COUNTERPOINTS

Mathematics: A Tool for Questioning

by

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Garry Kasparov

In the preceding article, the man who defeated the world's best chess champions and IBM's formidable "Deep Blue" computer has done *π in the Sky* an invaluable favour: by using mathematics to examine the world around him, past and present, he is greatly contributing to our mission of raising mathematical awareness, stimulating analytical thinking, and encouraging critical questioning of widely-held beliefs. Mathematics (Greek for "learning") should be cultivated as a tool for systematic questioning, our primary defense against mumbo-jumbo and demagoguery.

Kasparov's message is simple: "Do not accept authority unquestioned—look for yourself." The first authority he questions is that of Edward Gibbon, whose Decline and Fall of the Roman Empire is a monument not only to history but also to English prose. But wherever numbers are involved, you can jump in and at least check the arithmetic. Adding up the cohorts of infantry and cavalry is probably not done by most readers of Gibbon, but it is easy (9 times 555 equals 5 times 999, etc.) and fun. In the end you come up with 6826 (Gibbon has five more, perhaps officers) and have to multiply that by 30. No calculators are allowed: your number lies just short of half-way between 6667 and 7000, hence the total will come to about 205 000. To get from there to his "standing force of 375 000," Gibbon has to add 170 000 "attendant auxiliaries," almost one per soldier. Why so many? Did the Romans never have government cutbacks? With one auxiliary for every five soldiers (is that reasonable?) the total force would be less than 250 000, the number given for Napoleon.

To play around with these numbers some more, you can try to visualize how big a square one-quarter million men would occupy if each man occupies one square meter. Or you can distribute them on the 4000 miles of paved highway the Romans had (according to Gibbon). How far apart would they stand? If the Empire had 50 million inhabitants, that size of army would comprise one percent of the male population. If their life expectancy was 50 years, how long would their military service have to be to arrive at that number? As you can see, historical writings can provide an almost endless source of such exercises. Why should arithmetic and history always be taught separately?

Thomas R. Malthus

After wondering about the feasibility of some of the Roman marvels reported by Gibbon (for instance, the steel required to equip each legionnaire with a "pilum"), Kasparov's curiosity turns to the work of another famous Englishman, whom he however does not name. In political circles, that name invariably unleashes heated and bitter debates, because its owner wrote in 1798 that "population increases in a geometric ratio, while the means of subsistence increases in an arithmetic ratio." We are, of course, talking about Thomas R. Malthus. What does he mean? Population grows by perpetual multiplication (exponentially), while food production grows only by repeated addition (linearly); in other words, humanity is doomed!

Malthus does not leave it at these vague pronouncements, but says in his *Essay on the Principles of Population* (Chapter 2) that "population, when unchecked, goes on doubling itself every twenty-five years," after citing "the United States of America, where the means of subsistence have been more ample, the manners of the people more pure…" The phrase "when unchecked" throws a big spanner into the works: we are now at 200 years (eight doubling periods after Malthus), but have not doubled the world population of his time (about one billion) eight times; otherwise we'd now be at 256 billion instead of "only" six. Going backward in time, where Malthus would reduce the population by 50 percent every 25 years, similar nonsense would result. In working with doubling or halving, it is convenient to remember that the 10th power of 2 is 1024. Going back in time 250 years (10 Malthusian doubling times), he would go from one billion to one million—two more such large steps (750 years in total), and he would arrive at Adam. That's why these calculations need the condition "unchecked."

There are situations where this condition is almost satisfied. If you take a culture of bacteria in plenty of nutrient solution—they have no wars and do not practise birth control—you can observe (almost) pure exponential growth. And in radio-active decay—because atoms don't make choices—you can see it in reverse: every so many years (always the same number, called the "half-life"), the remaining "population" of radio-active atoms is halved. For radio-active carbon, the half-life is about 5700 years. When a plant or animal ceases to take part in the great cycle of life, its carbon content remains static, and the radio-active part of it decays with that fixed half-life. So if you find a piece of wood with only one-quarter the "typical" amount of radioactive carbon, you would presume that it has been dead for about 11 000 years.

