On the Partial Differential Equations of Electrostatic MEMS Devices II: Dynamic Case

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Abstract

This paper is a continuation of [9], where we analyzed steady-states of the nonlinear parabolic problem $u_t = \Delta u - \frac{\lambda f(x)}{(1+u)^2}$ on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. Here u is modeled to describe dynamic deflection of the elastic membrane. When a voltage –represented here by λ– is applied, the membrane deflects towards the ground plate and a snap-through (touchdown) may occur when it exceeds a certain critical value λ^* (pull-in voltage), creating a so-called "pull-in instability" which greatly affects the design of many devices. In an effort to achieve better MEMS designs, the material properties of the membrane can be technologically fabricated with a spatially varying dielectric permittivity profile $f(x)$. We show that when $\lambda \leq \lambda^*$ the membrane globally converges to its unique maximal steady-state. On the other hand, if $\lambda > \lambda^*$ the membrane must touchdown at finite time T, and that touchdown cannot occur at a location where the permittivity profile vanishes. We establish upper and lower bounds on first touchdown times, and we analyze their dependence on f, λ and Ω by applying various analytical and numerical techniques. A refined description of MEMS touchdown profiles will be given in a forthcoming paper [10].

Key words: MEMS; pull-in voltage; steady-state; quenching.

Contents

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1 Introduction

Micro-Electromechanical Systems (MEMS) are often used to combine electronics with micro-size mechanical devices in the design of various types of microscopic machinery. MEMS devices have therefore become key components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches and chemical sensors and so on. The simplicity and importance of this technique have led many applied mathematicians and engineers to study mathematical models of electrostatic-elastic interactions. An overview of the physical phenomena of the mathematical models associated with the rapidly developing field of MEMS technology is given in [19].

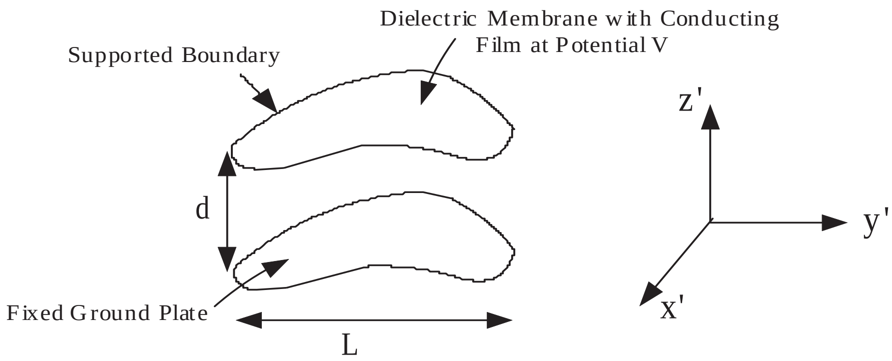

Figure 1: *The simple electrostatic MEMS device.*

The key component of many modern MEMS is the simple idealized electrostatic device shown in Fig. 1. The upper part of this device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage V is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate, and when V is increased beyond a certain critical value V^* –known as pull-in voltage– the steady-state of the elastic membrane is lost, and proceeds to touchdown or snap through at a finite time creating the so-called pull-in instability.

A mathematical model of the physical phenomena, leading to a partial differential equation for the dimensionless dynamic deflection of the membrane, was derived and analyzed in [8] and [13]. In the damping-dominated limit, and using a narrow-gap asymptotic analysis, the dimensionless dynamic deflection $u = u(x,t)$ of the membrane on a bounded domain Ω in $\mathbb{R}^N$, is found to satisfy the following parabolic problem

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda f(x)}{(1-u)^2} \quad \text{for} \quad x \in \Omega\,, \tag{1.1a}$$

$$u(x,t) \geq 0 \quad \text{for} \quad x \in \Omega\,, \tag{1.1b}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega\,, \tag{1.1c}$$

$$u(x,0) = 0 \quad \text{for} \quad x \in \Omega\,. \tag{1.1d}$$

An outline of the derivation of (1.1) was given in Appendix A of [13]. This initial condition in (1.1c) assumes that the membrane is initially undeflected and the voltage is suddenly applied to the upper surface of the membrane at time $t = 0$. The parameter $\lambda > 0$ in (1.1a) characterizes the relative strength of the electrostatic and mechanical forces in the system, and is given in terms of the applied voltage V by $\lambda = \frac{\varepsilon_0 V^2 L^2}{2T_e d^3}$, where d is the undeflected gap size (see Fig. 1), L is the length scale of the membrane, T_e is the tension of the membrane, and ε_0 is the permittivity of free space in the gap between the membrane and the bottom plate.

We shall use here the parameter λ (resp., λ^*) to represent the applied voltage V (resp., pull-in voltage V^*). Referred to as the *permittivity profile*, $f(x)$ in (1.1a) is defined by the ratio $f(x) = \frac{\varepsilon_0}{\varepsilon_2(x)}$ where $\varepsilon_2(x)$ is the dielectric permittivity of the thin membrane.

There are several issues that must be considered in the actual design of MEMS devices. Typically one of the primary goals is to achieve the maximum possible stable deflection before touchdown occurs, which is referred to as *pull-in distance* (cf. [13] and [18]). Another consideration is to increase the stable operating range of the device by improving the pull-in voltage λ^* subject to the constraint that the range of applied voltage is limited by the available power supply. Such improvements in the stable operating range are important for the design of certain MEMS devices such as microresonators. One way –studied in [18] and [13]– of achieving larger values of λ^*, while simultaneously increasing the pull-in distance, is to introduce a spatially varying dielectric permittivity $\varepsilon_2(x)$ of the membrane. The idea is to locate the region where the membrane deflection would normally be largest under a spatially uniform permittivity, and then make sure that a new dielectric permittivity $\varepsilon_2(x)$ is largest –and consequently the profile $f(x)$ smallest– in that region.

This latter approach requires the membrane having varying dielectric properties, a framework investigated recently in [18] and [13]. In [18] J. Pelesko studied the steady-states of (1.1), when $f(x)$ is assumed to be bounded away from zero. He established in this case an upper bound $\bar\lambda_1$ for λ^*, and derived numerical results for the power-law permittivity profile, from which the larger pull-in voltage and thereby the larger pull-in distance, the existence and multiplicity of the steady-states were observed. Recently, Y. Guo, Z. Pan and M. Ward studied in [13] the dynamic behavior of (1.1), which is also of great practical interest. They considered a more general class of profiles $f(x)$, where the membrane is allowed to be perfectly conducting, i.e., $0 \le f(x) \le 1$ on Ω with $f(x) > 0$ on a subset of positive measure. By using both analytical and numerical techniques, they obtained larger pull-in voltage λ^* and larger pull-in distance for different classes of varying permittivity profiles. These results were extended and sharpened in [9], where we focussed on the steady-state solutions of (1.1), *i.e*,

$$\begin{aligned} -\Delta u &= \frac{\lambda f(x)}{(1-u)^2} \quad && x \in \Omega, \\ u(x) &= 0 && x \in \partial\Omega. \end{aligned} \tag{$(S)_\lambda$}$$

with $0 < u < 1$ on Ω. We establish in particular the following lower and upper bound estimates on the pull-in voltage. Here we write $|\Omega|$ for the volume of a domain Ω in $\mathbb{R}^N$ and $P(\Omega) := \int_{\partial\Omega} ds$ for its "perimeter", with ω_N referring to the volume of the unit ball $B_1(0)$ in $\mathbb{R}^N$. We denote by μ_Ω the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and by ϕ_Ω the corresponding positive eigenfunction normalized with $\int_\Omega \phi_\Omega dx = 1$.

Theorem A (Theorem 1.1 in [9]) *Assume f is a non-negative continuous function on a bounded domain Ω in $\mathbb{R}^N$, then there exists a finite pull-in voltage $\lambda^* := \lambda^*(\Omega, f) > 0$ with the following properties:*

1. *If $0 \le \lambda < \lambda^*$, there exists at least one solution for $(S)_\lambda$.*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*
3. *The following bounds on λ^* hold for any bounded domain Ω:*

$$\max\left\{\frac{8N}{27}, \frac{6N-8}{9}\right\} \frac{1}{\sup_\Omega f}\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \le \lambda^*(\Omega) \le \min\left\{\bar\lambda_1 := \frac{4\mu_\Omega}{27 \inf\limits_{x\in\Omega} f(x)}, \bar\lambda_2 := \frac{\mu_\Omega}{3\int_\Omega f\phi_\Omega\, dx}\right\} \tag{1.2}$$

4. *If Ω is a strictly star-shaped domain with $x \cdot \nu(x) \ge a > 0$ for all $x \in \partial\Omega$, where $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$, and if $f \equiv 1$, then*

$$\lambda^*(\Omega) \le \bar\lambda_3 = \frac{(N+2)^2 P(\Omega)}{8aN|\Omega|}. \tag{1.3}$$

   In particular, if $\Omega = B_1(0) \subset \mathbb{R}^N$ then we have the bound $\lambda^*(B_1(0)) \le \frac{(N+2)^2}{8}$.

