On the Hénon-Lane-Emden conjecture

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July 27, 2011

Abstract

We consider the problem of non-existence of solutions for the following Hénon-Lane-Emden system

$$\begin{cases} -\Delta u = |x|^a v^p \text{ in } \mathbb{R}^N, \\ -\Delta v = |x|^b u^q \text{ in } \mathbb{R}^N, \end{cases}$$

when $pq > 1$, $p, q, a, b \geq 0$, and (p, q) are *under* the critical hyperbola, i.e. $\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2$. We show that there is no positive bounded solution in dimension $N = 3$, extending a result established recently by Phan-Souplet in the scalar case. This solves the *Hénon-Lane-Emden conjecture* in dimension $N = 3$ for bounded positive solutions. For the scalar cases, whether of second order ($a = b$ and $p = q$) or of fourth order ($a \geq 0 = b$ and $p > 1 = q$), we show that for all dimensions $N \geq 3$ (resp., $N \geq 5$), there is no positive solution with a finite Morse index, whenever p is below the corresponding critical exponent, i.e $1 < p < \frac{N+2+2a}{N-2}$ (resp., $1 < p < \frac{N+4+2a}{N-4}$).

2010 Mathematics Subject Classification. 35J47; 35B33; 35B45; 35B08.

Key words. Liouville-type theorems, Non-linear elliptic systems, Finite Morse index solutions, Hénon-Lane-Emden conjecture.

1 Introduction and main results

We consider the following weighted system

$$\begin{cases} -\Delta u = |x|^a v^p \text{ in } \Omega, \\ -\Delta v = |x|^b u^q \text{ in } \Omega, \end{cases} \tag{1}$$

where $pq > 1$ and $p, q, a, b \geq 0$ and Ω is a subset of $\mathbb{R}^N$, $N \geq 1$.

We start by noting that in the case of the Lane-Emden equation (i.e., when $p = q$ and $a = b = 0$), the Pohozaev inequality shows that there is no positive solution satisfying the Dirichlet boundary condition, whenever Ω is a bounded star-shaped domain and $p \geq \frac{N+2}{N-2}$, the critical Sobolev exponent. On the other hand, a celebrated theorem by Gidas-Spruck [11] states that there is no solution whenever $\Omega = \mathbb{R}^N$ and $p < \frac{N+2}{N-2}$ for $N \geq 3$. This non-existence result is also optimal as shown by Gidas, Ni and Nirenberg in [10] under the assumption that $u = O(|x|^{2-N})$, and by Caffarelli, Gidas and Spruck in [3] without the growth assumption. See also Chen and Li [4] for an easier proof based on the moving planes method. Also, Lin [13]

*Research partially supported by a University Graduate Fellowship at the University of British Columbia.
†Partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

using moving plane methods proved similar optimal non-existence results for $p < \frac{N+4}{N-4}, N > 4$ in the case of the fourth order Lane-Emden equation (i.e., when $p > 1 = q$ and $a = b = 0$).

Note that while there is no positive solution (for the Sobolev critical and super-critical exponents) in a bounded star-shaped domain, the non-existence result (for Sobolev sub-critical exponents) on the whole space is sharper as it does not assume the positivity of the solution.

In the case of the system (1), one can again use the Pohozaev identity whenever Ω is a bounded star-shaped domain in $\mathbb{R}^N$, to establish the following non-existence result.

Theorem A. [8, 19] *Let $N \geq 3$ and let $\Omega \subset \mathbb{R}^N$ be a star-shaped bounded domain. If*

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} \leq N-2, \tag{2}$$

then there is no positive solution for (1) that satisfy the Dirichlet boundary conditions.

By noting that the curve $\frac{N+a}{p+1} + \frac{N+b}{q+1} = N-2$ is the *critical hyperbola* or *Sobolev hyperbola*, the above theorem states that the Liouville-type result for positive solutions on bounded star-shaped domain holds when (p, q) is *above* the critical hyperbola. It is therefore expected that – just like the case of the scalar Lane-Emden equation ($p = q$ and $a = b = 0$) – the non-existence of solutions on the whole space $\mathbb{R}^N$ should occur exactly when (p, q) is in the complementary domain, that is when it is *under* the critical hyperbola.

This is the statement of the following *Hénon-Lane-Emden conjecture.*

Conjecture 1. *Suppose (p, q) is under the critical hyperbola, i.e.,*

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N-2. \tag{3}$$

If $\Omega = \mathbb{R}^N$, then there is no positive solution for system (1).

Proving such a non-existence result seems to be challenging even for the Lane-Emden conjecture (i.e., when $a = b = 0$) for systems. The case of radial solutions was solved by Mitidieri [14] in any dimension, and both Mitidieri [14] and Serrin-Zou [23] constructed positive radial solutions *on* and *above* the critical hyperbola, i.e. $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}$, which means that the non-existence theorem is optimal for radial solutions. For non-radial solutions of the Lane-Emden system, there are the results of Souto [24], Mitidieri [14] and Serrin-Zou [22] who proved the non-existence of solutions in dimensions $N = 1, 2$, while in dimension $N = 3$, Serrin-Zou [22] gave a proof for the non-existence of polynomially bounded solutions, an assumption that was removed later by Poláčik, Quittner and Souplet [18]. More recently, Souplet [21] settled completely the conjecture in dimension $N = 4$, while providing in dimensions $N \geq 5$, a more restrictive new region for the exponents (p, q) that insures non-existence.

Theorem B. (Souplet [21]) Assume $a = b = 0$.

(i) Let $N = 4$ and $p, q > 0$. If (p, q) satisfies

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \tag{4}$$

then system (1) with $\Omega = \mathbb{R}^N$ has no positive classical solutions.

(ii) Let $N \geq 5$, and $p, q > 0$ with $pq > 1$. If (p, q) satisfies (4), along with

$$max\{2\frac{p+1}{pq-1}, 2\frac{q+1}{pq-1}\} > N-3, \tag{5}$$

then system (1) with $\Omega = \mathbb{R}^N$ has no positive classical solutions.

The Lane-Emden conjecture in dimensions $N \geq 5$ is still open. The Hénon-Lane-Emden conjecture is even less understood. Even for the scalar case $a = b$ and $p = q$ (i.e., the Hénon equation), Gidas and Spruck in [11] solved the conjecture only for radial solutions, also showing that in this case, the non-existence result is optimal. For non-radial solutions, they proved some partial results such as the non-existence of positive solutions for $a \geq 2$ and $p \leq \frac{N+2}{N-2}$ (the Sobolev critical exponent for $a = 0$). Recently, Phan and Souplet [17] showed among other things that the Hénon-Lane-Emden conjecture for the scalar case holds for bounded positive solutions in dimension $N = 3$.

Theorem C. (Phan-Souplet [17]) *Let $N = 3$, $a = b > 0$ and $p = q > 1$. Assume (p, q) satisfies (3), then there is no positive bounded solution for the Hénon equation, i.e.,*

$$-\Delta u = |x|^a u^p \quad in \ \ \mathbb{R}^N. \tag{6}$$

For systems, Mitidieri [14] gave a partial solution to the conjecture for radial solutions by showing the following.

Theorem D. (Miditieri [14]) *Let $N \geq 3$. Assume $p, q > 1$ satisfy*

$$\frac{N + \min\{a, b\}}{p + 1} + \frac{N + \min\{a, b\}}{q + 1} > N - 2,$$

then there is no positive radial solution for (1) in $\Omega = \mathbb{R}^N$.

Recently, Bidaut-Veron-Giacomini [2] used a Pohozaev type argument and a suitable change of variables to improve the above result by proving the following result.

Theorem E. (Bidaut-Veron-Giacomini [2]) *For $N \geq 3$, System (1) admits a positive radial solution (u, v) such that $u, v \in C^2(0, \infty) \cap C([0, \infty))$ if and only if (p, q) is above or on the critical hyperbola, i.e., when (2) holds.*

In this note, we shall prove that the Hénon-Lane-Emden conjecture holds in dimension $N = 3$ for bounded positive solutions, extending the result of Phan-Souplet [17] mentioned in Theorem C above[1]. We also give a few partial results for the Hénon equation whether of second order or fourth order in all dimensions $N \geq 3$ or $N \geq 5$. Here are our main results:

Theorem 1. *Suppose $N = 3$ and (p, q) satisfy (3). Then, there is no positive bounded solution for (1) with $\Omega = \mathbb{R}^N$.*

We note that Miditieri and Pohozaev [16] have shown that the above result holds in higher dimension provided the following stronger condition holds:

$$\max\{\alpha, \beta\} \geq N - 2,$$

where $\alpha := \frac{(b+2)p+(a+2)}{pq-1}$ and $\beta := \frac{(a+2)q+(b+2)}{pq-1}$. For that they used a rescaled test-function method (as in Lemma 1 below) to prove the result for $p, q \geq 1$. More recently, Armstrong and Sirakov [1] proved –among other things– similar results for $p, q > 0$, by developing new maximum principle type arguments. We are thankful to P. Souplet for informing us of these latest developments by Armstrong and Sirakov.