But let us get back to human populations, where growth is apparently not "unchecked." It does not help, in the long run, to assume a greater doubling time: whatever length of step you choose, after 30 such steps back in time, you'll knock off nine zeroes, going from the present six billion to a mere six individuals—the Garden of Eden. In the medium run, you might observe something resembling exponential growth—but don't count on it. Look at the recent past: in 1800 we were one billion, in 1935 we were two billion, in 1975 we were four billion. The sad truth is that our doubling time seems to be shrinking. Pretty soon, it will be at the 25-year level assumed by Malthus—it looks as though the Old Man was not pessimistic enough.

Kasparov's inquisitiveness is not random but has a theme: exactly how long ago was it that the Romans had their Empire? At first glance, this question is surprising (don't we all know about those 2000 years?), but on second thought it is entirely legitimate. Anyone with a scientific bent of mind will put more trust in directly accessible data (e.g., the movement of stars) than in stories told by knights and monks—especially if these are vague and contradictory. According to people who study old manuscripts, medieval European record-keeping was a mess, and so it seems that some scrupulous revision is in order. The same scientific spirit that allows the question, however, compels us to question any answer—in this case, the one proposed by Fomenko's Moscow team. Since everyone seems to agree that time-keeping was fairly good from Caesar until about 400 AD and then again since Galileo (at least!), we have only about 1 200 possibly "sloppy" years to straighten out. If Islamic history, which is "modern" compared to most others, turns out to be as reliable as it looks, these uncertain years might shrink to a mere 200. For instance, the idea suggested in the article by Krawcewicz on page 12 of this issue, that "pagan" Egyptian frescoes could have been painted 600 years ago, would itself become rather questionable, if it were shown that Egypt was solidly Islamic at the time. That does not invalidate the author's study—it only shows that history is less certain than we sometimes think. Until the dust has settled, it is advisable not to pass judgment.

If the Roman Empire is really so far removed from us in time, why is it that Roman numerals were still in commercial use until the 14th century? Before we throw our own guess into the debate, let us look at the nature of these much maligned numerals. How could anyone calculate with them? Well, how can anyone compute "three hundred and seventy-six times two hundred and thirty-seven." You type these data into your pocket calculator and press the "×" button, that's how. You certainly would not fill page after page with number words. Neither did the Romans. They would load CCCLXXVI and CCXXXVII onto their counting board or abacus and manipulate the pebbles and beads until they had the result. We shall do such a multiplication, but first we'll look at addition and subtraction.

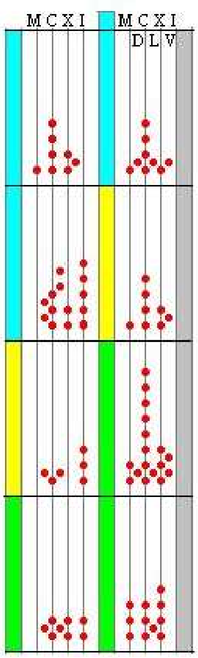


Figure 1

The counting board shown in Figure 1 is divided into two vertical strips; the left one is for subtraction and the right one is for addition. Let's do addition first. The number shown in the top-right field is MDCCCCLXV; the number immediately below is MCCCCXXV. To add them, we just pile everything together into the mess shown in the third field on the right. To make it readable, we have to reduce it—any five "beads" on a line are converted to one "button" in the space to the left of that line, and any two buttons in a space are converted to a single bead on the line immediately to the left. The answer is MMMCCCLXXXX, as shown in the bottom right field.

Note: we use the term "beads" to remind you of an abacus; our "buttons" would be found in the separate top compartment (called "heaven" by the Chinese) of the abacus. We are ignoring the medieval convention of writing IV, XL, CD instead of the longer but clearer IIII, XXXX, CCCC notation used by the ancients.

