phi(\boldsymbol{r}_t) = \text{E}\{\boldsymbol{x}|\boldsymbol{r}_t\}, \end{aligned}$$

where $\boldsymbol{r}_t^{\text{Onsager}} = \sum_{\tau=0}^{t-1} \left[\prod_{t'=\tau}^{t-1} \langle \phi'(\boldsymbol{r}_{t'}) \rangle \right] (\theta_{t-\tau} \boldsymbol{A}^{\text{T}} \boldsymbol{A} - g_{t-\tau})(\boldsymbol{r}_\tau - \boldsymbol{x}_\tau)$.

✓ Bayes optimal (replica), if converges

✓ Unitarily-invariant $\boldsymbol{A}$

✓ Low-complexity

× Fails to converge for $\boldsymbol{A}$ with high condition numbers

□ K. Takeuchi, "Bayes-optimal convolutional AMP," *IEEE Trans. Inf. Theory*, 2021.

Memory AMP

- Memory AMP (MAMP):

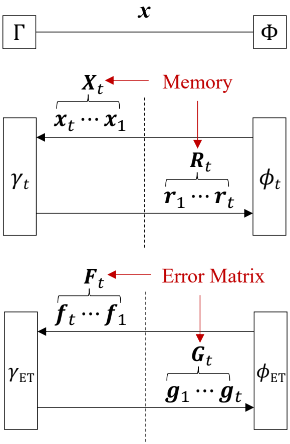
$$\begin{aligned} \text{LE}: \quad & \boldsymbol{r}_t = \gamma_t(\boldsymbol{X}_t) = \boldsymbol{Q}_t \boldsymbol{y} + \textstyle\sum_{i=1}^{t} \boldsymbol{P}_{t,i} \boldsymbol{x}_i, \\ \text{NLE}: \quad & \boldsymbol{x}_{t+1} = \phi_t(\boldsymbol{R}_t), \end{aligned}$$

  under orthogonality: $\forall t \geq 1$,

$$\tfrac{1}{N} \boldsymbol{g}_t^{\mathrm{H}} \boldsymbol{x} \overset{\text{a.s.}}{=} 0, \quad \tfrac{1}{N} \boldsymbol{F}_t^{\mathrm{H}} \boldsymbol{g}_t \overset{\text{a.s.}}{=} \boldsymbol{0}, \quad \tfrac{1}{N} \boldsymbol{G}_t^{\mathrm{H}} \boldsymbol{f}_{t+1} \overset{\text{a.s.}}{=} \boldsymbol{0},$$

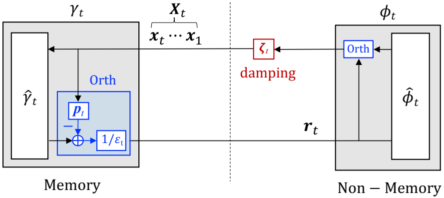
  where $\boldsymbol{G}_t$ and $\boldsymbol{F}_t$ are error matrices.

- AMP, OAMP/VAMP, CAMP can be unified under MAMP.



☐ L. Liu, S. Huang, and B. M. Kurkoski, "Memory AMP," *IEEE Trans. Inf. Theory*, 2022.

Bayes-Optimal MAMP (BO-MAMP) — Principle



- The LE has memory and consists of a local estimator $\hat{\gamma}_t$ and orthogonalization.
  1. $\hat{\gamma}_t$ approaches $(\rho_t \boldsymbol{I} + \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}})^{-1}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t)$ in OAMP/VAMP.
  2. The error of $\boldsymbol{r}_t$ is orthogonal to $\boldsymbol{x}$ and the errors of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_t$.
- Damping ζ is added. ζ is analytically optimized to guarantee convergence. Also improves convergence speed.
- NLE: Same as OAMP/VAMP and has no memory.