We shall also consider in the scalar case the question of existence of solutions with finite Morse index solutions (as opposed to bounded solutions). We get the following counterpart to the Phan-Souplet result in higher dimensions ($N \geq 3$).

[1] Upon receiving our preprint, P. Souplet informed us that Q.H. Phan has also proved the same result in dimension $N = 3$, as well as other interesting results in higher dimensions. Our proofs are quite similar since both are essentially refinements of those of P. Souplet in his groundbreaking work on the Lane-Emden conjecture for systems.

Theorem 2. *Let $a \geq 0$, $p > 1$ and $N \geq 3$. Then, for any Sobolev sub-critical exponent, i.e.,*

$$1 < p < \frac{N+2+2a}{N-2},$$

equation (6) has no positive solution with finite Morse index.

We also have the following result for the fourth order equation,

$$\Delta^2 u = |x|^a u^p \quad \text{in } \mathbb{R}^N. \tag{7}$$

Theorem 3. *Let $a \geq 0$, $p > 1$ and $N \geq 5$. Then, for any Sobolev sub-critical exponent, i.e.,*

$$1 < p < \frac{N+4+2a}{N-4},$$

equation (7) has no positive solution with finite Morse index.

There are also various results for the cases where $-2 < a, b < 0$ and $pq \leq 1$. For that we refer to [2, 8, 16, 9, 11, 12, 17]. In [8], similar Liouville-type theorems in the notion of stability have been proved for positive solutions of system (1) on $\mathbb{R}^N$ provided $p > q = 1$ and $a = b$ in dimensions

$$N < 8 + 3a + \frac{8+4a}{p-1}.$$

Note that this range of dimensions falls under the corresponding critical hyperbola, i.e. $N < 4 + a + \frac{8+4a}{p-1}$.

2 Proofs

The main tools used in our proof are Pohozaev-type identities for both systems and equations as well as various integral estimates.

2.1 Proof of Theorem 1

The proof of Theorem 1 is heavily inspired by ideas of Souplet [21] and Serrin-Zou [22]. We use Pohozaev-type identities, various integral estimates, as well as some elliptic estimates on the sphere. Throughout this subsection, all norms refer to functions defined on the unit sphere, i.e. $||u||_m := ||u||_{L^m(S^{N-1})}$.

We start with the following estimate on the non-linear terms. Note that for $a = b = 0$, this was proved by Serrin and Zou [22] via ODE techniques, and by Miditieri and Pohozaev [16] who used the following rescaled test functions approach for $a, b > -2$. For the sake of convenience of readers, we recall the proof. Interested readers can find more details for both scalar and system cases in [20].

Lemma 1. *For any positive entire solution (u, v) of (1) and $R > 1$, there holds*

$$\int_{B_R} |x|^a v^p \leq C\, R^{N-2-\frac{(b+2)p+(a+2)}{pq-1}}, \tag{8}$$

$$\int_{B_R} |x|^b u^q \leq C\, R^{N-2-\frac{(a+2)q+(b+2)}{pq-1}}, \tag{9}$$

where the positive constant C does not depend on R.

Proof: Fix the following function $\zeta_R \in C_c^2(\mathbb{R}^N)$ with $0 \leq \zeta_R \leq 1$;

$$\zeta_R(x) = \begin{cases} 1, & \text{if } |x| < R; \\ 0, & \text{if } |x| > 2R; \end{cases}$$

where $||\nabla\zeta_R||_\infty \leq \frac{C}{R}$ and $||\Delta\zeta_R||_\infty \leq \frac{C}{R^2}$. For fixed $m \geq 2$, we have

$$|\Delta\zeta_R^m(x)| \leq C \begin{cases} 0, & \text{if } |x| < R \text{ or } |x| > 2R; \\ R^{-2}\zeta_R^{m-2}, & \text{if } R < |x| < 2R; \end{cases}$$

For $m \geq 2$, test the first equation of (1) by ζ_R^m and integrate to get

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^a v^p \zeta_R^m &= -\int_{\mathbb{R}^N} \Delta u \zeta_R^m \\ &= -\int_{\mathbb{R}^N} u \Delta\zeta_R^m \leq CR^{-2} \int_{B_{2R}\setminus B_R} u\zeta_R^{m-2}. \end{aligned}$$

Applying Hölder's inequality we get

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^a v^p \zeta_R^m &\leq C\ R^{-2} \left(\int_{B_{2R}\setminus B_R} |x|^{\frac{-b}{q}q'} \right)^{\frac{1}{q'}} \left(\int_{B_{2R}\setminus B_R} |x|^b u^q \zeta_R^{(m-2)q} \right)^{1/q} \\ &\leq C\ R^{(N-\frac{b}{q}q')\frac{1}{q'}-2} \left(\int_{B_{2R}\setminus B_R} |x|^b u^q \zeta_R^{(m-2)q} \right)^{1/q}. \end{aligned}$$

By a similar calculation for $k \geq 2$, we obtain

$$\int_{\mathbb{R}^N} |x|^b u^q \zeta_R^k \leq C\ R^{(N-\frac{a}{p}p')\frac{1}{p'}-2} \left(\int_{B_{2R}\setminus B_R} |x|^a v^p \zeta_R^{(k-2)p} \right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Since $pq > 1$, for large enough k we have $2 + \frac{k}{q} < (k-2)p$. So, we can choose m such that $2 + \frac{k}{q} \leq m \leq (k-2)p$ which means that $m \leq (k-2)p$ and $k \leq (m-2)q$. By collecting the above inequalities we get for $pq > 1$,

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |x|^a v^p \zeta_R^m \right)^{pq} &\leq C\ R^{[(N-\frac{b}{q}q')\frac{1}{q'}-2]pq} \left(\int_{B_R} |x|^b u^q \zeta_R^k \right)^p \\ &\leq C\ R^{(N-2)(pq-1)-[(b+2)p+(a+2)]} \int_{B_{2R}\setminus B_R} |x|^a v^p \zeta_R^m, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |x|^b u^q \zeta_R^k \right)^{pq} &\leq C\ R^{[(N-\frac{a}{p}p')\frac{1}{p'}-2]pq} \left(\int_{B_R} |x|^a v^p \zeta_R^m \right)^q \\ &\leq C\ R^{(N-2)(pq-1)-[(a+2)q+(b+2)]} \int_{B_{2R}\setminus B_R} |x|^b u^q \zeta_R^k. \end{aligned} \tag{11}$$

□

By using Hölder's inequality, we can now get the following L^1-estimates.

Corollary 1. *With the same assumptions as Lemma 1, we have*

$$\begin{aligned} \int_{B_R} v &\leq CR^{N-\frac{(a+2)q+(b+2)}{pq-1}}, \\ \int_{B_R} u &\leq CR^{N-\frac{(b+2)p+(a+2)}{pq-1}}, \end{aligned}$$

where the positive constant C does not depend on R.

We now recall the following fundamental estimates (see [21, 17]).

Lemma 2. *(Sobolev inequalities on the sphere S^{N-1}) Let $N \geq 2$, integer $j \geq 1$ and $1 < k < m \leq \infty$. For $z \in W^{j,k}(S^{N-1})$, we have*

$$||z||_{L^m(S^{N-1})} \leq C(||D_\theta^j z||_{L^k(S^{N-1})} + ||z||_{L^1(S^{N-1})}),$$

where

$$\begin{cases} \frac{1}{k} - \frac{1}{m} = \frac{j}{N-1}, & \text{if } k < (N-1)/j, \\ m = \infty, & \text{if } k > (N-1)/j, \end{cases}$$

and $C = C(j,k,N) > 0$.

Lemma 3. *(Elliptic L^p-estimate on B_R). Let $1 < k < \infty$ and $R > 0$. For $z \in W^{2,k}(B_{2R})$, we have*

$$\int_{B_R \setminus B_{R/2}} |D_x^2 z|^k \leq C \left(\int_{B_{2R} \setminus B_{R/4}} |\Delta z|^k + R^{-2k} \int_{B_{2R} \setminus B_{R/4}} |z|^k \right),$$

where $C = C(k,N) > 0$.