In the subtraction on the left strip, the first number MCCCCXXV must be expanded in order to have enough beads on each line and buttons in each space to allow the second number DCLIII, depicted in the third field, to be subtracted. The expansion, which is reduction in reverse, is shown in the second field from the top. It need not be done all at once, but can be performed as needed for subtraction. Answer: DCCLXXII.

The power and flexibility of the Roman numeral system is best demonstrated in how it handles multiplication: because of the numbers V, L, D, etc., you need not memorize any multiplication table beyond five. But five itself is just 10 halves, and halving is an easy operation. Doubling is another easy operation, and quadrupling is just doubling twice—so the hardest multiplier is three. If you do happen to know the 10-by-10 table, you can read every line together with its preceding space as a single decimal digit, and thus increase your speed.

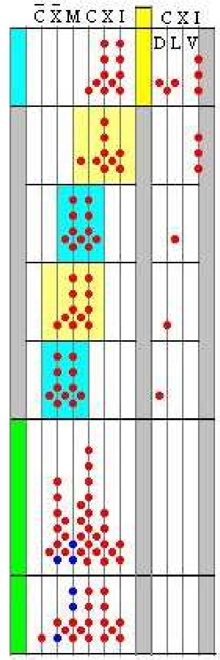


Figure 2

The multiplication shown in Figure 2 is CLXXXXVIIII times DCLIII. There are four partial products (in the blue and yellow fields) corresponding to the four digits of the multiplier: three, five (shifted), one (shifted twice), and five (shifted twice). As you pile all that into the first of the fields marked green, something special happens on the M-line: three sets of four. Since there is no space for that many, you turn them into a 12 (cf. blue beads) and carry on. After reducing this, you get $\overline{\text{CXXV}}$MMMMDCCCCXXXXVII, as shown in the bottom field. If you find this too long, compare it to "one hundred twenty-nine thousand nine hundred and forty-seven."

A Roman wine merchant would have done this in his head: CLXXXXVIIII is one less than CC, so double DCLIII to MCCCVI, shift to $\overline{\text{CXXX}}$DC, and subtract DCLIII, and that'll be LIII short of $\overline{\text{CXXX}}$—factus est.

After all of this, you must be dying to see a division, and here it is: MMMMDCXXVIIII divided by XIII (the divisor is not entered in). It goes just as you expect. Since XIII takes up two lines, you look at the first two lines (plus spaces) of the number to be divided, and you see XXXXVI, which can accommodate three times XIII.

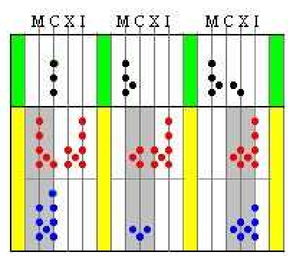


Figure 3

So you write a III on the line where your XXXXVI had its I. Then you subtract III times XIII and are left with VII, which is really DCC in disguise. Then you repeat the game, this time taking aim at what looks like LXXII—and so on, always wandering toward the smaller values on the right (see Figure 3).

To appreciate the ease and freedom of this simple gadget, you owe it to yourself to try one. For starters, why not take a chessboard and a supply of pennies? You can start your calculations on the right or on the left, change direction when you spot an opportunity for an easy move—as long as you keep track of where you are in the calculation, it cannot go wrong. You can add or subtract tokens to undo a lousy move—you never need an eraser.

The Indo-Arabic numeral system was supposedly introduced to Europe in the early 13th century with a book called Liber Abaci (book of the abacus) written by the widely travelled Leonardo da Pisa (alias Fibonacci), himself no mean mathematician. Present-day scholars say that it was known in the West much earlier—though still regarded as a Levantine curiosity—but that the 13th century introduction of paper from China, as a cheap medium for writing, made it the system of choice for all auditors and tax-collectors who wanted to see the details of every calculation.

The pen-on-paper computation with Indo-Arabic numerals—including the famous zero (originally a punctuation mark)—made it possible to check calculations for errors, but also penalized false starts and other trivial mistakes with ugly and confusing erasures. To avoid these, you had to follow certain very tight algorithms, which to this day make elementary arithmetic an incomprehensible and unpleasant discipline to many people. As Scott Carlson points out in the article preceding Kasparov's, the paper method makes little sense when a calculator is at hand—although mental arithmetic is something he evidently likes. To build the bridge between the two, how about re-introducing the counting board?