5. *If $f(x) \equiv |x|^\alpha$ with $\alpha \geq 0$ and Ω is a ball of radius R, then*

$$\lambda^*(B_R, |x|^\alpha) \geq \max\{\frac{4(2+\alpha)(N+\alpha)}{27}, \frac{(2+\alpha)(3N+\alpha-4)}{9}\}R^{-(2+\alpha)}. \tag{1.4}$$

Moreover, if $N \geq 8$ and $0 \leq \alpha \leq \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$, then

$$\lambda^*(B_1, |x|^\alpha) = \frac{(2+\alpha)(3N+\alpha-4)}{9}. \tag{1.5}$$

Fine properties of steady states –such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results– were also shown in [9] and [6] to depend on the dimension of the ambient space and on the permittivity profile. In particular, the following properties of positive minimal solutions of $(S)_\lambda$ were established.

Definition 1.1. A solution u_λ of $(S)_\lambda$ is said to be a minimal solution, if $u_\lambda(x) \leq u(x)$ in Ω whenever u is any solution of $(S)_\lambda$.

For any solution u of $(S)_\lambda$, one can introduce the linearized operator at u defined by $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$, and its corresponding eigenvalues $\{\mu_{k,\lambda}(u); k = 1, 2, ...\}$. The following was also proved in [9].

Theorem B (Theorem 1.2 in [9]) *Suppose f is a non-negative continuous function on a bounded domain Ω, and consider $\lambda^* := \lambda^*(\Omega, f)$ as defined in Theorem A. Then,*

1. *For any $0 \leq \lambda < \lambda^*$, there exists a unique minimal solution u_λ of $(S)_\lambda$ such that $\mu_{1,\lambda}(u_\lambda) > 0$. Moreover for each $x \in \Omega$, the function $\lambda \to u_\lambda(x)$ is strictly increasing and differentiable on $(0, \lambda^*)$.*

2. *If $1 \leq N \leq 7$ then –by means of energy estimates– one has $\sup_{\lambda \in (0,\lambda^*)} \| u_\lambda \|_\infty < 1$ and consequently, $u^* = \lim\limits_{\lambda \uparrow \lambda^*} u_\lambda$ exists in $C^{1,\alpha}(\bar{\Omega})$ with $0 < \alpha < 1$ and is a solution for $(S)_{\lambda^*}$ such that $\mu_{1,\lambda^*}(u^*) = 0$. In particular, u^* –often referred to as the extremal solution of problem $(S)_\lambda$– is unique.*

3. *On the other hand, if $N \geq 8$, $f(x) = |x|^\alpha$ with $0 \leq \alpha \leq \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$ and Ω is the unit ball, then the extremal solution is necessarily $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ and is therefore singular.*

In this paper, we deal with issues of global convergence as well as finite and infinite time "touchdown" in the dynamic problem (1.1).

Recall that a point $x_0 \in \bar{\Omega}$ is said to be a *touchdown point* for a solution $u(x,t)$ of (1.1), if for some $T \in (0, +\infty]$, we have $\lim\limits_{t_n \to T} u(x_0, t_n) = 1$. T is then said to be a –finite or infinite– touchdown time. For each such solution, we define its corresponding –possibly infinite– "first touchdown time":

$$T_\lambda(\Omega, f, u) = \inf\big\{t \in (0, +\infty]; \sup_{x \in \Omega} u(x,t) = 1\big\}.$$

We shall analyze the relationship between the applied voltage λ, the permittivity profile f, and the dynamic deflection of the elastic membrane. It is already known that solutions corresponding to large voltages λ necessarily touchdown in finite time (See [13]). The following theorem proved in section 2, completes the picture.

Theorem 1.1. *Suppose $\lambda^* := \lambda^*(\Omega, f)$ is as in Theorem A, then the following hold:*

1. *If $\lambda \leq \lambda^*$, then there exists a unique solution $u(x,t)$ for (1.1) which globally converges as $t \to +\infty$, monotonically and pointwise to its unique minimal steady-state.*

2. *If $\lambda > \lambda^*$, then the solution $u(x,t)$ of (1.1) must touchdown at a finite time.*

This "touchdown" phenomenon is referred to sometimes as *quenching*. Note that in the case where the unique minimal steady-state of (1.1) at $\lambda = \lambda^*$ is non-regular − which can happen if $N \geq 8$ − the above result means that the corresponding dynamic solution must touchdown but that quenching occurs here in infinite time.
In section 3 we shall establish that –an isolated– touchdown cannot occur at a point in Ω where the permittivity profile is zero, a fact that was observed numerically and conjectured to hold in [13]. More precisely, we prove the following.

Theorem 1.2. *Suppose $u(x,t)$ is a touchdown solution of* (1.1) *at a finite time T, then $u_t > 0$ for all $0 < t < T$. Furthermore,*

1. *The permittivity profile f cannot vanish on an isolated set of touchdown points in Ω.*

2. *On the other hand, zeroes of the permittivity profile can be locations of touchdown in infinite time.*

In §4 we shall provide upper and lower estimates for touchdown times. Uniqueness considerations lead to a first touchdown time $T_\lambda(\Omega, f)$ that only depend on the domain Ω and on the profile f. These touchdown times translate into useful information concerning the speed of the operation for many MEMS devices, such as Radio Frequency (RF) switches and microvalves. Estimates 1.8 and 1.9 below were already established in [13] for large λ. Considering that $\lambda^* < \min\{\bar\lambda_1, \bar\lambda_2\}$, the estimate (1.7) below gives an upper bound on the first touchdown time as soon as we exceed the pull-in voltage λ^*.

Theorem 1.3. *Suppose f is a non-negative continuous function on a bounded domain Ω, and let $T_\lambda(\Omega, f)$ be the first –possibly infinite– touchdown time corresponding to a voltage λ.*

1. *The following lower estimate then holds for any $\lambda > 0$:*

$$\frac{1}{3\lambda \sup_{x\in\Omega} f(x)} \leq T_\lambda(\Omega, f). \tag{1.6}$$

2. *If $\inf_{x\in\Omega} f(x) > 0$, then the following upper estimate holds for any $\lambda > \lambda^*$:*

$$T_\lambda(\Omega, f) \leq T_{0,\lambda}(\Omega, f) := \frac{8(\lambda+\lambda^*)^2}{3\inf_{x\in\Omega} f(x)(\lambda-\lambda^*)^2(\lambda+3\lambda^*)}\Big[1 + \Big(\frac{\lambda+3\lambda^*}{2\lambda+2\lambda^*}\Big)^{1/2}\Big]. \tag{1.7}$$

3. *If $\inf_{x\in\Omega} f(x) > 0$, and $\lambda > \bar\lambda_1 := \frac{4\mu_\Omega}{27\inf_{x\in\Omega} f(x)}$, then*

$$T_\lambda(\Omega, f) \leq T_{1,\lambda}(\Omega, f) := \int_0^1 \Big[\frac{\lambda \inf_{x\in\Omega} f(x)}{(1-s)^2} - \mu_\Omega s\Big]^{-1} ds. \tag{1.8}$$

4. *If $\lambda > \bar\lambda_2 := \frac{\mu_\Omega}{3\int_\Omega f\phi_\Omega\, dx}$, then*

$$T_\lambda(\Omega, f) \leq T_{2,\lambda}(\Omega, f) := -\frac{1}{\mu_\Omega}\log\Big[1 - \frac{\mu_\Omega}{3\lambda}\Big(\int_\Omega f\phi_\Omega\, dx\Big)^{-1}\Big]. \tag{1.9}$$

Note that the upper bounds $T_{0,\lambda}$ and $T_{1,\lambda}$ are relevant only when f is bounded away from 0, while the upper bound $T_{2,\lambda}$ is valid for all permittivity profiles provided of course that $\lambda > \lambda_2$.

In a forthcoming paper [10], the second-named author will give a refined description of the touchdown behavior of a MEMS device, including some touchdown estimates, touchdown rates, as well as some information on the location of touchdown points and on the shape of the touchdown set.

2 Global Convergence or Touchdown at Finite or Infinite Time

In this section, we discuss the dynamic deflection $u = u(x,t)$ satisfying (1.1) and establish the claims in Theorem 1.1. We first prove in section §2.1 global convergence in the case $\lambda < \lambda^*$. In section §2.2 we study finite-time touchdown for the case $\lambda > \lambda^*$. Finally we discuss the case $\lambda = \lambda^*$ in section §2.3.

First, we note the following uniqueness result.

Lemma 2.1. *Suppose u_1 and u_2 are solutions of* (1.1) *on the interval* $[0,T]$ *such that* $\|u_i\|_{L^\infty(\Omega\times[0,T])} < 1$ *for $i = 1,2$, then $u_1 = u_2$.*

Proof: Indeed, the difference $U = u_1 - u_2$ then satisfies

$$U_t - \Delta U = \alpha U \quad \text{in} \quad \Omega \tag{2.1}$$

with initial data $U(x,0) = 0$ and zero boundary condition. Here

$$\alpha(x,t) = \frac{\lambda(2 - u_1 - u_2)f(x)}{(1-u_1)^2(1-u_2)^2}\,.$$

The assumption on u_1, u_2 implies that $\alpha(x,t) \in L^\infty(\Omega \times [0,T])$. We now fix $T_1 \in [0,T]$ and consider the solution ϕ of the problem

$$\begin{cases} \phi_t + \Delta\phi + \alpha\phi = 0 \qquad x \in \Omega,\ 0 < t < T_1, \\ \qquad \phi(x,T_1) = \theta(x) \in C_0(\Omega), \\ \qquad \phi(x,t) = 0 \qquad x \in \partial\Omega, \end{cases} \tag{2.2}$$

The standard linear theory (cf. Theorem 8.1 of [16]) gives that the solution of (2.2) is unique and bounded. Now multiplying (2.1) by ϕ, and integrating it on $\Omega \times [0,T_1]$, together with (2.2), yield that

$$\int_\Omega U(x,T_1)\theta(x)dx = 0$$

for arbitrary T_1 and $\theta(x)$, which implies that $U \equiv 0$, and we are done.