Lemma 4. *(An interpolation inequality on B_R). Let $R > 0$. For $z \in W^{2,1}(B_{2R})$, we have*

$$\int_{B_R \setminus B_{R/2}} |D_x z| \leq C \left(R \int_{B_{2R} \setminus B_{R/4}} |\Delta z| + R^{-1} \int_{B_{2R} \setminus B_{R/4}} |z| \right),$$

where $C = C(N) > 0$.

By applying Lemma 1, Corollary 1 and Lemma 4, we obtain the following estimates on the derivatives of u and v.

Lemma 5. *We have*

$$\begin{aligned} \int_{B_R} |D_x v| &\leq C\ R^{N-1-\frac{(a+2)q+(b+2)}{pq-1}}, \\ \int_{B_R} |D_x u| &\leq C\ R^{N-1-\frac{(b+2)p+(a+2)}{pq-1}}, \end{aligned}$$

where the positive constant C does not depend on R.

For $a = b = 0$, the following Pohozaev identity has been obtained by Mitidieri [15], Serrin and Zou [22]. It has also been used by Souplet in [21].

Lemma 6. *(Pohozaev identity). Suppose $\lambda, \gamma \in \mathbb{R}$ satisfy $\lambda + \gamma = N - 2$. If (u,v) is a positive solution of (1), then it necessarily satisfy*

$$\begin{aligned} \left(\frac{N+a}{p+1} - \lambda\right) \int_{B_R} |x|^a v^{p+1} &+ \left(\frac{N+b}{q+1} - \gamma\right) \int_{B_R} |x|^b u^{q+1} \\ &= R^{N+a} \int_{S^{N-1}} \frac{v^{p+1}}{p+1} + R^{N+b} \int_{S^{N-1}} \frac{u^{q+1}}{q+1} + R^N \int_{S^{N-1}} \left(u_r v_r - R^{-2} u_\theta v_\theta\right) \\ &\quad + R^{N-1} \int_{S^{N-1}} (\lambda u_r v + \gamma v_r u). \end{aligned}$$

Now, we are in the position to prove Theorem 1.

Proof of Theorem 1: Since (p,q) satisfy (3), then we can choose λ and γ such that $\frac{N+a}{p+1} > \lambda$ and $\frac{N+b}{q+1} > \gamma$. Now, for all $R>0$ define

$$F(R) := \left(\frac{N+a}{p+1} - \lambda\right) \int_{B_R} |x|^a v^{p+1} + \left(\frac{N+b}{q+1} - \gamma\right) \int_{B_R} |x|^b u^{q+1}.$$

From Lemma 6, we have

$$F(R) \le C\left(G_1(R) + G_2(R)\right), \tag{12}$$

where

$$G_1(R) := R^{N+a} \int_{S^{N-1}} v^{p+1} + R^{N+b} \int_{S^{N-1}} u^{q+1},$$

and

$$G_2(R) := R^N \int_{S^{N-1}} \left(|D_x u(R)| + R^{-1} u(R)\right)\left(|D_x v(R)| + R^{-1} v(R)\right).$$

Step 1. Upper bounds for G_1 and G_2. Set $m=\infty$ in Lemma 2 to get for either $t=p+1$ or $t=q+1$

$$||u||_t \le ||u||_\infty \le C(||D_\theta^2 u||_{1+\epsilon} + ||u||_1) \le C(R^2 ||D_x^2 u||_{1+\epsilon} + ||u||_1),$$

where $\epsilon > 0$ is small enough and will be chosen later. So,

$$\begin{aligned} G_1(R) \quad &\le \quad R^{N+a+2(p+1)} \left(||D_x^2 v||_{1+\epsilon} + R^{-2}||v||_1\right)^{1+p} \\ &\quad + R^{N+b+2(q+1)} \left(||D_x^2 u||_{1+\epsilon} + R^{-2}||u||_1\right)^{1+q}. \end{aligned} \tag{13}$$

We now look for the same type bounds for G_2. Apply Schwarz's inequality to get

$$\begin{aligned} G_2(R) \quad &\le \quad R^N \left(\int_{S^{N-1}} \left(|D_x u(R)| + R^{-1}u(R)\right)^2\right)^{1/2} \left(\int_{S^{N-1}} \left(|D_x v(R)| + R^{-1}v(R)\right)^2\right)^{1/2} \\ &\le \quad R^N \left(||D_x u||_2 + R^{-1}||u||_1\right)\left(||D_x v||_2 + R^{-1}||v||_1\right). \end{aligned}$$

Then, using Lemma 2 we obtain the following upper bounds.

$$\begin{aligned} ||D_x u||_2 \quad &\le \quad C\left(||D_\theta D_x u||_{1+\epsilon} + ||D_x u||_1\right) \le C\left(R||D_x^2 u||_{1+\epsilon} + ||D_x u||_1\right), \\ ||D_x v||_2 \quad &\le \quad C\left(||D_\theta D_x v||_{1+\epsilon} + ||D_x v||_1\right) \le C\left(R||D_x^2 v||_{1+\epsilon} + ||D_x v||_1\right). \end{aligned}$$

It follows that

$$G_2(R) \quad \le \quad R^{N+2}\left(||D_x^2 u||_{1+\epsilon} + R^{-1}||D_x u||_1 + R^{-2}||u||_1\right)\left(||D_x^2 v||_{1+\epsilon} + R^{-1}||D_x v||_1 + R^{-2}||v||_1\right). \tag{14}$$

Step 2. The following L^t-estimates hold in the annulus domain $B_R \setminus B_{R/2}$;

$$\int_{R/2}^R ||v(r)||_1 r^{N-1} dr \quad \le \quad C\ R^{N - \frac{(a+2)q+(b+2)}{pq-1}}, \tag{15}$$

$$\int_{R/2}^R ||u(r)||_1 r^{N-1} dr \quad \le \quad C\ R^{N - \frac{(b+2)p+(a+2)}{pq-1}}, \tag{16}$$

$$\int_{R/2}^R ||D_x v||_1 r^{N-1} dr \quad \le \quad C\ R^{N-1 - \frac{(a+2)q+(b+2)}{pq-1}}, \tag{17}$$

$$\int_{R/2}^R ||D_x u||_1 r^{N-1} dr \quad \le \quad C\ R^{N-1 - \frac{(b+2)p+(a+2)}{pq-1}}, \tag{18}$$

$$\int_{R/2}^R ||D_x^2 v||_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \quad \le \quad C\ R^{N-2 - \frac{(a+2)q+(b+2)}{pq-1} + b\epsilon}, \tag{19}$$

$$\int_{R/2}^R ||D_x^2 u||_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \quad \le \quad C\ R^{N-2 - \frac{(b+2)p+(a+2)}{pq-1} + a\epsilon}. \tag{20}$$

To prove (15)-(18), we just apply Corollary 1 and Lemma 5. Here is for example the proof for (20). Apply Lemma 3, Corollary 1 and Lemma 1 to get

$$\begin{aligned}
\int_{R/2}^{R} ||D_x^2 u||_{1+\epsilon}^{1+\epsilon} r^{N-1} dr &= \int_{R/2}^{R} |D_x^2 u|^{1+\epsilon} dx \\
&\leq C \int_{B_{2R}\setminus B_{R/4}} |\Delta u|^{1+\epsilon} dx + C\ R^{-2(1+\epsilon)} \int_{B_{2R}\setminus B_{R/4}} u^{1+\epsilon} dx \\
&\leq C\ R^{a\epsilon} \int_{B_{2R}\setminus B_{R/4}} |x|^a v^{p(1+\epsilon)} dx + C\ R^{-2(1+\epsilon)} \int_{B_{2R}\setminus B_{R/4}} u \\
&\leq C\ R^{N-2-\frac{(b+2)p+(a+2)}{pq-1}+a\epsilon} + C\ R^{N-\frac{(b+2)p+(a+2)}{pq-1}-2(1+\epsilon)} \\
&\leq C\ R^{N-2-\frac{(b+2)p+(a+2)}{pq-1}+a\epsilon}.
\end{aligned}$$

The proof of (19) is similar.