This ancient and user-friendly tool was still being used in Europe long after people had begun writing numbers in the more compact Indo-Arabic style. As late as 1550, a German textbook was published by one Adam Ries, in which the multiplication shown above would be written as 199 times 653 equals 129 947, but the intermediate steps would be left as unnamed patterns on the board. Even the Chinese and Japanese use this style to write input and output of their abacus work, and this would probably be the right way to bridge the gap between mental arithmetic and the calculator.

In conclusion: the counting board survived (at least) until the 16th century, and for a while (we guess) just carried the Roman numerals along with it. The fact that they are harder to falsify may also have helped.

The last major question raised by Kasparov concerns Diophantus of Alexandria. This Greek working in Roman times, considered the "father" of number theory, is indeed an enigma for anyone interested in chronology—the guesses about his dates range from 150 BC to 350 AD. If he lived that long ago, at a time when equations were allowed only one unknown (called the "arithm"), how could he have solved equations like "y cubed minus x cubed equals y minus x"? Here is what the Master himself says in Book IV, Problem 11 of his *Arithmetica*, according to the French translation by Paul Ver Eeke (1959), here rendered in English:

"To find two cubes having a difference equal to the difference of their sides. Suppose the sides to be 2 arithms and 3 arithms. Then the difference of the cubes with these sides is 19 cube arithms, and the difference of their sides is 1 arithm. Consequently, 1 arithm equals 19 cube arithms, and the arithm cannot be rational, because the ratio between these quantities is not like that of one square to another. We are thus led to look for cubes such that their difference is to the difference of their sides as one square number is to another."

If his first arithm was x, he then boldly grabs another arithm—let's call it z—and imagines cubes with sides $(z+1)x$ and zx, respectively. A bit of standard algebra shows $(3zz+3z+1)xx = 1$, and therefore $3zz+3z+1$ should be a square number. Diophantus assumes it to be the square of $(2z-1)$—how does he get away with that?—and then finds $z = 7$. He now repeats his initial argument with 7 arithms and 8 arithms, and finds the arithm to be 1/13. In our language: $x = 7/13$ and $y = 8/13$.

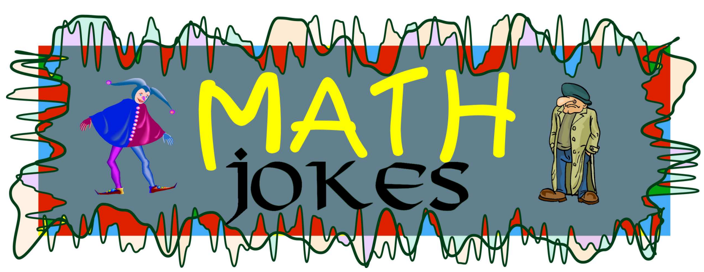
Is this a solution? Yes. Is it the general solution? No. But it points to a technique: had he taken $(z+2)x$ and zx, he would, in the same way, have obtained $6zz + 12z + 8$ and concluded that it should be twice a square number. Setting it equal to twice the square of $(3z - 2)$ would have yielded $z = 3$ and the arithm 1/7. In modern language: $x = 3/7$ and $y = 5/7$. There is method in this madness. Can you discover it?

We've discussed enough for today, but this is not the end of Kasparov's intellectual challenges to scholars and his questioning of widely accepted theories. They certainly have taken us on an interesting journey—and left us much to ponder.

If you are interested in learning more about issues relating to chronology, we invite you to visit the discussion forum at the web site

http://www.revisedhistory.org/forum.

Garry Kasparov, the author of the article "Mathematics of the Past" on page 5, will check this site periodically and try to respond to your questions. Submissions will be moderated before publication in the Forum.



Q: What does the math PhD with a job say to the math PhD without a job?

A: "Paper or plastic?"