2.1 Global convergence when $\lambda < \lambda^*$

Theorem 2.2. *Suppose $\lambda^* := \lambda^*(\Omega, f)$ is the pull-in voltage defined in Theorem A, then for every $\lambda < \lambda^*$ there exists a unique global solution $u(x,t)$ for* (1.1) *which monotonically converges as $t \to +\infty$ to the minimal solution u_λ of $(S)_\lambda$.*

Proof:: This is standard and follows from the maximum principle combined with the existence of regular minimal steady-state solutions at this range of λ. Indeed, fix $0 < \lambda < \lambda^*$, and use Theorem (B) to obtain the existence of a unique minimal solution $u_\lambda(x)$ of $(S)_\lambda$. It is clear that the pair $\tilde{u} \equiv 0$ and $\hat{u} = u_\lambda(x)$ are sub- and super-solutions of (1.1). This implies that the unique global solution $u(x,t)$ of (1.1) satisfies $1 > u_\lambda(x) \geq u(x,t) \geq 0$ in $\Omega \times (0,\infty)$.

By differentiating in time and setting $v = u_t$, we get for any fixed $t_0 > 0$

$$v_t = \Delta v + \frac{2\lambda f(x)}{(1-u)^3}v \qquad (x,t) \in \Omega \times (0,t_0) \tag{2.3}$$

$$v(x,t) = 0 \qquad (x,t) \in \partial\Omega \times (0,t_0) \tag{2.4}$$

$$v(x,0) \geq 0 \qquad x \in \Omega. \tag{2.5}$$

Here $\frac{2\lambda f(x)}{(1-u)^3}$ is a locally bounded non-negative function, and by the strong maximum principle, we get that $u_t = v > 0$ for $(x,t) \in \Omega \times (0,t_0)$ or $u_t = 0$. The second case is impossible because otherwise $u(x,t) = u_\lambda(x)$ for any $t > 0$. It follows that $u_t > 0$ holds for all $(x,t) \in \Omega \times (0,\infty)$, and since $u(x,t)$ is bounded, this monotonicity property implies that the unique global solution $u(x,t)$ converges to some function $u_s(x)$ as $t \to \infty$. Hence, $1 > u_\lambda(x) \geq u_s(x) > 0$ in Ω.

Next we claim that the limit $u_s(x)$ is a solution of $(S)_\lambda$. Indeed, consider a solution u_1 of the linear stationary boundary problem

$$-\Delta u_1 = \frac{\lambda f(x)}{(1-u_s)^2} \quad x \in \Omega \tag{2.6}$$

$$u_1 = 0 \quad x \in \partial\Omega. \tag{2.7}$$

Let $w(x,t) = u(x,t) - u_1(x)$, then w satisfies

$$w_t - \Delta w = \lambda f(x)\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_s)^2}\Big] \quad (x,t) \in \Omega \times (0,T) \tag{2.8}$$

$$w(x,t) = 0 \quad x \in \partial\Omega \times (0,T) \tag{2.9}$$

$$w(x,0) = -u_1(x) \quad x \in \Omega \tag{2.10}$$

Since the right side of (2.8) converges to zero in $L^2(\Omega)$ as $t \to \infty$, a standard eigenfunction expansion implies that the solution w of (2.8) also converges to zero in $L^2(\Omega)$ as $t \to \infty$. This shows that $u(x,t) \to u_1(x)$ in $L^2(\Omega)$ as $t \to \infty$. But since $u(x,t) \to u_s(x)$ pointwise in Ω as $t \to \infty$, we deduce that $u_1(x) \equiv u_s(x)$ in $L^2(\Omega)$, which implies that $u_s(x)$ is also a solution for $(S)_\lambda$. The minimal property of $u_\lambda(x)$ then yields that $u_\lambda(x) \equiv u_s(x)$ on Ω from which follows that for every $x \in \Omega$, we have $u(x,t) \uparrow u_\lambda(x)$ as $t \to \infty$. ■

2.2 Touchdown at finite time when $\lambda > \lambda^*$

In this case, we know from Theorem (A) that there is no solution for $(S)_\lambda$ as soon as $\lambda > \lambda^*$. Since the solution $u(x,t)$ of (1.1) –whenever it exists– is strictly increasing in time t (see preceeding theorem), then there must be $T \leq \infty$ such that $u(x,t)$ reaches 1 at some point of $\overline{\Omega}$ as $t \to T^-$. Otherwise, a proof similar to Theorem 2.2 would imply that $u(x,t)$ will converge to its steady-state which is then the unique minimal solution u_λ of $(S)_\lambda$, contrary to the hypothesis that $\lambda > \lambda^*$. Therefore for this case, it only remains to know whether the touchdown time is finite or infinite. It was actually proved in [13] –via energy methods– that the touchdown time T must be finite whenever λ is large enough, but whether it is the case for any $\lambda > \lambda^*$ was left open. This is exactly what we prove in the following.

Theorem 2.3. *Suppose $\lambda^* := \lambda^*(\Omega, f)$ is the pull-in voltage defined in Theorem A, then for $\lambda > \lambda^*(\Omega)$, there exists a finite time $T_\lambda(\Omega, f)$ at which the unique solution $u(x,t)$ of* (1.1) *must touchdown. Moreover, if $\inf_{x\in\Omega} f(x) > 0$, then we have the bound*

$$T_\lambda(\Omega, f) \leq T_{0,\lambda} := \frac{8(\lambda+\lambda^*)^2}{3\inf_{x\in\Omega} f(x)(\lambda-\lambda^*)^2(\lambda+3\lambda^*)}\Big[1 + \Big(\frac{\lambda+3\lambda^*}{2\lambda+2\lambda^*}\Big)^{1/2}\Big]. \tag{2.11}$$

We start by transforming the problem from a touchdown situation (i.e. quenching) into a blow-up problem where a concavity method can be used. For that, we set $V = 1/(1-u)$ which reduces (1.1) to the following parabolic problem

$$V_t = \Delta V - \frac{2|\nabla V|^2}{V} + \lambda f(x)V^4 \quad \text{for} \quad x \in \Omega\,, \tag{2.12a}$$

$$V(x,t) = 1 \quad \text{for} \quad x \in \partial\Omega\,, \tag{2.12b}$$

$$V(x,0) = 1 \quad \text{for} \quad x \in \Omega\,. \tag{2.12c}$$

This transformation implies that when $\lambda > \lambda^*$, the solution of (2.12) must blow up (in finite or infinite time) and that there is no solution for the corresponding stationary equation:

$$\Delta V - \frac{2|\nabla V|^2}{V} + \lambda f(x)V^4 = 0\,, \quad x \in \Omega\,; \qquad V = 1\,, \quad x \in \partial\Omega\,. \tag{2.13}$$

Therefore, proving finite touchdown time of u for (1.1) is equivalent to showing finite blow-up time of the solution V for (2.12).

In the case where $\inf_{x\in\Omega} f(x) = 0$, we will also need to consider the stationary problem on a subset $\Omega_\epsilon := \{x \in \Omega : f(x) > \epsilon\}$ of Ω, where $\epsilon > 0$ is small enough. We recall from [9] the following properties for the corresponding pull-in voltage $\lambda^*(\Omega_\epsilon, f)$:

$$\lambda^*(\Omega_\epsilon, f) \geq \lambda^* = \lambda^*(\Omega, f) \text{ and } \lim_{\epsilon\to 0} \lambda^*(\Omega_\epsilon, f) = \lambda^*.$$

For the proof, we shall first analyze the following auxiliary parabolic equation

$$\begin{aligned} v_t &= \Delta v - \frac{2|\nabla v|^2}{v} + \lambda a^2 t^2 f(x) v^4 && \text{for} \quad x \in \Omega\,, && \text{(2.14a)} \\ v &= 1 && \text{for} \quad x \in \partial\Omega\,, && \text{(2.14b)} \\ v(x,0) &= 1 && \text{for} \quad x \in \Omega\,, && \text{(2.14c)} \end{aligned}$$

where $a > 0$ is a given constant.