Step 3 For large enough M, define following sets;

$$\begin{aligned}
\Gamma_1(R) &:= \{r\ \in (R,2R);\ ||v(r)||_1 > MR^{-\frac{(a+2)q+(b+2)}{pq-1}}\}, \\
\Gamma_2(R) &:= \{r\ \in (R,2R);\ ||u(r)||_1 > MR^{-\frac{(b+2)p+(a+2)}{pq-1}}\}, \\
\Gamma_3(R) &:= \{r\ \in (R,2R);\ ||D_x v||_1 > MR^{-1-\frac{(a+2)q+(b+2)}{pq-1}}\}, \\
\Gamma_4(R) &:= \{r\ \in (R,2R);\ ||D_x u||_1 > MR^{-1-\frac{(b+2)p+(a+2)}{pq-1}}\}, \\
\Gamma_5(R) &:= \{r\ \in (R,2R);\ ||D_x^2 v||_{1+\epsilon}^{1+\epsilon} > MR^{-2-\frac{(a+2)q+(b+2)}{pq-1}+b\epsilon}\}, \\
\Gamma_6(R) &:= \{r\ \in (R,2R);\ ||D_x^2 u||_{1+\epsilon}^{1+\epsilon} > MR^{-2-\frac{(b+2)p+(a+2)}{pq-1}+a\epsilon}\}.
\end{aligned}$$

Using (20), we get

$$\begin{aligned}
C &\geq R^{-N+2+\frac{(b+2)p+(a+2)}{pq-1}-a\epsilon} \int_R^{2R} ||D_x^2 u||_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \\
&\geq R^{-N+2+\frac{(b+2)p+(a+2)}{pq-1}-a\epsilon} |\Gamma_6(R)| R^{N-1} M R^{-2-\frac{(b+2)p+(a+2)}{pq-1}+a\epsilon} = M|\Gamma_6(R)|R^{-1}.
\end{aligned}$$

Therefore, choosing large enough M, we get $|\Gamma_6(R)| \leq R/7$. Similarly, using (15)-(19), one can see $|\Gamma_i(R)| \leq R/7$ for $1 \leq i \leq 5$. Hence, for each $R \geq 1$, we can find

$$\hat{R} \in (R,2R) \setminus \bigcup_{i=1}^{i=6} \Gamma_i(R) \neq \phi. \tag{21}$$

We now have the following upper bounds on (13) and (14) for the radius $\hat{R}$ given by (21);

$$\begin{aligned}
G_1(\hat{R}) &\leq C\ \hat{R}^{N+a+2(p+1)} \left(\hat{R}^{\left(-\frac{(a+2)q+(b+2)}{pq-1}-2+b\epsilon\right)\frac{1}{1+\epsilon}} + \hat{R}^{-2-\frac{(a+2)q+(b+2)}{pq-1}} \right)^{p+1} \\
&\quad +C\ \hat{R}^{N+b+2(q+1)} \left(\hat{R}^{\left(-\frac{(b+2)p+(a+2)}{pq-1}-2+a\epsilon\right)\frac{1}{1+\epsilon}} + \hat{R}^{-2-\frac{(b+2)p+(a+2)}{pq-1}} \right)^{q+1}, \\
&\leq C\left(\hat{R}^{-a_1(\epsilon)} + \hat{R}^{-a_1'(\epsilon)}\right),
\end{aligned}$$

where

$$\begin{aligned}
a_1(\epsilon) = (p+1)\left[\left(2 + \frac{(a+2)q+(b+2)}{pq-1} - b\epsilon\right)\frac{1}{1+\epsilon} - 2 - \frac{N+a}{p+1}\right], \\
a_1'(\epsilon) = (q+1)\left[\left(2 + \frac{(b+2)p+(a+2)}{pq-1} - a\epsilon\right)\frac{1}{1+\epsilon} - 2 - \frac{N+b}{q+1}\right].
\end{aligned}$$

Also,

$$\begin{aligned} G_2(\hat{R}) &\leq C\ \hat{R}^{N+2}\left(\hat{R}^{\left(-\frac{(b+2)p+(a+2)}{pq-1}-2+a\epsilon\right)\frac{1}{1+\epsilon}}+\hat{R}^{-2-\frac{(b+2)p+(a+2)}{pq-1}}\right) \\ &\quad \left(\hat{R}^{\left(-\frac{(a+2)q+(b+2)}{pq-1}-2+b\epsilon\right)\frac{1}{1+\epsilon}}+\hat{R}^{-2-\frac{(a+2)q+(b+2)}{pq-1}}\right), \\ &\leq C\ \hat{R}^{-a_2(\epsilon)}, \end{aligned}$$

where

$$a_2(\epsilon) = -N-2+\frac{1}{1+\epsilon}\left(4-(a+b)\epsilon+\frac{(b+2)(p+1)+(a+2)(q+1)}{pq-1}\right).$$

Hence, from (12) we get

$$F(R) \leq C\left(G_1(\hat{R})+G_2(\hat{R})\right) \leq C\ R^{-\eta_\epsilon},$$

where $\eta_\epsilon := \min\{a_1(\epsilon), a_1'(\epsilon), a_2(\epsilon)\}$ and the positive constant C does not depend on R. By a straightforward calculation, we have

$$a_2(0) = -N+2+\frac{(b+2)(p+1)+(a+2)(q+1)}{pq-1} > 0 \ \text{ iff }\ \frac{N+a}{p+1}+\frac{N+b}{q+1} > N-2.$$

Also,

$$a_1(0) > 0, \ \text{ iff }\ \frac{(a+2)q+(b+2)}{pq-1} > \frac{N+a}{p+1}, \tag{22}$$

$$a_1'(0) > 0, \ \text{ iff }\ \frac{(b+2)p+(a+2)}{pq-1} > \frac{N+b}{q+1}. \tag{23}$$

Now, if p and q satisfy (3), then (50) and (51) hold, and we can therefore choose $\eta_\epsilon > 0$ for small enough $\epsilon > 0$. We now conclude by sending $R \to \infty$ and get the contradiction.

□

2.2 Proof of Theorem 2

We recall that a critical point $u \in C^2(\Omega)$ of the energy functional

$$I(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|x|^a u^{p+1}.$$

is said to be

- a *stable* solution of (6) if for any $\phi \in C_c^1(\Omega)$, we have

$$I_{uu}(\phi) := \int_\Omega |\nabla\phi|^2 - p\int_\Omega |x|^a u^{p-1}\phi^2 \geq 0.$$

- a *stable solution outside a compact set* $\Sigma \subset \Omega$ if $I_{uu}(\phi) \geq 0$ for all $\phi \in C_c^1(\Omega \setminus \Sigma)$, also u has a *Morse index* equal to $m \geq 1$ if m is the maximal dimension of a subspace X_m of $C_c^1(\Omega)$ such that $I_{uu}(\phi) < 0$ for all $\phi \in X_m \setminus \{0\}$.

- a *solution with Morse index* m if there exist $\phi_1, ..., \phi_m$ such that $X_m = Span\{\phi_1, ..., \phi_m\} \subset C_c^1(\Omega)$ and $I_{uu}(\phi) < 0$ for all $\phi \in X_m \setminus \{0\}$.

Note that if u is of Morse index m, then for all $\phi \in C_c^1(\Omega\setminus\Sigma)$ we have $I_{uu}(\phi) \geq 0$, where $\Sigma = \cup_{i=1}^m supp(\phi_i)$, and therefore u is stable outside the compact set $\Sigma \subset \Omega$.

We shall need the following lemma.

Lemma 7. *Let $\Omega \subset \mathbb{R}^N$ and let $u \in C^2(\Omega)$ be a positive stable solution of (6). Set $f(x) = |x|^a, a > 0$, then, for any $1 \le t < -1 + 2p + 2\sqrt{p(p-1)}$ we have*

$$\int_\Omega \left(|\nabla u|^2 u^{t-1} + f(x) u^{t+p}\right) \phi^{2m} \quad \le \quad C \int_\Omega f(x)^{-\frac{t+1}{p-1}} |\nabla \phi|^{2\frac{t+p}{p-1}}, \tag{24}$$

for all $\phi \in C_c^1(\Omega)$ with $0 \le \phi \le 1$ and for large enough m. The constant C does not depend on Ω and u.