MAMP with Gradient Descent

GD-MAMP:

$$
\begin{aligned}
\mathrm{LE}: \quad \boldsymbol{u}_t &= \theta_t \boldsymbol{B} \boldsymbol{u}_{t-1} + \xi_t (\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t), \\
\boldsymbol{r}_t &= \gamma_t(\boldsymbol{X}_t) = \tfrac{1}{\varepsilon_t^\gamma}\big(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{u}_t + \textstyle\sum_{i=1}^{t} p_{t,i}\boldsymbol{x}_i\big), \\
\mathrm{NLE}: \boldsymbol{x}_{t+1} &= \big[\boldsymbol{x}_1 \cdots \boldsymbol{x}_t \ \phi_t(\boldsymbol{r}_t)\big] \cdot \zeta_{t+1},
\end{aligned}
$$

- $\boldsymbol{u}_t$ is an estimate of $(\rho_t \boldsymbol{I} + \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}})^{-1}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t)$.
- The parameters θ_t and ξ_t are optimized, $p_{t,i}$ and ε_t^γ are chosen to ensure orthogonality. Computation not shown, but we'll more talk about these later.
- ζ_{t+1} is the optimized damping vector
- $\phi_t(\cdot)$ is the same as that in OAMP/VAMP,

Gradient descent (GD) is used to approximate $\frac{\xi_t}{\theta_t}(\rho_t \boldsymbol{I} + \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}})^{-1}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t)$ by $\boldsymbol{u}_t$.

Why Memory? Intuition 1: Gradient Descent Avoids Matrix Inverse

Want to eliminate matrix inverse in OAMP: $\frac{\xi_t}{\theta_t}(\rho_t \boldsymbol{I} + \boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})^{-1}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t)$.
Solve $\boldsymbol{W}\boldsymbol{u} = \boldsymbol{b}$ without finding $\boldsymbol{W}^{-1}$.

Intuition 1 Solving $\boldsymbol{u} = \boldsymbol{W}^{-1}\boldsymbol{b}$ is equivalent to

$$\arg\min f(\boldsymbol{u}) = \tfrac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{u} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{u}$$

when $\boldsymbol{W}$ is positive definite. Find solution using gradient descent with step:

$$\begin{aligned}\boldsymbol{u}_i &= \boldsymbol{u}_{i-1} - \alpha \nabla f(\boldsymbol{u}_{i-1}) \\ &= \boldsymbol{u}_{i-1} + \alpha(\boldsymbol{b} - \boldsymbol{W}\boldsymbol{u}_{i-1}).\end{aligned}$$

Then $\boldsymbol{u}_i$ approaches the correct value $\boldsymbol{u}$.

Choosing $\alpha = \frac{2}{\lambda_{\max} + \lambda_{\min}}$ is close to optimal. Shown for real-valued case.

ChatGPT: Is it possible to find a matrix inverse using gradient descent?

ChatGPT

While there might be unconventional methods or iterative approaches that can approximate a matrix inverse, gradient descent is not the typical choice for this particular problem. If you need to find the inverse of a matrix, it's recommended to use established linear algebra techniques for accuracy and efficiency.

Why Memory? Intuition 2: Neumann Series for Matrix Inverse

Let $\rho(\boldsymbol{C})$ denote the spectral radius of $\boldsymbol{C}$. If $\rho(\boldsymbol{C}) < 1$, then

$$(\boldsymbol{I} - \boldsymbol{C})^{-1} = \textstyle\sum_{i=0}^{\infty} \boldsymbol{C}^i.$$

Choose $\boldsymbol{C} = \boldsymbol{I} - \boldsymbol{W}$. When $\rho(\boldsymbol{C}) \geq 1$, let $\boldsymbol{C}' = \boldsymbol{I} - \theta(\boldsymbol{I} - \boldsymbol{C})$, where θ ensures $\rho(\boldsymbol{C}') < 1$. With $\boldsymbol{u}_0 = \boldsymbol{0}$:

$$\boldsymbol{u}_i = \boldsymbol{C}'\boldsymbol{u}_{i-1} + \theta\boldsymbol{b}.$$

Then $\boldsymbol{u}_i$ approaches $\boldsymbol{u} = \boldsymbol{W}^{-1}\boldsymbol{b}$. This iteration is identical to gradient descent.

To accelerate convergence, we can minimize $\rho(\boldsymbol{C}')$ by:

$$\theta = \tfrac{2}{\lambda_1 + \lambda_2},$$

where λ_1 and λ_2 denote the maximum and minimum eigenvalues of $\boldsymbol{I} - \boldsymbol{C}$.