Lemma 2.4. *Suppose v is a solution of* (2.14) *up to a finite time $\bar{T}$, then $\big(\frac{v_t}{v^4}\big)_t \geq 0$ for all $t < \bar{T}$.*

Proof: Dividing (2.14a) by v^4, we obtain

$$\frac{v_t}{v^4} = \frac{\Delta v}{v^4} - \frac{2|\nabla v|^2}{v^5} + \lambda a^2 t^2 f(x)\,.$$

Setting $w = v^{-3}$, then direct calculations show that

$$w_t - \Delta w + \frac{2|\nabla w|^2}{3w} + 3\lambda a^2 t^2 f(x) = 0\,. \tag{2.15}$$

Differentiate (2.15) twice with respect to t, we obtain

$$\begin{aligned} \Big(\frac{|\nabla w|^2}{w}\Big)_{tt} &= \Big(\frac{2\nabla w \nabla w_t}{w} - \frac{|\nabla w|^2 w_t}{w^2}\Big)_t \\ &= \frac{2\nabla w \nabla w_{tt}}{w} + \frac{2|\nabla w_t|^2}{w} - \frac{4\nabla w \nabla w_t w_t}{w^2} - \frac{|\nabla w|^2 w_{tt}}{w^2} + \frac{2|\nabla w|^2 w_t^2}{w^3}\,, \end{aligned}$$

which means that the function

$$z = w_{tt} = -3\Big(\frac{v_t}{v^4}\Big)_t \tag{2.16}$$

satisfies

$$\begin{aligned} L(z) :&= z_t - \Delta z + \frac{4\nabla w}{3w}\nabla z - \frac{2|\nabla w|^2}{3w^2} z \\ &= -6\lambda a^2 f(x) - \frac{2}{3}\Big[\frac{2|\nabla w_t|^2}{w} + \frac{2|\nabla w|^2 w_t^2}{w^3} - \frac{4\nabla w \nabla w_t w_t}{w^2}\Big] \\ &\leq -6\lambda a^2 f(x)\,, \end{aligned}$$

after an application of Cauchy-Schwarz inequality. Hence we have

$$L(z) \leq -6\lambda a^2 f(x) \leq 0\,. \tag{2.17}$$

Now from (2.14) and the definition of z, we have $z(x,0) = 0$ and $z = 0$ on $\partial\Omega$. Since the coefficients of L remain bounded as long as v is bounded, we conclude from the maximum principle ([7], p.. 369) that $z(x,t) \leq 0$ holds for all $t < \bar{T}$. This completes the proof of Lemma 2.3. ∎

Proof of Theorem 2.3: Let $\lambda > \lambda^*$ and let $\epsilon > 0$ be small enough so that $\lambda > \lambda^*(\Omega_\epsilon, f) \geq \lambda^*$. Let $\lambda' = \lambda - \lambda^* > 0$, and set

$$a_\epsilon = \frac{3\epsilon\lambda'(4\lambda^* + \lambda')}{4(2\lambda^* + \lambda')}\Big[1 - \Big(\frac{4\lambda^* + \lambda'}{2(2\lambda^* + \lambda')}\Big)^{1/2}\Big], \tag{2.18a}$$

and

$$T^\epsilon_{0,\lambda} = \frac{1}{a_\epsilon} = \frac{8(\lambda + \lambda^*)^2}{3\epsilon(\lambda - \lambda^*)^2(\lambda + 3\lambda^*)}\Big[1 + \Big(\frac{\lambda + 3\lambda^*}{2\lambda + 2\lambda^*}\Big)^{1/2}\Big] < +\infty\,. \tag{2.18b}$$

Consider now a solution v of (2.14) corresponding to $\lambda = \lambda^* + \lambda'$ and a_ϵ as defined in (2.18a). We first establish the following

Claim: There exists $x_\epsilon \in \Omega$ with $f(x_\epsilon) > \epsilon$ such that $v(x_\epsilon, t) \to \infty$ as $t \nearrow T^\epsilon_{0,\lambda}$.
Indeed, let $t_\epsilon = \frac{1}{a_\epsilon}\Big[\frac{4\lambda^*+\lambda'}{2(2\lambda^*+\lambda')}\Big]^{1/2}$ in such a way that

$$t_\epsilon < T^\epsilon_{0,\lambda} \quad \text{and} \quad a_\epsilon^2 t_\epsilon^2(\lambda^* + \frac{\lambda'}{2}) = \lambda^* + \frac{\lambda'}{4}.$$

We claim that there exists $x_\epsilon \in \Omega_\epsilon$ such that

$$\Delta v(x_\epsilon, t_\epsilon) - \frac{2|\nabla v(x_\epsilon, t_\epsilon)|^2}{v(x_\epsilon, t_\epsilon)} + (\lambda^* + \frac{\lambda'}{4})f(x_\epsilon)|v(x_\epsilon, t_\epsilon)|^4 > 0\,. \tag{2.19}$$

Indeed, otherwise we get that for all $x \in \Omega_\epsilon$

$$\Delta v(x, t_\epsilon) - \frac{2|\nabla v(x, t_\epsilon)|^2}{v(x, t_\epsilon)} + (\lambda^* + \frac{\lambda'}{4})f(x_\epsilon)|v(x, t_\epsilon)|^4 \leq 0\,. \tag{2.20}$$

Since $v(x, t_\epsilon) \geq 1$ on Ω and hence on Ω_ϵ, this means that the function $\bar{v}(x) = v(x, t_\epsilon)$ is a supersolution for the equation

$$\Delta V - \frac{2|\nabla V|^2}{V} + \lambda f(x)V^4 = 0\,, \quad x \in \Omega_\epsilon\,; \qquad V = 1\,, \quad x \in \partial\Omega_\epsilon\,. \tag{2.21}$$

Since $\underline{v} \equiv 1$ is obviously a subsolution of (2.21), it follows that the latter has a solution which contradicts the fact that $\lambda = \lambda^* + \frac{\lambda'}{4} > \lambda^*(f, \Omega_\epsilon) \geq \lambda^*$. Hence assertion (2.19) is verified.

On the other hand, we do get from (2.14) that for $t = t_\epsilon$ and every $x \in \Omega$,

$$v_t = \Delta v - \frac{2|\nabla v|^2}{v} + (\lambda^* + \frac{\lambda'}{4})f(x)v^4 + \frac{\lambda'}{2}a_\epsilon^2 t_\epsilon^2 f(x)v^4\,. \tag{2.22}$$

We then deduce from (2.22) and (2.19) that at the point (x_ϵ, t_ϵ), we have

$$\frac{v_t}{v^4} \geq \frac{\lambda'}{2}a_\epsilon^2 t_\epsilon^2 f(x_\epsilon) > 0.$$

Applying Lemma 2.3, we then get for all (x_ϵ, t), $t_\epsilon \leq t < T^\epsilon_{0,\lambda}$ that:

$$\frac{v_t}{v^4} \geq \frac{\lambda'}{2}a_\epsilon^2 t_\epsilon^2 f(x_\epsilon) > 0. \tag{2.23}$$

Integrating (2.23) with respect to t in $(t_\epsilon, T^\epsilon_{0,\lambda})$, we obtain since $f(x_\epsilon) \geq \epsilon$ that:

$$\frac{1}{3}\big(1 - v^{-3}(x_\epsilon, T^\epsilon_{0,\lambda})\big) \geq \frac{\lambda'}{2}a_\epsilon^2 t_\epsilon^2 f(x_\epsilon)(T^\epsilon_{0,\lambda} - t_\epsilon) \geq \frac{\lambda'}{2}a_\epsilon^2 t_\epsilon^2 \epsilon(T^\epsilon_{0,\lambda} - t_0) = \frac{1}{3}.$$

It follows that $v(x_\epsilon, t) \to \infty$ as $t \nearrow T^\epsilon_{0,\lambda}$, and the claim is proved.

To complete the proof of Lemma 2.4, we note that since $a_\epsilon^2 t^2 \le 1$ for all $t \le T^\epsilon_{0,\lambda}$, we obtain from (2.14) that

$$v_t \le \Delta v - \frac{2|\nabla v|^2}{v} + \lambda f(x) v^4\,, \quad (x,t) \in \Omega \times (0, T^\epsilon_{0,\lambda}).$$

Setting $w = V - v$, where V is the solution of (2.12), then w satisfies

$$w_t - \Delta w - \frac{2\nabla(V+v)}{V}\nabla w + \Big[\lambda(V^2+v^2)(V+v)f(x) + \frac{2|\nabla v|^2}{Vv}\Big] w \ge 0\,, \quad (x,t) \in \Omega \times (0, T^\epsilon_{0,\lambda}).$$

Here the coefficients of ∇w and w are bounded functions as long as V and v are both bounded. It is also clear that $w = 0$ on $\partial\Omega$ and $w(x,0) = 0$. Applying the maximum principle, we reduce that $w \ge 0$ and thus $V \ge v$. Consequently, V must also blow up at some finite time $T \le T^\epsilon_{0,\lambda}$, which means that u must touchdown at some finite time prior to $T^\epsilon_{0,\lambda}$.

Note that we have really proved that for any $\epsilon > 0$, there exists $\lambda^*_\epsilon \ge \lambda^*$ such that for any $\lambda > \lambda^*_\epsilon$, the solution of (1.1) touches down at a time prior to

$$T^\epsilon_{0,\lambda} = \frac{1}{3\max\{\epsilon, \inf_\Omega f\}} \frac{8(\lambda+\lambda^*)^2}{(\lambda-\lambda^*)^2(\lambda+3\lambda^*)} \Big[1 + \Big(\frac{\lambda+3\lambda^*}{2\lambda+2\lambda^*}\Big)^{1/2}\Big] < +\infty\,. \tag{2.24}$$

Moreover $\lambda^*_\epsilon \to \lambda^*$ as $\epsilon \to 0$. In the case where $\inf_{x\in\Omega} f(x) > 0$, formula (2.24) reduces to our second claim in Theorem 2.3.