Proof: The following proof also holds true for weak solutions. The ideas are adapted from [5, 6, 7]. Note first that for any stable solution of (6) and $\eta \in C_c^1(\Omega)$, we have the following:

$$p \int_\Omega |x|^a u^{p-1} \eta^2 \quad \le \quad \int_\Omega |\nabla \eta|^2, \tag{25}$$

$$\int_\Omega |x|^a u^p \eta \quad = \quad \int_\Omega \nabla u \cdot \nabla \eta. \tag{26}$$

Test (26) on $\eta = u^t \phi^2$ for $\phi \in C_c^1(\Omega)$ for an appropriate $t \in \mathbb{R}$ that will be chosen later, to get

$$\begin{aligned} \int_\Omega |x|^a u^{t+p} \phi^2 &= \int_\Omega \nabla u \cdot \nabla \left(u^t \phi^2\right) \\ &= t \int_\Omega |\nabla u|^2 u^{t-1} \phi^2 + 2 \int_\Omega u^t \nabla u \cdot \nabla \phi \phi. \end{aligned}$$

Apply Young's inequality[2] to $\left(|\nabla u| u^{\frac{t-1}{2}} \phi\right) \left(u^{\frac{t+1}{2}} |\nabla \phi|\right)$ to obtain

$$(t - \epsilon) \int_\Omega |\nabla u|^2 u^{t-1} \phi^2 \le C_\epsilon \int_\Omega u^{t+1} |\nabla \phi|^2 + \int_\Omega |x|^a u^{t+p} \phi^2. \tag{27}$$

Now, test (25) on $u^{\frac{t+1}{2}} \phi$ to get

$$\begin{aligned} p \int_\Omega |x|^a u^{t+p} \phi^2 &\le \frac{(t+1)^2}{4} \int_\Omega |\nabla u|^2 u^{t-1} \phi^2 + \int_\Omega u^{t+1} |\nabla \phi|^2 \\ &+ (t+1) \int_\Omega u^t \nabla u \cdot \nabla \phi \phi \\ &\le \left(\frac{(t+1)^2}{4} + 2\epsilon\right) \int_\Omega |\nabla u|^2 u^{t-1} \phi^2 + (C'_{\epsilon,t} + C''_{\epsilon,t}) \int_\Omega u^{t+1} |\nabla \phi|^2, \end{aligned}$$

where again we have used Young's inequality in the last estimate. Combine now this inequality with (27) to see

$$\left(p - \frac{\frac{(t+1)^2}{4} + 2\epsilon}{t - \epsilon}\right) \int_\Omega |x|^a u^{t+p} \phi^2 \le \left(\frac{\frac{(t+1)^2}{4} + 2\epsilon}{t - \epsilon} C_\epsilon + C'_{\epsilon,t} + C''_{\epsilon,t}\right) \int_\Omega u^{t+1} |\nabla \phi|^2. \tag{28}$$

For an appropriate choice of t, given in the assumption, we see that the coefficient in L.H.S. is positive for ϵ small enough. Therefore, replacing ϕ with ϕ^m for large enough m and applying Hölder's inequality with exponents $\frac{t+p}{t+1}$ and $\frac{t+p}{p-1}$ we obtain

$$\int_\Omega |x|^a u^{t+p} \phi^{2m} \le D_{\epsilon,t,m} \int_\Omega |x|^{-\frac{t+1}{p-1} a} |\nabla \phi|^{2\frac{t+p}{p-1}}. \tag{29}$$

Note that both exponents are greater than 1 for t given in (i) and (ii).

[2] For any $a, b, \epsilon > 0$, $ab \le \epsilon a^2 + C(\epsilon) b^2$, for some $C(\epsilon)$.

On the other hand, combining (27) and (28) gives us

$$\int_{\Omega} |\nabla u|^2 u^{t-1}\phi^2 \le D'_{\epsilon,t}\int_{\Omega} u^{t+1}|\nabla\phi|^2.$$

Similarly, replace ϕ by ϕ^m and apply Hölder's inequality with exponents $\frac{t+p}{t+1}$ and $\frac{t+p}{p-1}$ to get

$$\int_{\Omega} |\nabla u|^2 u^{t-1}\phi^{2m} \le D''_{\epsilon,t,m}\int_{\Omega} |x|^{-\frac{t+1}{p-1}a}|\nabla\phi|^{2\frac{t+p}{p-1}}.$$

This inequality and (29) finish the proof of (24).

□

Now, we are in the position to prove the theorem.

Proof of Theorem 2: We proceed in the following steps.

Step 1: We have the following standard Pohozaev type identity on any $\Omega \subset \mathbb{R}^N$.

$$\frac{N+a}{p+1}\int_{\Omega}|x|^a u^{p+1} - \frac{N-2}{2}\int_{\Omega}|\nabla u|^2 = \frac{1}{p+1}\int_{\partial\Omega}|x|^a u^{p+1} x\cdot\nu + \int_{\partial\Omega} x\cdot\nabla u\nu\cdot\nabla u - \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2 x\cdot\nu. \quad (30)$$

To get (30), just multiply both sides of (6) by $x\cdot\nabla u$, do integration by parts and collect terms.

Step 2: The following estimates hold:

$$\begin{aligned} |\nabla u| &\in L^2(\mathbb{R}^N), \\ |x|^a u^{p+1} &\in L^1(\mathbb{R}^N). \end{aligned}$$

First recall that u is stable outside a compact set $\Sigma \subset \Omega$. To prove our claim, we use (24) with the following test function $\xi_R \in C_c^1(\mathbb{R}^N \setminus \Sigma)$ for $R > R_0 + 3$ and $\Sigma \subset B_{R_0}$;

$$\xi_R(x) := \begin{cases} 0, & \text{if } |x| < R_0 + 1; \\ 1, & \text{if } R_0 + 2 < |x| < R; \\ 0, & \text{if } |x| > 2R; \end{cases}$$

which satisfies $0 \le \xi_R \le 1$, $||\nabla\xi_R||_{L^\infty(B_{2R}\setminus B_R)} < \frac{C}{R}$ and $||\nabla\xi_R||_{L^\infty(B_{R_0+2}\setminus B_{R_0+1})} < C_{R_0}$. Therefore,

$$\int_{R_0+2<|x|<R}(|\nabla u|^2 u^{t-1} + |x|^a u^{t+p}) \le C_{R_0} + \hat{C}\ R^{N-\frac{2(t+p)}{p-1}-\frac{t+1}{p-1}a},$$

for all $1 \le t < -1 + 2p + 2\sqrt{p(p-1)}$.
Now, set $t = 1$ and send $R \to \infty$. Since $N < \frac{2(p+a+1)}{p-1}$, we see $\int_{\mathbb{R}^N}|\nabla u|^2 < \infty$ and $\int_{\mathbb{R}^N}|x|^a u^{p+1} < \infty$.

Step 3: The following equality holds

$$\int_{\mathbb{R}^N}|\nabla u|^2 = \int_{\mathbb{R}^N}|x|^a u^{p+1}. \quad (31)$$

Multiply (6) with $u\zeta_R$ for $\zeta_R \in C_c^1(\mathbb{R}^N)$ which satisfies $0 \le \zeta_R \le 1$, $||\nabla\zeta_R||_\infty < \frac{C}{R}$ and

$$\zeta_R(x) := \begin{cases} 1, & \text{if } |x| < R; \\ 0, & \text{if } |x| > 2R. \end{cases}$$

Then, integrate over B_{2R} to get

$$\int_{B_{2R}}|x|^a u^{p+1}\zeta_R - \int_{B_{2R}}|\nabla u|^2\zeta_R = \int_{B_{2R}}\nabla\zeta_R\cdot\nabla u\ u. \quad (32)$$

By Hölder's inequality, we have the following upper bound for R.H.S. of (32),

$$
\begin{aligned}
|\int_{B_{2R}} \nabla\zeta_R \cdot \nabla u \ u| &\leq R^{-1} \int_{B_{2R}} |\nabla u|(|x|^{\frac{a}{p+1}} u) \ |x|^{-\frac{a}{p+1}} \\
&\leq R^{-1} \left(\int_{B_{2R}} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{B_{2R}} |x|^a u^{p+1} \right)^{\frac{1}{p+1}} \left(\int_{B_{2R}} |x|^{-\frac{2a}{p-1}} \right)^{\frac{p-1}{2(p+1)}} \\
&= R^{\frac{N(p-1)}{2(p+1)} - \frac{a}{p+1} - 1} \left(\int_{B_{2R}} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{B_{2R}} |x|^a u^{p+1} \right)^{\frac{1}{p+1}}.
\end{aligned}
$$

Therefore, from Step 2, there exists a positive constant C independent of R such that

$$
|\int_{B_{2R}} \nabla\zeta_R \cdot \nabla u \ u| \quad \leq \quad C \ R^{\frac{N(p-1)-2(a+p+1)}{2(p+1)}}.
$$

Since $N < \frac{2(p+a+1)}{p-1}$, we have $\lim_{R\to\infty} |\int_{B_{2R}} \nabla\zeta_R \cdot \nabla u \ u| = 0$. Hence (32) implies (31).