Overview of AMP-Type Algorithms

Table 1: Overview of AMP-Type Algorithms

Algorithm	Matrix $\boldsymbol{A}$	Convergence	Time complexity	Optimality
AMP	IID	Converges	Low: $\mathcal{O}(MN)$	Bayes-optimal
OAMP/VAMP	Right unitarily invariant	Converges	High: $\mathcal{O}(M^2N)$	Bayes-optimal
CAMP	Right unitarily invariant	Diverges in high condition numbers	Low: $\mathcal{O}(MN)$	Bayes-optimal
GD-MAMP	Right unitarily invariant	Converges	Low: $\mathcal{O}(MN)$	Bayes-optimal

- D. L. Donoho, A. Maleki, and A. Montanari, "Message-passing algorithms for compressed sensing," in *Proc. Nat. Acad. Sci.*, 2009.
- J. Ma and L. Ping, "Orthogonal AMP," *IEEE Access*, 2017.
- S. Rangan, P. Schniter, and A. Fletcher, "Vector approximate message passing," *IEEE Trans. Inf. Theory*, 2019.
- K. Takeuchi, "Bayes-optimal convolutional AMP," *IEEE Trans. Inf. Theory*, 2021.
- L. Liu, S. Huang, and B. M. Kurkoski, "Memory AMP," *IEEE Trans. Inf. Theory*, 2022.

Comparison of AMP-Type Algorithms

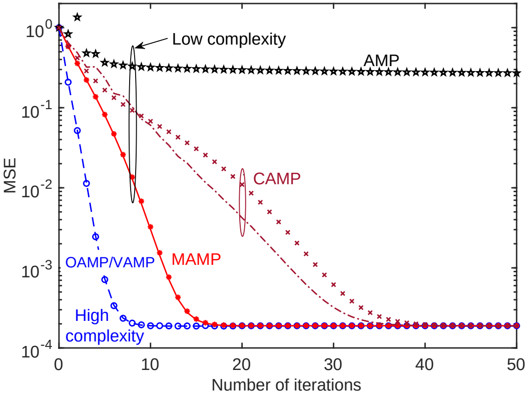


Figure 1: $M = 2^{13}, N = 2^{14}, \kappa(\boldsymbol{A}) = 10, \mathrm{SNR} = 30\mathrm{dB}$

- AMP:
  low complexity
  poor MSE
- OAMP/VAMP:
  fastest convergence
  high complexity
- CAMP:
  low complexity
  slow convergence
  incorrect state evolution
- GD-MAMP:
  low complexity
  fast convergence
  correct state evolution

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Overflow Problem in GD-MAMP

- Representation of floating point numbers from the IEEE 754 technical standard:
  - binary32 (single precision): $\pm 1.18 \times 10^{-38} \sim \pm 3.4 \times 10^{38}$
  - binary64 (double precision): $\pm 2.23 \times 10^{-308} \sim \pm 1.8 \times 10^{308}$

  Double-precision is widely used, including in Matlab and Python
- The dreaded NaN will appear for values that are too large.
- Overflow problem: In GD-MAMP, some intermediate variables may increase exponentially, and exceed the maximum value of double precision.

Which intermediate variables cause overflow in GD-MAMP?

- In iteration t, the parameters (1) ξ_t (2) $p_{t,i}$ and (3) $v_{t,t}^{\gamma}$ (variance of $\boldsymbol{r}_t$) require w_t.
- $\lambda^{\dagger} = (\lambda_{\max} + \lambda_{\min})/2$ and $\boldsymbol{B} = \lambda^{\dagger}\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$, where $\lambda_{\max}, \lambda_{\min}$ denotes the maximum and minimum eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$.

$$b_k \equiv \tfrac{1}{N}\mathrm{tr}\{\boldsymbol{B}^k\},$$
$$w_k \equiv \tfrac{1}{N}\mathrm{tr}\{\boldsymbol{A}^{\mathrm{H}}\boldsymbol{B}^k\boldsymbol{A}\} = \lambda^{\dagger}b_k - b_{k+1}.$$

- b_k can be computed if the eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$ are known. Otherwise, there are simple methods to approximate $\lambda_{\max}, \lambda_{\min}$ and b_k.
- If $\lambda_{\max} > \lambda_{\min} + 2$, the spectral radius $\rho(\boldsymbol{B}) > 1$, b_{2k} increases exponentially.
- Even w_k may increase exponentially.