■

2.3 Global convergence or touchdown in infinite time for $\lambda = \lambda^*$

We now discuss the dynamic behavior of (1.1) at $\lambda = \lambda^*$. For this critical case, there exists a unique steady-state w^* of (1.1) obtained as a pointwise limit of the minimal solution u_λ as $\lambda \uparrow \lambda^*$. If w^* is regular (i.e, if it is a classical solution such as in the case when $N \le 7$) a similar proof as in the case where $\lambda < \lambda^*$, yields the existence of a unique solution $u^*(x,t)$ which globally converges to the unique steady-state w^* as $t \to \infty$. On the other hand, if w^* is a non-regular steady-state, *i.e.* if $\|w^*\|_\infty = 1$, the situation is complicated as we shall still prove global convergence to the extremal solution, which then amounts to a touchdown in infinite time.

Throughout this subsection, we shall consider the unique solution $0 \le u^* = u^*(x,t) < 1$ for the problem

$$u^*_t - \Delta u^* = \frac{\lambda^* f(x)}{(1-u^*)^2} \qquad \text{for} \quad (x,t) \in \Omega \times [0, t^*)\,, \tag{2.25a}$$

$$u^*(x,t) = 0 \qquad \text{for} \quad x \in \partial\Omega \times [0,t^*)\,, \tag{2.25b}$$

$$u^*(x,0) = 0 \qquad \text{for} \quad x \in \Omega\,, \tag{2.25c}$$

where t^* is the maximal time for existence. We shall use techniques developed in [2] to establish the following

Theorem 2.5. *If w^* is a non-regular minimal steady-state of* (2.25)*, then there exists a unique global solution u^* of* (2.25) *such that $u^*(x,t) \le w^*(x)$ for all $t < \infty$, while $u^*(x,t) \to w^*(x)$ as $t \to \infty$. In particular,* $\lim_{t\to+\infty} \|u^*(x,t)\|_\infty = 1$.

We shall use the following fact which is essentially Lemma 7 of [2].

Lemma 2.6. *Consider the function $\delta(x) := \mathrm{dist}(x, \partial\Omega)$, then for any $0 < T < \infty$, there exists $\varepsilon_1 = \varepsilon_1(T)$ such that for $0 < \varepsilon \le \varepsilon_1$ the solution Z^ϵ of the problem*

$$\begin{aligned} Z_t - \Delta Z &= -\varepsilon f(x) && \text{in} \quad \Omega \times (0,\infty)\,, \\ Z(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0,\infty)\,, \\ Z(x,0) &= \delta(x) && \text{in} \quad \Omega \end{aligned}$$

satisfies $Z^\epsilon \ge 0$ on $[0,T] \times \bar\Omega$.

Proof of Theorem 2.5: We proceed in four steps.

Claim 1. We have that $u^*(x,t) \le w^*(x)$ for all $(x,t) \in \Omega \times [0,t^*)$. Indeed, fix any $T < t^*$ and let ξ be the solution of the backward heat equation:

$$\begin{aligned} \xi_t - \Delta\xi = h(x,t) \qquad &\text{in} \quad \Omega \times (0,T)\,, \\ \xi|_{\partial\Omega} = 0\,, \qquad &\xi(T) = 0\,, \end{aligned}$$

where $h(x,t) \ge 0$ is in $\Omega \times (0,T)$. Multiplying (2.25) by ξ and integrating on $\Omega \times (0,T)$ we find that

$$\int_0^T \int_\Omega u^* h\, dxdt = \int_0^T \int_\Omega \frac{\lambda^* \xi f(x)}{(1-u^*)^2}\, dxdt\,.$$

On the other hand,

$$-\int_0^T \int_\Omega w^* \xi_t\, dxdt = \int_\Omega w^* \xi(0) dx \quad \text{ and } \quad -\int_0^T \int_\Omega w^* \Delta\xi\, dxdt = \int_0^T \int_\Omega \frac{\lambda^* \xi f(x)}{(1-w^*)^2} \text{ dx dt}.$$

Therefore, we have

$$\begin{aligned} \int_0^T \int_\Omega (u^*-w^*) h\, dxdt \le \int_\Omega w^* \xi(0) dx + \int_0^T \int_\Omega (u^*-w^*) h\, dxdt &= \int_0^T \int_\Omega \Big(\frac{1}{(1-u^*)^2} - \frac{1}{(1-w^*)^2}\Big) \lambda^* \xi f(x)\, dxdt \\ &\le C \int_0^T \int_{\{u^* \ge w^*\}} \Big(\frac{1}{(1-u^*)^2} - \frac{1}{(1-w^*)^2}\Big) \xi\, dxdt \\ &\le C \int_0^T \int_\Omega (u^*-w^*)^+ \xi\, dxdt, \end{aligned}$$

since $\|u^*\|_\infty < 1$ for $t \in [0,T)$. Therefore, we have

$$\int_0^T \int_\Omega (u^*-w^*) h\, dxdt \le C \Big(\int_0^T \int_\Omega [(u^*-w^*)^+]^2\, dxdt\Big)^{1/2} \Big(\int_0^T \int_\Omega \xi^2\, dxdt\Big)^{1/2}.$$

On the other hand, $\xi(x,t) = \int_t^T T(s-t) h(x,s) ds$, where $T(t)$ is the heat semigroup with Dirichlet boundary condition, and hence

$$\|\xi(x,t)\|_{L^2}^2 \le \Big(\int_t^T \|h(x,s)\|_{L^2} ds\Big)^2 \le (T-t) \int_0^T \int_\Omega h^2\, dxdt\,.$$

Therefore,

$$\int_0^T \int_\Omega \xi^2\, dxdt \le \frac{T^2}{2} \int_0^T \int_\Omega h^2\, dxdt,$$

and so,

$$\int_0^T \int_\Omega (u^*-w^*) h\, dxdt \le \frac{CT}{\sqrt{2}} \Big(\int_0^T \int_\Omega [(u^*-w^*)^+]^2\, dxdt\Big)^{1/2} \Big(\int_0^T \int_\Omega h^2\, dxdt\Big)^{1/2}.$$

Letting h converge to $(u^*-w^*)^+$ in L^2, and since $u^*-w^* \in L^1(\Omega)$ we have

$$\int_0^T \int_\Omega [(u^*-w^*)^+]^2\, dxdt \le \frac{CT}{\sqrt{2}} \int_0^T \int_\Omega [(u^*-w^*)^+]^2\, dxdt\,,$$

which gives that $u^* \le w^*$ provided $C^2T^2 < 2$, and our first claim follows.

Claim 2. There exist $0 < \tau_1 < t^*$, and C_0, $c_0 > 0$ such that for all $x \in \Omega$

$$u^*(x, \tau_1) \leq \min\{C_0\delta(x);\, w^*(x) - c_0\delta(x)\}\,. \tag{2.26}$$

Fix $0 < \tau < t^*$ sufficiently small, and let v be the solution of

$$v_t - \Delta v = \frac{\lambda^* f(x)}{(1-v)^2} \qquad \text{for} \quad (x,t) \in \Omega \times [0, \bar{T})\,, \tag{2.27a}$$

$$v(x,t) = 0 \qquad \text{for} \quad x \in \partial\Omega \times [0, \bar{T})\,, \tag{2.27b}$$

$$v(x,0) = v_0 = u^*(x,\tau) \qquad \text{for} \quad x \in \Omega\,, \tag{2.27c}$$

where $[0, \bar{T})$ is the maximal interval of existence for v. Similarly to Claim 1, we can show that $0 \leq v \leq w^*$. Choose now $K > 1$ sufficiently large such that the path $z(x,t) := u^*(x,t) + \frac{1}{K}T(t)v_0$ satisfies $\|z(x,t)\|_\infty \leq 1$ for $0 \leq t < \bar{T}$. We then have

$$\begin{aligned} z_t - \Delta z &= \frac{\lambda^* f(x)}{(1-u^*)^2} \leq \frac{\lambda^* f(x)}{(1-z)^2} && \text{in} \quad \Omega \times (0, \bar{T})\,,\\ z(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0, \bar{T})\,,\\ z(x,0) &= \frac{v_0(x)}{K} && \text{in} \quad \Omega\,, \end{aligned}$$

and the maximum principle gives that $z \leq v$. Consider now a function $\gamma : [0, \infty) \to R$ such that $\gamma(t) > 0$ and

$$T(t)v_0 \geq K\gamma(t)\delta \text{ on } \Omega. \tag{2.28}$$

We then get

$$u^* \leq v - \frac{1}{K}T(t)v_0 \leq w^* - \frac{1}{K}T(t)v_0 \leq w^* - \gamma(t)\delta \quad \text{for} \quad 0 \leq t < \bar{T}\,. \tag{2.29}$$

Consider now the solution ξ_0

$$-\Delta\xi_0 = 1 \qquad \text{in} \quad \Omega\,; \quad \xi_0 = 0 \qquad \text{on} \quad \partial\Omega$$

in such a way that $\xi_0 = T(t)\xi_0 + \int_0^t T(s)1_\Omega ds$ for all $0 \leq t \leq T < \min\{\bar{T}, t^*\}$. Since $T(t)\xi_0 \geq 0$ it follows that $\int_0^t T(s)1_\Omega ds \leq \xi_0 \leq C\delta$. On the other hand, for any $0 \leq t \leq T < t^*$, u^* is bounded by some constant $M < 1$ on $\bar{\Omega} \times [0, T]$ such that

$$u^* \leq MT(t)1_\Omega + \frac{C}{(1-M)^2}\int_0^t T(s)1_\Omega ds\,.$$

Consider now a function $C : [0, \infty) \to R$ such that $T(t)1_\Omega \leq C(t)\delta$ for $t \geq 0$, which means that

$$u^* \leq MC(t)\delta + C(M)C\delta$$

for any $0 \leq t \leq T$. This combined with (2.29) conclude the proof of Claim (2.26).