Step 4: we have

$$
(\frac{N+a}{p+1} - \frac{N-2}{2}) \int_{\mathbb{R}^N} |x|^a u^{p+1} = 0.
$$

Apply Lemma 7 for $t = 1$ with the following test function $\phi_R \in C_c^1(\mathbb{R}^N \setminus \Sigma)$ for $R > 2R_0$;

$$
\phi_R(x) := \begin{cases} 0, & \text{if } |x| < R/2; \\ 1, & \text{if } R < |x| < 2R; \\ 0, & \text{if } |x| > 3R; \end{cases}
$$

which satisfies $0 \leq \phi_R \leq 1$, $||\nabla\phi_R||_{L^\infty(B_{3R}\setminus B_{R/2})} < \frac{C}{R}$ to get

$$
\int_{B_{2R}\setminus B_R} \left(|\nabla u|^2 + |x|^a u^{p+1}\right) \quad \leq \quad CR^{N-\frac{2(p+a+1)}{p-1}}. \tag{33}
$$

Now, define the following sets for large enough M;

$$
\begin{aligned}
\theta_1(R) &:= \{r \ \in (R, 2R); \ ||D_x u(r)||_2^2 > MR^{-\frac{2(p+a+1)}{p-1}}\}, \\
\theta_2(R) &:= \{r \ \in (R, 2R); \ ||u(r)||_{p+1}^{p+1} > MR^{-\frac{2(p+a+1)}{p-1} - a}\}.
\end{aligned}
$$

From (33), we have

$$
\begin{aligned}
C &\geq R^{-N+\frac{2(p+a+1)}{p-1}+a} \int_R^{2R} ||u(r)||_{p+1}^{p+1} r^{N-1} dr \\
&\geq R^{-N+\frac{2(p+a+1)}{p-1}+a} |\theta_2(R)| R^{N-1} M R^{-\frac{2(p+a+1)}{p-1}-a} = M|\theta_2(R)|R^{-1}.
\end{aligned}
$$

Similarly, one can show $|\theta_1(R)| \leq R/M$. By choosing M large enough we conclude $|\theta_i(R)| \leq R/3$ for $i = 1, 2$. Therefore, for each $R \geq 1$, we can find

$$
\tilde{R} \in (R, 2R) \setminus \bigcup_{i=1}^{i=2} \Lambda_i(R) \neq \phi.
$$

Now, apply Pohozaev identity, (30), with $\Omega = B_{\tilde{R}}$ to see that R.H.S. converges to zero if $R \to \infty$ for subcritical p, i.e. $N < \frac{2(p+a+1)}{p-1}$. Hence,

$$
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 = \frac{N+a}{p+1} \int_{\mathbb{R}^N} |x|^a u^{p+1}.
$$

From this and (31), we finish the proof of Step 4.

□

Remark: For the Sobolev critical case $p = \frac{N+2+2a}{N-2}$, using the change of variable $w := u(r^{1+\frac{a}{2}})$ and applying well-known classifying-type results mentioned in the introduction for the Lane-Emden equation, one can see all radial solutions of (6) are of the following form

$$u_\epsilon(r) := k(\epsilon)(\epsilon + r^{2+a})^{\frac{2-N}{2+a}}, \tag{34}$$

where $k(\epsilon) = (\epsilon(N+a)(N-2))^{\frac{N-2}{2(2+a)}}$. Then, from the classical Hardy's inequality it is straightforward to see u_ϵ is stable outside a compact set $\overline{B_{R_0}}$, for an appropriate R_0. Note that for $-2 < a \le 0$, by Schwarz symmetrization (or rearrangement), it is shown in [9] that all radial solutions of (6) with $p = \frac{N+2+2a}{N-2}$ and $N > 2$ are of the form (34).

2.3 Proof of Theorem 3:

We recall that a critical point $u \in C^4(\Omega)$ of the energy functional

$$I(u) := \int_\Omega \frac{1}{2}|\Delta u|^2 - \frac{1}{p+1}|x|^a u^{p+1},$$

is said to be a *stable* solution of (7), if for any $\phi \in C_c^4(\Omega)$, we have

$$I_{uu}(\phi) := \int_\Omega |\Delta\phi|^2 - p\int_\Omega |x|^a u^{p-1}\phi^2 \ge 0.$$

Similarly to the second order case, one can define the notion of *stability outside a compact set*, which contains the notion of solutions with *finite Morse index*.

We first prove the following estimate.

Lemma 8. *Let $\Omega \subset \mathbb{R}^N$ and let $u \in C^4(\Omega)$ be a positive stable solution of (7). Then, for large enough m, we have for all $\phi \in C_c^4(\Omega)$ with $0 \le \phi \le 1$,*

$$\int_\Omega \left(|\Delta u|^2 + |x|^a u^{p+1}\right)\phi^{2m} \quad \le \quad C\int_\Omega |x|^{-\frac{2}{p-1}a}|T(\phi)|^{\frac{p+1}{p-1}}, \tag{35}$$

where $T(\phi) := |\Delta\phi|^2 + |\nabla\phi|^4 + |\Delta|\nabla\phi|^2| + |\nabla\phi\cdot\nabla\Delta\phi|$. The constant C does not depend on Ω and u.

Proof: For any stable solution of (7) and $\eta \in C_c^4(\Omega)$, we have the followings:

$$p\int_\Omega |x|^a u^{p-1}\eta^2 \quad \le \quad \int_\Omega |\Delta\eta|^2, \tag{36}$$

$$\int_\Omega |x|^a u^p \eta \quad = \quad \int_\Omega \Delta u \Delta\eta. \tag{37}$$

Test (37) on $\eta = u\phi^2$ for $\phi \in C_c^4(\Omega)$ to get

$$\int_\Omega |x|^a u^{p+1}\phi^2 \quad = \quad \int_\Omega \Delta u\ \Delta\left(u\phi^2\right) \tag{38}$$

Also, test (36) on $u\phi$ and use (38) to get

$$\begin{aligned}(p-1)\int_\Omega |x|^a u^{p+1}\phi^2 \quad &\le \quad \int_\Omega |\Delta(u\phi)|^2 - \int_\Omega |x|^a u^{p+1}\phi^2 \\ &= \quad \int_\Omega |\Delta(u\phi)|^2 - \int_\Omega \Delta u\Delta(u\phi^2).\end{aligned}$$

By a straightforward calculation, one can see that the following identity holds:

$$|\Delta(u\phi)|^2 - \Delta u\Delta(u\phi^2) = 4|\nabla u\cdot\nabla\phi|^2 + u^2|\Delta\phi|^2 - 2u\Delta u|\nabla\phi|^2 + 2\nabla u^2\cdot\nabla\phi\Delta\phi. \tag{39}$$

Therefore, we have

$$\begin{aligned}(p-1)\int_\Omega |x|^a u^{p+1}\phi^2 &\leq 4\int_\Omega |\nabla u|^2|\nabla\phi|^2 + \int_\Omega u^2|\Delta\phi|^2 - 2\int_\Omega u\Delta u|\nabla\phi|^2 \\ &\quad +2\int_\Omega \nabla u^2\cdot\nabla\phi\Delta\phi.\end{aligned}$$

A simple integration by parts yields

$$\int_\Omega |\nabla u|^2|\nabla\phi|^2 = \int_\Omega u(-\Delta u)|\nabla\phi|^2 + \frac{1}{2}\int_\Omega u^2\Delta|\nabla\phi|^2, \tag{40}$$

which then simplifies the previous inequality to become

$$(p-1)\int_\Omega |x|^a u^{p+1}\phi^2 \leq 6\int_\Omega u(-\Delta u)|\nabla\phi|^2 + \int_\Omega u^2(-|\Delta\phi|^2 + 2\Delta|\nabla\phi|^2 - 2\nabla\phi\cdot\nabla\Delta\phi).$$

Therefore,

$$\int_\Omega |x|^a u^{p+1}\phi^2 \leq C\int_\Omega u|\Delta u||\nabla\phi|^2 + \int_\Omega u^2 L(\phi), \tag{41}$$

where $L(\phi) := |\Delta\phi|^2 + 2|\Delta|\nabla\phi|^2| + 2|\nabla\phi\cdot\nabla\Delta\phi|$.