Overflow of w_k

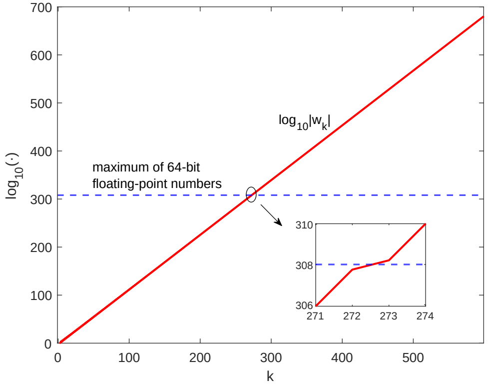

Figure 2: $\log_{10}|w_k|$ versus k

- $w_1, \cdots, w_{2T}$ are required, where T is the maximum number of iterations.
- As shown in Figure 2, $|w_k|$ increases exponentially as k increases.

Overflow Problem in GD-MAMP

- To compute ξ_t, $p_{t,i}$ and $v^{\gamma}_{t,t}$, we need w_k:

$$b_k \equiv \tfrac{1}{N}\mathrm{tr}\{\boldsymbol{B}^k\},$$
$$w_k \equiv \tfrac{1}{N}\mathrm{tr}\{\boldsymbol{A}^{\mathrm{H}}\boldsymbol{B}^k\boldsymbol{A}\} = \lambda^{\dagger} b_k - b_{k+1}.$$

- While w_k increases exponentially, it always appears in the product ϑw_k, which is bounded (i.e. ϑ is small).
- For any $\vartheta \in \mathbb{R}\backslash\{0\}$, the following holds:

$$\vartheta w_k = \frac{\mathrm{sgn}(\vartheta)}{N}\mathbf{1}^{\mathrm{T}}\big[(\lambda^{\dagger}\mathbf{1} - \boldsymbol{\lambda}_{\boldsymbol{B}}) \circ \boldsymbol{s}_{\lambda}^{\circ k} \circ \mathrm{e}^{\circ \log|\vartheta|\mathbf{1} + k\boldsymbol{\lambda}_B^{\log}}\big],$$

where $\boldsymbol{\lambda}_B$ denotes the eigenvalues of $\boldsymbol{B}$, $\boldsymbol{s}_{\lambda} \equiv \mathrm{sgn}(\boldsymbol{\lambda}_B)$, $\boldsymbol{\lambda}_B^{\log} \equiv \log^{\circ}|\boldsymbol{\lambda}_B|$ and $\circ$ is component-wise operation.

Overflow-Avoiding GD-MAMP with Eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$

- ϑw_k requires $\mathcal{O}(M)$ computations, GD-MAMP requires to compute $\mathcal{O}(T^3)$ terms involving w_k, where T is the number of iterations. The overall complexity is $\mathcal{O}(MT^3)$. How to reduce the complexity?
- Define

$$\chi_k \equiv \theta_0^k w_k,$$

where $\theta_0 = (\lambda^\dagger + \sigma^2)^{-1} > 0$. We pre-compute $\chi_1, \cdots, \chi_{2T-1}$ before the iterations. Computing ϑw_k can be reduced to a scalar operation:

$$\vartheta w_k = \mathrm{sgn}(\vartheta)\mathrm{e}^{\log|\alpha| - k\log\vartheta}\chi_k.$$

- Pre-computing $\chi_1, \cdots, \chi_{2T-1}$ costs $\mathcal{O}(MT)$, and computing terms involving w_k in iterations costs $\mathcal{O}(T^3)$. The overall complexity is reduced to $\mathcal{O}(MT + T^3)$.