Claim 3. For $0 < \varepsilon < 1$ there exists w_ε satisfying $\|w_\varepsilon\|_\infty < 1$ and

$$\int_\Omega \nabla w_\varepsilon \nabla\varphi \geq \int_\Omega \Big(\frac{1}{(1-w_\varepsilon)^2} - \varepsilon\Big)\lambda^*\varphi f(x) \tag{2.30}$$

for all $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$ on Ω. Moreover, there exists $0 < \varepsilon_1 \leq 1$ such that for $0 < \varepsilon < \varepsilon_1$, we also have

$$0 \leq w_\varepsilon(x) - \tfrac{c_0}{2}\delta(x) \quad \text{ for } x \in \Omega \tag{2.31}$$

c_0 being as in (2.26).

To prove (2.30), we set

$$g(w^*) = \frac{1}{(1-w^*)^2}\,, \quad h(w^*) = \int_0^{w^*} \frac{ds}{g(s)}\,, \quad 0 \le w^* < 1\,. \tag{2.32}$$

For any $\varepsilon \in (0,1)$ we also set

$$\widetilde{g}(w^*) = \frac{1}{(1-w^*)^2} - \varepsilon\,, \quad \widetilde{h}(w^*) = \int_0^{w^*} \frac{ds}{\widetilde{g}(s)}\,, \quad 0 \le w^* < 1\,, \tag{2.33}$$

and $\phi_\varepsilon(w^*) := \widetilde{h}^{-1}\big(h(w^*)\big)$. It is easy to check that $\phi_\varepsilon(0) = 0$ and $0 \le \phi_\varepsilon(s) < s$ for $s \ge 0$, and ϕ_ε is increasing and concave with

$$\phi_\varepsilon'(s) = \frac{g(\phi_\varepsilon(s)) - \varepsilon}{g(s)} > 0\,.$$

Setting $w_\varepsilon = \phi_\varepsilon(w^*)$, we have for any $\varphi \in H_0^1(\Omega)$ with $\varphi \ge 0$ on Ω,

$$\begin{aligned}\int_\Omega \nabla w_\varepsilon \nabla \varphi &= \int_\Omega \phi_\varepsilon'(w^*) \nabla w^* \nabla \varphi = \int_\Omega \nabla w^* \nabla\left(\phi_\varepsilon'(w^*)\varphi\right) - \int_\Omega \phi_\varepsilon''(w^*)\varphi |\nabla w^*|^2 \\ &\ge \int_\Omega \frac{\lambda^* f(x)}{(1-w^*)^2} \phi_\varepsilon'(w^*)\varphi = \int_\Omega \Big(\frac{1}{(1-w_\varepsilon)^2} - \varepsilon\Big)\lambda^* \varphi f(x)\,,\end{aligned}$$

which gives (2.30) for any $\varepsilon \in (0, \varepsilon_0)$.

In order to prove (2.31), we set

$$\eta(x) = \min\{w^*(x), (C_0 + c_0)\delta(x)\} \quad \text{and} \quad \eta_\varepsilon = \phi_\varepsilon \circ \eta\,,$$

where $\phi_\varepsilon(\cdot)$ is defined above, and C_0 and c_0 are as in (2.26). Since $\eta \le w^*$ and ϕ_ε is increasing, we have $\eta_\varepsilon \le \phi_\varepsilon(w^*) = w_\varepsilon$. Applying (2.26) we get that

$$0 \le \eta(x) - c_0\delta(x) \text{ on } \Omega\,. \tag{2.34}$$

We also note that $\eta_\varepsilon = \phi_\varepsilon(\eta) \le \eta \le M$ with $M = (C_0 + c_0)\delta(x)$, and $\phi_\varepsilon'(s) \to 1$ as $\varepsilon \to 0$ uniformly in $[0,1]$. Therefore, for some $\theta \in (0,1)$ we have

$$\begin{aligned}\eta - \eta_\varepsilon = \eta - \big(\phi_\varepsilon(\eta) - \phi_\varepsilon(0)\big) &= \eta\big(1 - \phi_\varepsilon'(\theta\eta)\big) \le \eta \sup_{\{0 \le s \le 1\}} (1 - \phi_\varepsilon'(s)) \\ &\le (C_0 + c_0)\delta \sup_{\{0 \le s \le 1\}} (1 - \phi_\varepsilon'(s)) \le \frac{c_0}{2}\delta\end{aligned}$$

provided ε small enough, which gives

$$\eta \le \eta_\varepsilon + \frac{c_0}{2}\delta\,. \tag{2.35}$$

We now conclude from (2.34) and (2.35) that

$$0 \le \eta - c_0\delta \le \eta_\varepsilon - \frac{c_0}{2}\delta \le w_\varepsilon - \frac{c_0}{2}\delta$$

for small $\varepsilon > 0$, and (2.31) is therefore proved.

To complete the proof of Theorem 2.5, we assume that $t^* < \infty$ and we shall work towards a contradiction. In view of Claim 3), we let $\varepsilon > 0$ be small enough so that $0 \le w_\varepsilon - \frac{c_0}{2}\delta$. Use Lemma 2.6 and choose $K > 2$ large enough such that the solution Z of the problem

$$\begin{aligned} Z_t - \Delta Z &= -\varepsilon \lambda^* f(x) && \text{in} \quad \Omega \times (0, t^*)\,, \\ Z(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0, t^*)\,, \\ Z(x,0) &= \frac{c_0}{K}\delta && \text{in} \quad \Omega \end{aligned}$$

satisfies $0 \le Z < 1 - u^*$ on $\bar{\Omega} \times (0, t^*)$. Let v be the solution of

$$\begin{aligned} v_t - \Delta v &= \big(\frac{1}{(1-|v|)^2} - \varepsilon\big)\lambda^* f(x) && \text{in} \quad \Omega \times (0, s^*)\,, \\ v(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0, s^*)\,, \\ v(x,0) &= w_\varepsilon && \text{in} \quad \Omega\,, \end{aligned}$$

where $[0, s^*)$ is the maximal interval of existence for v. Setting $z(x,t) = Z(x,t) + u^*(x,t)$ for $0 \le t < t^*$, we then have $0 \le u^* \le z < 1$ and

$$\begin{aligned} z_t - \Delta z = \Big(\frac{1}{(1-u^*)^2} - \varepsilon\Big)\lambda^* f(x) &\le \Big(\frac{1}{(1-z)^2} - \varepsilon\Big)\lambda^* f(x) && \text{in} \quad \Omega \times (0, t^*)\,, \\ z(x,t) &= 0 && \text{on} \quad \partial\Omega \times (0, t^*)\,, \\ z(x,0) &= \frac{c_0}{K}\delta(x) \le w_\varepsilon(x) && \text{in} \quad \Omega\,. \end{aligned}$$

Now the maximum principle gives that $z \le v$ on $\Omega \times (0, \min\{s^*, t^*\})$, and in particular we have $0 \le v$ on $\Omega \times (0, \min\{s^*, t^*\})$. Furthermore, the maximum principle and (2.30) also yield that $v \le w_\varepsilon$. Since $\|w_\varepsilon\|_\infty < 1$ we necessarily have $t^* < s^* = \infty$. Therefore, $u^* \le z \le v \le w_\varepsilon$ on $[0, t^*)$, which implies that $\|u^*\|_\infty < 1$ at $t = t^*$, which contradicts to our initial assumption that u^* is not a regular solution. ∎

3 The location of touchdown points

We first present a couple of numerical simulations for different domains, different permittivity profiles, and various values of λ, by applying an implicit Crank-Nicholson scheme (see [13] for details), on the problem

$$\frac{\partial u}{\partial t} = \Delta u - \frac{\lambda f(x)}{(1+u)^2} \quad \text{for} \quad x \in \Omega\,, \tag{3.1a}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega\,, \tag{3.1b}$$

$$u(x,0) = 0 \quad \text{for} \quad x \in \Omega\,, \tag{3.1c}$$

in the following two choices for the domain Ω

$$\Omega : [-1/2, 1/2] \quad (\text{Slab})\,; \qquad \Omega : x^2 + y^2 \le 1 \quad (\text{Unit Disk})\,. \tag{3.2}$$

Simulation 1: We consider $f(x) = |2x|$ for a permittivity profile in the slab domain $-1/2 \le x \le 1/2$. Here the number of the meshpoints is chosen as $N = 2000$ for the plots u versus x at different times. Fig. 2(a) shows, for $\lambda = 4.38$, a typical sequence of solutions u for (3.1) approaching to the maximal negative steady-state. In Fig. 2(b) we take $\lambda = 4.50$, and a touchdown behavior is observed at two different nonzero points $x = \pm 0.14132$. These numerical results and Theorem 1.1 point to a pull-in voltage $4.38 \le \lambda^* < 4.50$.

Simulation 2: Here we consider $f(r) = r$ for a permittivity profile in the unit disk domain. The number of meshpoints is again chosen to be $N = 2000$ for the plots u versus r at different times. Fig. 3(a) shows how for $\lambda = 1.70$, a typical sequence of solutions u for (3.1) approach to the maximal negative steady-state. In Fig. 3(b) we take $\lambda = 1.80$ and a touchdown behavior is observed at the nonzero points $r = 0.21361$. Again these numerical results point to a pull-in voltage $1.70 \le \lambda^* < 1.80$.