On the other hand, from (39) and (40), one can see

$$\begin{aligned}\int_\Omega |\Delta(u\phi)|^2 &= \int_\Omega \Delta u\Delta(u\phi^2) + 4\int_\Omega |\nabla u\cdot\nabla\phi|^2 + \int_\Omega u^2|\Delta\phi|^2 - 2u\Delta u|\nabla\phi|^2 - 2\int_\Omega u^2 div(\nabla\phi\Delta\phi) \\ &= \int_\Omega |x|^a u^{p+1}\phi^2 + 6\int_\Omega u(-\Delta u)|\nabla\phi|^2 + \int_\Omega u^2\left(-|\Delta\phi|^2 + 2\Delta|\nabla\phi|^2 - 2\nabla\phi\cdot\nabla\Delta\phi\right).\end{aligned}$$

By combining (41), the identity $\Delta(u\phi) = \phi\Delta u + 2\nabla u\cdot\nabla\phi + u\Delta\phi$ and Young's inequality, we get the following estimate

$$\int_\Omega |\Delta u|^2\phi^2 \leq C\int_\Omega u|\Delta u||\nabla\phi|^2 + C\int_\Omega u^2 L(\phi).$$

Therefore,

$$\int_\Omega \left(|x|^a u^{p+1} + |\Delta u|^2\right)\phi^2 \leq C\int_\Omega u|\Delta u||\nabla\phi|^2 + C\int_\Omega u^2 L(\phi).$$

Now, replacing ϕ with ϕ^m for large enough $m>0$ and applying Young's inequality we end up with

$$\begin{aligned}\int_\Omega \left(|x|^a u^{p+1} + |\Delta u|^2\right)\phi^{2m} &\leq C\int_\Omega u|\Delta u||\nabla\phi|^2\phi^{2(m-1)} + C\int_\Omega u^2 L(\phi^m) \\ &\leq \epsilon\int_\Omega |\Delta u|^2\phi^{2m} + C_\epsilon\int_\Omega u^2|\nabla\phi|^4\phi^{2(m-2)} + C\int_\Omega u^2 L(\phi^m).\end{aligned}$$

Then, for large enough m

$$\int_\Omega \left(|x|^a u^{p+1} + |\Delta u|^2\right)\phi^{2m} \leq C\int_\Omega u^2\phi^{2(m-2)}T(\phi), \tag{42}$$

where $T(\phi) := |\Delta\phi|^2 + |\nabla\phi|^4 + |\Delta|\nabla\phi|^2| + |\nabla\phi\cdot\nabla\Delta\phi|$. Now, apply Hölder's inequality to get

$$\begin{aligned}\int_\Omega u^2\phi^{2(m-2)}T(\phi) &= \int_\Omega |x|^{\frac{2a}{p+1}}u^2\phi^{2(m-2)}|x|^{-\frac{2a}{p+1}}T(\phi)\\ &\le \left(\int_\Omega |x|^a u^{p+1}\phi^{2(m-2)\frac{p+1}{2}}\right)^{\frac{2}{p+1}}\left(\int_\Omega |x|^{-\frac{2a}{p-1}}T^{\frac{p+1}{p-1}}(\phi)\right)^{\frac{p-1}{p+1}}\end{aligned}$$

Choosing m large enough, say $2(m-2)\frac{p+1}{2}\ge 2m$, from (42) we finally get the desired inequality

$$\int_\Omega (|x|^a u^{p+1} + |\Delta u|^2)\phi^{2m} \le C\int_\Omega |x|^{-\frac{2a}{p-1}}T^{\frac{p+1}{p-1}}(\phi).$$

□

Proof of Theorem 3: We proceed in the following steps.

Step 1: We have the following standard Pohozaev type identity on any $\Omega\subset\mathbb{R}^N$.

$$\begin{aligned}\frac{N+a}{p+1}\int_\Omega |x|^a u^{p+1} - \frac{N-4}{2}\int_\Omega|\Delta u|^2 &= \frac{1}{p+1}\int_{\partial\Omega}|x|^a u^{p+1}x\cdot\nu - \frac12\int_{\partial\Omega}|\Delta u|^2 x\cdot\nu\\ &\quad -\int_{\partial\Omega}\nabla\Delta u\cdot\nu x\cdot\nabla u + \int_{\partial\Omega}\Delta u\nabla(x\cdot\nabla u)\cdot\nu. \qquad (43)\end{aligned}$$

To get (43), just multiply both sides of (7) by $x\cdot\nabla u$, do integration by parts and collect terms.

Step 2: we have

$$\begin{aligned}|\Delta u| &\in L^2(\mathbb{R}^N),\\ |x|^a u^{p+1} &\in L^1(\mathbb{R}^N).\end{aligned}$$

Since u is stable outside a compact set $\Sigma\subset\Omega$, using (35) with the following test function $\xi_R\in C_c^1(\mathbb{R}^N\setminus\Sigma)$ for $R>R_0+3$ and $\Sigma\subset B_{R_0}$;

$$\xi_R(x) := \begin{cases}0, & \text{if } |x|<R_0+1;\\ 1, & \text{if } R_0+2<|x|<R;\\ 0, & \text{if } |x|>2R;\end{cases}$$

which satisfies $0\le\xi_R\le 1$, $||D^i\xi_R||_{L^\infty(B_{2R}\setminus B_R)} < \frac{C}{R^i}$ and $||D^i\xi_R||_{L^\infty(B_{R_0+2}\setminus B_{R_0+1})} < C_{R_0}$ for $i=1,\cdots,4$, we get

$$\int_{R_0+2<|x|<R}(|\Delta u|^2 + |x|^a u^{p+1}) \le C_{R_0} + \hat{C}\ R^{N-\frac{4(p+1)}{p-1}-\frac{2}{p-1}a}.$$

For subcritical exponents, $N<\frac{2(2p+a+2)}{p-1}$, we see $\int_{\mathbb{R}^N}|\Delta u|^2<\infty$ and $\int_{\mathbb{R}^N}|x|^a u^{p+1}<\infty$.

Step 3: The following equality holds

$$\int_{\mathbb{R}^N}|x|^a u^{p+1} = \int_{\mathbb{R}^N}|\Delta u|^2. \qquad (44)$$

Multiply (7) with $u\zeta_R$ for $\zeta_R\in C_c^4(B_{2R})$ which satisfies $0\le\zeta_R\le 1$, $||D^i\zeta_R||_\infty < \frac{C}{R^i}$ for $i=1,\dots,4$ and

$$\zeta_R(x) := \begin{cases}1, & \text{if } |x|<R;\\ 0, & \text{if } |x|>2R.\end{cases}$$

Then, integrate over B_{2R} to get

$$\int_{B_{2R}}|x|^a u^{p+1}\zeta_R - \int_{B_{2R}}|\Delta u|^2\zeta_R = \int_{B_{2R}}u\Delta u\Delta\zeta_R + 2\int_{B_{2R}}\Delta u\nabla u\cdot\nabla\zeta_R =: I_1(R)+I_2(R). \qquad (45)$$

By Hölder's inequality, we have the following upper bound for $I_1(R)$,

$$\begin{aligned} |I_1(R)| &\leq R^{-2}\int_{B_{2R}} |\Delta u|(|x|^{\frac{a}{p+1}}u)\ |x|^{-\frac{a}{p+1}} \\ &\leq R^{-2}\left(\int_{B_{2R}}|\Delta u|^2\right)^{\frac{1}{2}}\left(\int_{B_{2R}}|x|^a u^{p+1}\right)^{\frac{1}{p+1}}\left(\int_{B_{2R}}|x|^{-\frac{2a}{p-1}}\right)^{\frac{p-1}{2(p+1)}} \\ &= R^{\frac{N(p-1)}{2(p+1)}-\frac{a}{p+1}-2}\left(\int_{B_{2R}}|\Delta u|^2\right)^{\frac{1}{2}}\left(\int_{B_{2R}}|x|^a u^{p+1}\right)^{\frac{1}{p+1}}. \end{aligned}$$

Therefore, from Step 2, there exists a positive constant C independent of R such that

$$|I_1(R)| \leq C\ R^{\frac{N(p-1)-2(a+p+1)}{2(p+1)}}.$$

Since $N < \frac{2(p+a+1)}{p-1}$, we have $\lim_{R\to\infty}|I_1(R)| = 0$. Now, we consider the second term in R.H.S. of (45). Apply Young's inequality for a given $\epsilon > 0$ (we choose it later) to get

$$|I_2(R)| \leq \epsilon\int_{\mathbb{R}^N}|\Delta u|^2 + C_\epsilon\int_{B_{2R}}|\nabla u|^2|\nabla\zeta_R|^2,$$

Using Green's theorem we get

$$\int_{B_{2R}}|\nabla u|^2|\nabla\zeta_R|^2 = \int_{B_{2R}}u(-\Delta u)|\nabla\zeta_R|^2 + \frac{1}{2}\int_{B_{2R}}u^2\Delta|\nabla\zeta_R|^2 =: I_3(R) + I_4(R).$$

By the same discussion as given for $I_1(R)$ one can see $\lim_{R\to\infty}|I_3(R)| = 0$. For the term $I_4(R)$, we apply Hölder's inequality again

$$\begin{aligned} |I_4| &\leq R^{-4}\int_{B_{2R}}|x|^{\frac{a}{p+1}}u^2\ |x|^{-\frac{a}{p+1}} \\ &\leq R^{-4}\left(\int_{B_{2R}}|x|^a u^{p+1}\right)^{\frac{2}{p+1}}\left(\int_{B_{2R}}|x|^{-\frac{2a}{p-1}}\right)^{\frac{p-1}{p+1}} \\ &= R^{\frac{N(p-1)}{(p+1)}-\frac{2a}{p+1}-4}\left(\int_{B_{2R}}|x|^a u^{p+1}\right)^{\frac{2}{p+1}}. \end{aligned}$$

By Step 2 and sending R to infinity we get, $\lim_{R\to\infty}|I_4(R)| = 0$. Since $\lim_{R\to\infty}|I_2(R)| \leq \epsilon\int_{\mathbb{R}^N}|\Delta u|^2$ for any $\epsilon > 0$, we have $\lim_{R\to\infty}|I_2(R)| = 0$. Therefore, (44) follows.