Overflow-Avoiding GD-MAMP with Eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$

Theorem (2)

For any $k \geq 0$,

$$|\mathcal{X}_k| \leq \delta(\lambda^{\dagger} + \theta_0^{-1}),$$

where $\delta = M/N$.

- Theorem 2 shows that $\mathcal{X}_k$ is bounded. In other words, computing $\mathcal{X}_k$ has no risks of overflow.

Overflow-Avoiding GD-MAMP without Eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$

In large-scale systems, computation of eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$ may be impractical.

- A method to estimate the maximum and minimum eigenvalue $\lambda_{\max}$ and $\lambda_{\min}$ was given in [LHK22].
- For $k \geq 0$, χ_k can be estimated by

$$\chi_k = \bar{\boldsymbol{h}}_i^{\mathrm{H}} \bar{\boldsymbol{h}}_{k-i}$$

where $i = \lceil k/2 \rceil$ and $\bar{\boldsymbol{h}}_i$ is given by a recursion

$$\bar{\boldsymbol{h}}_i = \theta_0 (\lambda^{\dagger} \boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}) \bar{\boldsymbol{h}}_{i-1}$$

with $\bar{\boldsymbol{h}}_0 = \boldsymbol{A}\boldsymbol{h}_0$, $\boldsymbol{h}_0 \sim \mathcal{N}(\boldsymbol{0}, \frac{1}{N}\boldsymbol{I}_N)$.

Simulation Results

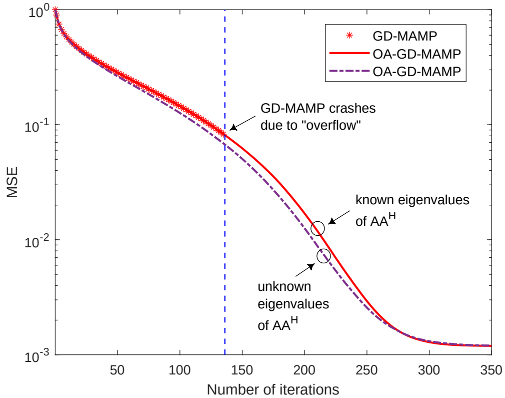

Figure 3: $M = 2^{13}, N = 2^{14}, \kappa(\boldsymbol{A}) = 1000, \mathrm{SNR} = 35\mathrm{dB}$

- When $t > 136$, the unmodified GD-MAMP does not reach the fixed point since b_{137} and w_{137} overflows.
- Both OA-GD-MAMP with and without eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$ work properly.

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Complexity Analysis of GD-MAMP

Let T be the number of iterations. The main complexity of GD-MAMP is $\mathcal{O}(MNT)$, dominated by the number of matrix-vector products, each with $\mathcal{O}(MN)$

$$\begin{aligned} \text{LE}: \quad \boldsymbol{u}_t &= \theta_t \boldsymbol{B} \boldsymbol{u}_{t-1} + \xi_t(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}_t), \\ \boldsymbol{r}_t &= \tfrac{1}{\varepsilon_t^\gamma}\big(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{u}_t + \textstyle\sum_{i=1}^t p_{t,i}\boldsymbol{x}_i\big), \\ \text{NLE}: \boldsymbol{x}_{t+1} &= \big[\boldsymbol{x}_1 \ \cdots \ \boldsymbol{x}_t \ \phi_t(\boldsymbol{r}_t)\big] \cdot \zeta_{t+1}. \end{aligned}$$

GD-MAMP requires 4 matrix-vector products per iteration?

- Computing $\boldsymbol{A}\phi_{t-1}(\boldsymbol{r}_{t-1})$ to estimate $v_{t,1}^{\phi}, \cdots, v_{t,t}^{\phi}$ requires one (hidden in ζ_t).
- Computing $\boldsymbol{B}\boldsymbol{u}_{t-1} = (\lambda^\dagger \boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}})\boldsymbol{u}_{t-1}$ requires two.
- Computing $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{u}_t$ requires one.

Easily eliminate one matrix-vector product:

- Two of the products are $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{u}_t$, these only need to be computed once.

GD-MAMP requires 3 matrix-vector products per iteration!