One can note that touchdown points at finite time are not the zero points of the varying permittivity profile f, a fact already observed and conjectured in [13]. Here we give a proof for this interesting phenomenon also stated in Theorem 1.2 of the introduction.

Theorem 3.1. *Let T be the first touchdown time for a solution $u(x,t)$ of* (1.1)*. If T is finite, then $u_t > 0$ for all $0 < t < T$. Moreover, if K is an isolated set of touchdown points in Ω, then necessarily $\inf_{x \in K} f(x) > 0$. On the other hand,* (1.1) *can have solutions that touchdown in infinite time at points $x \in \Omega$ where $f(x) = 0$.*

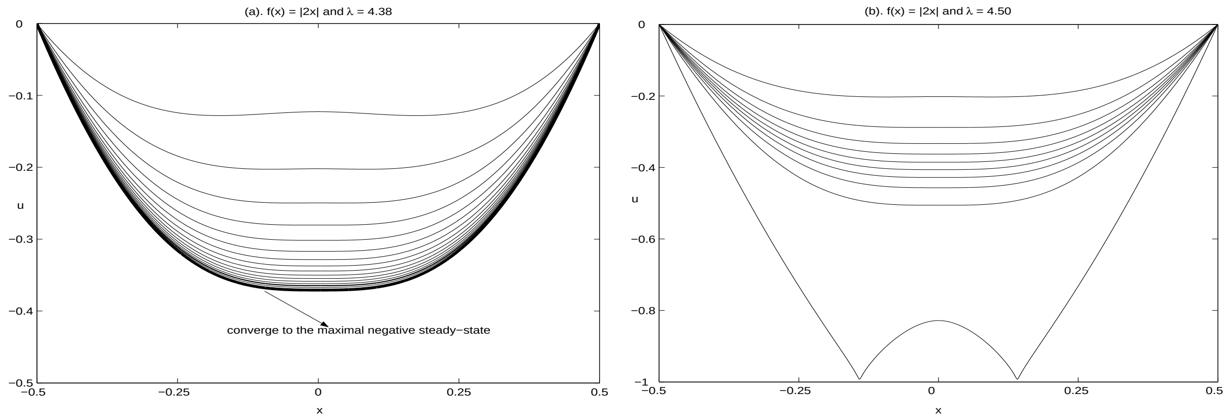

Figure 2: *Left figure: for* $\lambda = 4.38$ *we plot* u *versus* x *at different times showing the approach to the maximal negative steady-state. Right figure: for* $\lambda = 4.50$ *we plot* u *versus* x *at different times* $t = 0$, 0.1880, 0.3760, 0.5639, 0.7519, 0.9399, 1.1279, 1.3159, 1.5039, 1.6918, 1.879818, *from which touchdown is observed at* $x = \pm 0.14132$. *For both cases, we consider* (3.1) *with* $f(x) = |2x|$ *in the slab domain.*

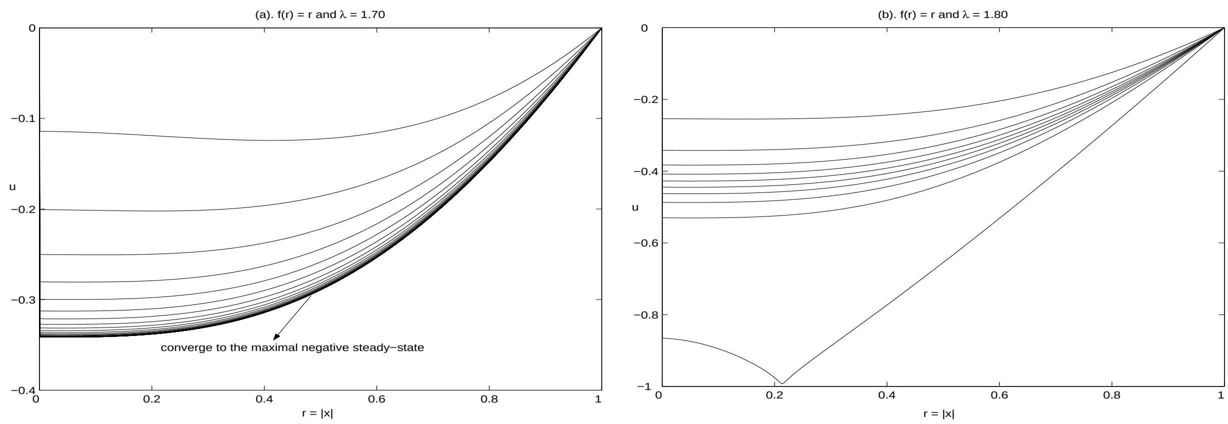

Figure 3: *Left figure: for* $\lambda = 1.70$ *we plot* u *versus* r *at different times showing the approach to the maximal negative steady-state. Right figure: for* $\lambda = 1.80$ *we plot* u *versus* x *at different times* $t = 0$, 0.4475, 0.8950, 1.3426, 1.7901, 2.2376, 2.6851, 3.1326, 3.5802, 4.0277, 4.4751942, *from which touchdown is observed at* $r = 0.21361$. *For both cases, we consider* (3.1) *with* $f(r) = r$ *in the unit disk domain.*

The proof of Theorem 3.1 is based on the following Harnack-type estimate.

Lemma 3.2. *For any compact subset K of Ω and any $m > 0$, there exists a constant $C = C(K, m) > 0$ such that $\sup_{x\in K} |u(x)| \le C < 1$ whenever u satisfies*

$$\begin{aligned} \Delta u &\ge \frac{m}{(1-u)^2} \qquad x \in \Omega, \\ 0 \le u &< 1 \qquad\quad x \in \Omega. \end{aligned} \tag{3.3}$$

Proof: Setting $v = 1/(1-u)$, then (3.3) gives that v satisfies

$$\frac{\Delta v}{v^2} - \frac{2|\nabla v|^2}{v^3} \ge mv^2 \quad \text{in } \Omega\,,$$

which means that v is a subsolution of the "linear" equation $\Delta v = 0$ in Ω. In order to apply the Harnack inequality on v, we need to show that for balls $B_r \subset \Omega$, we have that $v \in L^3(B_r)$ with an L^3-norm that only depends on m and the radius r.

Without loss of generality, we may assume $0 \in K \subset \overline{\Omega}$. Let $B_r = B_r(0) \subset K$ be the ball centered at $x = 0$ and radius r. For $0 < r_1 < r_2 \le 4r_1$, let $\eta(x) \in C_0^\infty(B_{r_2})$ be such that $\eta \equiv 1$ in B_{r_1}, $0 \le \eta \le 1$ in $B_{r_2} \setminus B_{r_1}$ and $|\nabla\eta| \le 2/(r_2 - r_1)$. Multiplying (3.3) by $\phi^2/(1-u)$, where $\phi = \eta^\alpha$ and $\alpha \ge 1$ is to be determined later, and integrating by parts we have

$$\int_{B_{r_2}} \frac{m\phi^2}{(1-u)^3} \le \int_{B_{r_2}} \frac{\phi^2\Delta u}{1-u} = -\int_{B_{r_2}} \frac{\phi^2|\nabla u|^2}{(1-u)^2} - \int_{B_{r_2}} \frac{2\phi\nabla\phi\cdot\nabla u}{1-u}\,. \tag{3.4}$$

From the fact,

$$\int_{B_{r_2}} \frac{2\phi\nabla\phi\cdot\nabla u}{1-u} \le \int_{B_{r_2}} \phi^2|\nabla u|^2 + 4\int_{B_{r_2}} \frac{|\nabla\phi|^2}{(1-u)^2} \le \int_{B_{r_2}} \frac{\phi^2|\nabla u|^2}{(1-u)^2} + 4\int_{B_{r_2}} \frac{|\nabla\phi|^2}{(1-u)^2}\,,$$

(3.4) gives that

$$\int_{B_{r_2}} \frac{m\phi^2}{(1-u)^3} \le 4\int_{B_{r_2}} \frac{|\nabla\phi|^2}{(1-u)^2}\,.$$

Now choose $\phi = \eta^{2\beta}$ with $\beta = \frac{3}{2}$. Then Hölder's inequality implies that

$$m\int_{B_{r_2}} \frac{\eta^{4\beta}}{(1-u)^3} \le 16\beta^2\Big[\int_{B_{r_2}} |\nabla\eta|^{4\beta}\Big]^{\frac{1}{2\beta}}\Big[\int_{B_{r_2}} \frac{\eta^{4\beta}}{(1-u)^3}\Big]^{\frac{2\beta-1}{2\beta}}\,.$$

This shows that

$$\int_{B_{r_1}} \frac{1}{(1-u)^3} \le \int_{B_{r_2}} \frac{\eta^{4\beta}}{(1-u)^3} < C(m, r_1). \tag{3.5}$$

By virtue of the one-sided Harnack inequality, we have

$$\Big\| \frac{1}{1-u} \Big\|_{L^\infty(B_{\frac{r_1}{2}})} = \| v \|_{L^\infty(B_{\frac{r_1}{2}})} \le C(r_1) \| v \|_{L^3(B_{r_1})} < C(r_1, m)\,.$$