Step 4: The following equality holds

$$\left(\frac{N+a}{p+1} - \frac{N-4}{2}\right)\int_{\mathbb{R}^N}|x|^a u^{p+1} = 0.$$

Apply Lemma 8 with the following test function $\phi_R \in C_c^1(\mathbb{R}^N\setminus\Sigma)$ for $R > 2R_0$;

$$\phi_R(x) := \begin{cases} 0, & \text{if } |x| < R/2; \\ 1, & \text{if } R < |x| < 2R; \\ 0, & \text{if } |x| > 3R; \end{cases}$$

where $0 \leq \phi_R \leq 1$, $||D^i\phi_R||_{L^\infty(B_{3R}\setminus B_{R/2})} < \frac{C}{R^i}$. Then, we get

$$\int_{B_{2R}\setminus B_R}|\Delta u|^2 + |x|^a u^{p+1} \leq CR^{N-\frac{2(2p+2+a)}{p-1}}. \tag{46}$$

On the other hand, we are interested in similar upper bounds for the following terms

$$J_1(R) := \int_{B_{2R}\setminus B_R} |\Delta u||\nabla u| \quad \text{and} \qquad J_2(R) := \int_{B_{2R}\setminus B_R} |\Delta u||D_x^2 u|.$$

For the first term, $J_1(R)$, using Schwarz's inequality we have

$$\int_{B_{2R}\setminus B_R} |\Delta u||\nabla u| < \left(\int_{B_{2R}\setminus B_R} |\Delta u|^2\right)^{1/2} \left(\int_{B_{2R}\setminus B_R} |\nabla u|^2\right)^{1/2}.$$

From standard elliptic interpolation estimates, L^2-norm version of Lemma 4, we have

$$\begin{aligned}
\int_{B_{2R}\setminus B_R} |\nabla u|^2 &\leq CR^2 \int_{B_{4R}\setminus B_{R/2}} |\Delta u|^2 + CR^{-2} \int_{B_{4R}\setminus B_{R/2}} u^2 \\
&\leq CR^{N-\frac{2(2p+2+a)}{p-1}+2} + R^{\frac{N(p-1)}{(p+1)}-\frac{2a}{p+1}-2} \left(\int_{\mathbb{R}^N} |x|^a u^{p+1}\right)^{\frac{2}{p+1}} \\
&= CR^{\frac{p-1}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}+2\right)+2} \left(R^{\frac{2}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}\right)} + \left(\int_{\mathbb{R}^N} |x|^a u^{p+1}\right)^{\frac{2}{p+1}}\right)
\end{aligned}$$

Since $\int_{\mathbb{R}^N} |x|^a u^{p+1} < \infty$ and $N < \frac{2(2p+2+a)}{p-1}$, for $R > 1$ we have

$$\int_{B_{2R}\setminus B_R} |\nabla u|^2 \leq CR^{\frac{p-1}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}+2\right)+2}$$

Therefore,

$$\int_{B_{2R}\setminus B_R} |\Delta u||\nabla u| < CR^{\frac{p}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}\right)+1}. \tag{47}$$

Similarly for the second term, $J_2(R)$, using Lemma 3, i.e.,

$$\int_{B_{2R}\setminus B_R} |D_x^2 u|^2 \leq C\left(\int_{B_{4R}\setminus B_{R/2}} |\Delta u|^2 + R^{-4} \int_{B_{4R}\setminus B_{R/2}} u^2\right),$$

and similar type discussions one can see

$$\int_{B_{2R}\setminus B_R} |\Delta u||D_x^2 u| < CR^{\frac{p}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}\right)}. \tag{48}$$

Now, define the following sets for large enough M;

$$\begin{aligned}
\Lambda_1(R) &:= \{r \in (R,2R);\ ||\Delta_x u(r)||_2^2 > MR^{-\frac{2(2p+2+a)}{p-1}}\}, \\
\Lambda_2(R) &:= \{r \in (R,2R);\ ||u(r)||_{p+1}^{p+1} > MR^{-\frac{2(2p+2+a)}{p-1}-a}\}, \\
\Lambda_3(R) &:= \{r \in (R,2R);\ ||\Delta_x u(r)|\nabla_x u(r)|||_1 > MR^{-\frac{p}{p+1}\left(\frac{N}{p}+\frac{2(2p+2+a)}{p-1}\right)+1}\}, \\
\Lambda_4(R) &:= \{r \in (R,2R);\ ||\Delta_x u(r) D_x^2 u(r)||_1 > MR^{-\frac{p}{p+1}\left(\frac{N}{p}+\frac{2(2p+2+a)}{p-1}\right)}\}.
\end{aligned}$$

In the following, we shall find a bound for the measure of the above sets. From (46), we have

$$\begin{aligned}
C &\geq R^{-N+\frac{2(2p+2+a)}{p-1}+a} \int_R^{2R} ||u(r)||_{p+1}^{p+1} r^{N-1} dr \\
&\geq R^{-N+\frac{2(2p+2+a)}{p-1}+a} |\Lambda_2(R)| R^{N-1} MR^{-\frac{2(2p+2+a)}{p-1}-a} = M|\Lambda_2(R)|R^{-1}.
\end{aligned}$$

Also, from (47)

$$\begin{aligned} C &\geq R^{\frac{p}{p+1}\left(-N+\frac{2(2p+2+a)}{p-1}\right)-1}\int_R^{2R}||\Delta_x u(r)|\nabla_x u(r)|||_1 r^{N-1}dr \\ &\geq R^{\frac{p}{p+1}\left(-N+\frac{2(2p+2+a)}{p-1}\right)-1}|\Lambda_3(R)|R^{N-1}MR^{-\frac{p}{p+1}\left(\frac{N}{p}+\frac{2(2p+2+a)}{p-1}\right)+1} = M|\Lambda_3(R)|R^{-1}. \end{aligned}$$

Similarly, from (48) and (46) we get $|\Lambda_1(R)|, |\Lambda_4(R)| \leq R/M$. By choosing M large enough we conclude $|\Lambda_i(R)| \leq R/5$ for $i = 1, \cdots, 4$. Therefore, for each $R \geq 1$, we can find

$$\tilde{R} \in (R, 2R) \setminus \bigcup_{i=1}^{i=4} \Lambda_i(R) \neq \phi. \tag{49}$$

Then, from the definition of $\tilde{R}$ and Λ_i for $i = 1, \cdots, 4$, we have

$$\int_{|x|=\tilde{R}} |\Delta_x u(\tilde{R})||D_x^2 u(\tilde{R})| \leq C\tilde{R}^{\frac{p}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}\right)-1} \tag{50}$$

$$\int_{|x|=\tilde{R}} |\Delta_x u(\tilde{R})||\nabla_x u(\tilde{R})| \leq C\tilde{R}^{\frac{p}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}\right)} \tag{51}$$

$$\int_{|x|=\tilde{R}} |\Delta_x u(\tilde{R})|^2 \leq C\tilde{R}^{\frac{p}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}\right)-1} \tag{52}$$

$$\int_{|x|=\tilde{R}} u^{p+1}(\tilde{R}) \leq C\tilde{R}^{\frac{p}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}\right)-a-1} \tag{53}$$

Using (43) with $\Omega = B_{2\tilde{R}} \setminus B_{\tilde{R}}$, one can see

$$\left|\int_{|x|=\tilde{R}} \nabla\Delta u \cdot \nu x \cdot \nabla u\right| < C\tilde{R}^{\frac{p}{p+1}\left(N-\frac{2(2p+2+a)}{p-1}\right)}. \tag{54}$$

Now, applying the Pohozaev identity, (43), with $\Omega = B_{\tilde{R}}$ and using (50)-(54), R.H.S. of (43), converges to zero if $R \to \infty$ for subcritical p, i.e. $N < \frac{2(2p+2+a)}{p-1}$. Hence,

$$\frac{N-4}{2}\int_{\mathbb{R}^N} |\Delta u|^2 = \frac{N+a}{p+1}\int_{\mathbb{R}^N} |x|^a u^{p+1}.$$

From this and (44), we finish the proof of Step 4.

□

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