Complexity-Reduced GD-MAMP Using Approximate ξ_t

Finding the damping vector ζ_t nominally requires one matrix-vector product:

1. Compute $\boldsymbol{z}_t = \boldsymbol{y} - \boldsymbol{A}\phi_{t-1}(\boldsymbol{r}_{t-1})$.
2. Estimate $v^{\phi}_{t,1}, \cdots, v^{\phi}_{t,t}$ by using $\boldsymbol{z}_t$, where $v^{\phi}_{t,i}$ denotes the covariance of $\phi_{t-1}(\boldsymbol{r}_{t-1})$ and $\boldsymbol{x}_i$ for $i < t$.
3. Compute ζ_t from the covariance matrix $\boldsymbol{V}^{\phi}_t$ of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{t-1}$ and $\phi_{t-1}(\boldsymbol{r}_{t-1})$.

To remove the above matrix-vector product:
(1) We move the damping from the NLE to the LE (details omitted).
(2) ξ_t nominally depends on $\boldsymbol{z}_t$ and $\boldsymbol{V}^{\phi}_t$. But we found that approximating ξ_t gave little to no performance loss:

$$\tilde{\xi}_t = 1/(v^{\phi}_{t,t} + \sigma^2).$$

where $v^{\phi}_{t,t}$ is the variance of $\boldsymbol{x}_t$, given as that in OAMP/VAMP.

- The resulting complexity-reduced GD-MAMP requires only 2 matrix-vector products per iteration.

Simulation Results

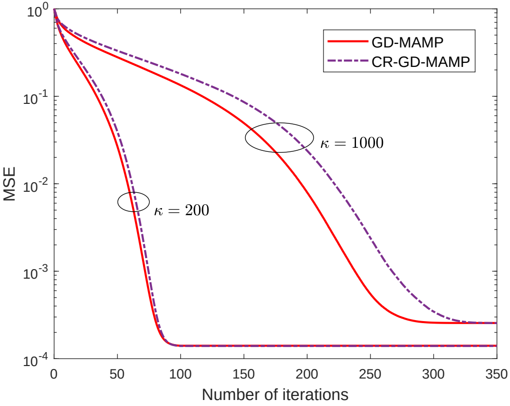

Figure 4: MSE versus number of iterations, $M = 2^{13}, N = 2^{14}, \mathrm{SNR} = 35\mathrm{dB}$

- CR-GD-MAMP requires a few more iterations to converge.

Simulation Results

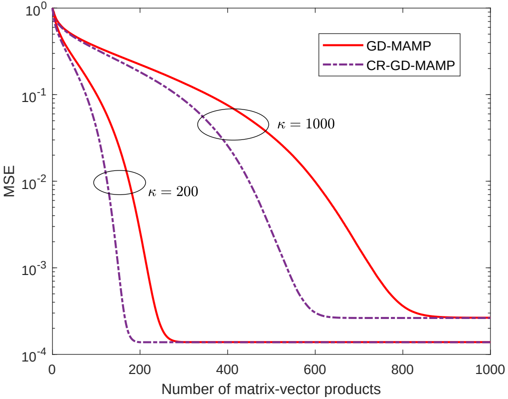

Figure 5: MSE versus number of matrix-vector products, $M = 2^{13}, N = 2^{14}, \mathrm{SNR} = 35\mathrm{dB}$

- CR-GD-MAMP achieves the same MSE as GD-MAMP while requiring about 2/3 matrix-vector products.

Conclusion

GD-MAMP is a memory AMP algorithm that:

- Converges for unitarily-invariant matrices
- is low complexity, avoiding matrix inverse
- converges for $\boldsymbol{A}$ of high condition number

overcoming the weakness of AMP, OAMP/VAMP and CAMP.

(1) To solve the overflow problem, we propose OA-GD-MAMP:

- With known eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$, OA-GD-MAMP is equivalent to GD-MAMP.
- Otherwise, OA-GD-MAMP can achieve nearly the same performance.

(2) GD-MAMP requires three matrix-vector products per iteration. To reduce it:

- We propose CR-GD-MAMP as a variant of GD-MAMP. It requires only two matrix-vector products per iteration with almost the same convergence speed.