The rest follows from a standard compactness argument. ■

Proof of Theorem 3.1: Proof: Set $v = u_t$, then we have for any $t_1 < T$ that

$$v_t = \Delta v + \frac{2\lambda f(x)}{(1-u)^3}v \quad (x,t) \in \Omega \times (0, t_1)\,; \tag{3.6a}$$

$$v(x,t) = 0 \text{ for } (x,t) \in \partial\Omega \times (0,t_1) \quad \text{and} \quad v(x,0) \geq 0 \text{ for } x \in \Omega. \tag{3.6b}$$

Note that the term $\frac{2\lambda f}{(1-u)^3}$ is locally bounded in $\Omega \times (0,t_1)$, so that by the strong maximum principle, we may conclude

$$u_t = v > 0 \text{ for } (x,t) \in \Omega \times (0,t_1) \tag{3.7}$$

and therefore, $u_t > 0$ holds for all $(x,t) \in \Omega \times (0,T)$. Since K is an isolated set of touchdown points, there exists an open set U such that $K \subset U \subset \bar{U} \subset \Omega$ with no touchdown points in $\bar{U} \setminus K$. Consider now $0 < t_0 < T$ such that $\inf_{x\in\bar{U}} u_t(x,t_0) = C_1 > 0$. We claim that there exists $\varepsilon > 0$ such that

$$J^\epsilon(x,t) = u_t - \tfrac{\varepsilon}{(1-u)^2} \geq 0 \text{ for all } (x,t) \in U \times (t_0,T), \tag{3.8}$$

Indeed, there exists $C_2 > 0$ such that $u_t(x,T) \geq C_2 > 0$ on U, and since ∂U has no touchdown points, there exists $\varepsilon > 0$ such that $J^\epsilon \geq 0$ on the parabolic boundary of $U \times (t_0,T)$. Also, direct calculations imply that

$$J^\epsilon_t - \Delta J^\epsilon = \frac{2\lambda f}{(1-u)^3} J^\epsilon + \frac{6\varepsilon|\nabla u|^2}{(1-u)^4} \geq \frac{2\lambda f}{(1-u)^3} J^\epsilon \,.$$

Since $\frac{\varepsilon}{(1-u)^2}$ is locally bounded on $U \times (t_0,T)$, we can apply the maximum principle to obtain (3.8).

If now $\inf_{x\in K} f(x) = 0$, then we may combine (3.8) and (1.1), to deduce that for a small neighborhood $B \subset U$ of some point $x_0 \in K$ where $f(x) \leq \varepsilon/2$, we have

$$\Delta u \geq \frac{\varepsilon}{2}\frac{1}{(1-u)^2} \quad \text{for } (x,t) \in B \times (t_0,T)\,.$$

In view of Lemma 3.2, this contradicts to the assumption that x_0 is a touchdown point.

For the second part, recall from Theorem (B) stated in the introduction that the unique extremal solution for the stationary problem on the ball in the case $N \geq 8$ and for a permittivity profile $f(x) = |x|^\alpha$, is $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ as long as α is small enough. Theorem 2.5 then implies that the origin 0 is a touchdown point of the solution even though it is also a root for the permittivity profile (i.e., $f(0) = 0$). This complements the statement of Theorem 3.1 above. In other words, zero points of f in Ω cannot be on the isolated set of touchdown points in finite time (which occur when $\lambda > \lambda^*$) but can very well be touchdown points in infinite time of (1.1), which can only happen when $\lambda = \lambda^*$. The proof of Theorem 3.1 fails for touchdowns in infinite time, simply because the maximum principle cannot be applied in the infinite cylinder $\Omega \times (0,\infty)$. ■

4 Estimates for Finite Touchdown Times

In this section we give comparaison results and explicit estimates on finite touchdown times of dynamic deflections $u = u(x,t)$ whenever $\lambda > \lambda^*$. This often translates into useful information concerning the speed of the operation for many MEMS devices such as RF switches or micro-valves.

4.1 Comparison results for finite touchdown time

We start by comparing the effect on the finite touchdown time of two different but comparable permittivity profiles $f(x)$, at a given voltage λ.

Theorem 4.1. *Suppose $u_1 = u_1(x,t)$ (resp., $u_2 = u_2(x,t)$) is a touchdown solution for (1.1) associated to a fixed voltage λ and permittivity profiles f_1 (resp., f_2) with a corresponding finite touchdown time $T_\lambda(\Omega, f_1)$ (resp., $T_\lambda(\Omega, f_2)$). If $f_1(x) \geq f_2(x)$ on Ω and if $f_1(x) > f_2(x)$ on a set of positive measure, then necessarily $T_\lambda(\Omega, f_1) < T_\lambda(\Omega, f_2)$.*

Proof: By making a change of variable $v = 1 - u$, we can assume to be working with solutions of the following equation:

$$\frac{\partial v}{\partial t} = \Delta v - \frac{\lambda f(x)}{v^2} \quad \text{for} \quad x \in \Omega\,, \tag{4.1a}$$

$$v(x,t) = 1 \quad \text{for} \quad x \in \partial\Omega\,, \tag{4.1b}$$

$$v(x,0) = 1 \quad \text{for} \quad x \in \Omega\,, \tag{4.1c}$$

where f is either f_1 or f_2. Suppose now that $T_\lambda(\Omega, f_1) > T_\lambda(\Omega, f_2)$ and let $\Omega_0 \subset \Omega$ be the set of touchdown points of u_2 at finite time $T_\lambda(\Omega, f_2)$. Setting $w = u_2 - u_1$, we get that

$$w_t - \Delta w - \frac{\lambda(f_2 u_1 + f_1 u_2)}{u_1^2 u_2^2} w = \frac{\lambda(f_1 - f_2)}{u_1 u_2} \geq 0 \quad (x,t) \in \Omega \times (0, T_\lambda(\Omega, f_2))\,. \tag{4.2}$$

Since $w = 0$ at $t = 0$ as well as on $\partial\Omega \times (0, T_\lambda(\Omega, f_2))$, we get from the maximum principle that w cannot attain a negative minimum in $\Omega \times (0, T_\lambda(\Omega, f_2))$, and therefore $w \geq 0$ in $\Omega \times (0, T_\lambda(\Omega, f_2))$. Since $u_2 \to 0$ in Ω_0 as $t \to T_\lambda(\Omega, f_2)$, and since our assumption is that $T_\lambda(\Omega, f_1) > T_\lambda(\Omega, f_2)$, we then have $u_1 > 0$ in Ω_0 as $t \to T_\lambda(\Omega, f_2)$. Therefore, $w < 0$ in Ω_0 as $t \to T_\lambda(\Omega, f_2)$, which is a contradiction and therefore $T_\lambda(\Omega, f_1) \leq T_\lambda(\Omega, f_2)$.

To prove the strict inequality, we note that the above proof shows that $w \geq 0$ in $\Omega \times (0, T_\lambda(\Omega, f_2))$, which once combined with (4.2) gives that

$$w_t - \Delta w \geq 0\,, \quad \mathit{in} \quad \Omega \times (t_1, T_\lambda(\Omega, f_2))\,,$$

where $t_1 > 0$ is chosen so that $w(x, t_1) \not\equiv 0$ in Ω. Now we compare w with the solution z of

$$z_t - \Delta z = 0 \quad \mathit{in} \quad \Omega \times (t_1, T_\lambda(\Omega, f_2))$$

subject to $z(x, t_1) = w(x, t_1)$ and $z(x,t) = 0$ on $\partial\Omega \times (t_1, T_\lambda(\Omega, f_2))$. Clearly, $w \geq z$ in $\Omega \times (t_1, T_\lambda(\Omega, f_2))$. On the other hand, for any $t_0 > t_1$ we have $z > 0$ in $\Omega \times (t_0, T_\lambda(\Omega, f_2))$. Consequently, $w > 0$ which means that $u_2 > u_1$ in $\Omega \times (t_0, T_\lambda(\Omega, f_2))$ and therefore $T_\lambda(\Omega, f_1) < T_\lambda(\Omega, f_2)$. ■

The second comparison result deals with different applied voltages but identical permittivity profiles.

Theorem 4.2. *Suppose $u_1 = u_1(x,t)$ (resp., $u_2 = u_2(x,t)$) is a solution for* (1.1) *associated to a voltage λ_1 (resp., λ_2) and which has a finite touchdown time $T_{\lambda_1}(\Omega, f)$ (resp., $T_{\lambda_2}(\Omega, f)$). If $\lambda_1 > \lambda_2$ then necessarily $T_{\lambda_1}(\Omega, f) < T_{\lambda_2}(\Omega, f)$.*

Proof: It is similar to the proof of Theorem 4.1, except that for $w = u_2 - u_1$, (4.2) is replaced by

$$w_t - \Delta w - \frac{\lambda_1(u_1 + u_2) f}{u_1^2 u_2^2} w = \frac{(\lambda_1 - \lambda_2) f}{u_2^2} \geq 0 \quad (x,t) \in \Omega \times (0, T)\,.$$

The details are left for the interested reader. ■

Remark 4.1. A reasoning similar to the one found in Proposition 2.5 of [9], gives some information on the dependence on the shape of the domain. Indeed, for any bounded domain Γ in $\mathbb{R}^N$ and any non-negative continuous function f on Γ, we have

$$\lambda^*(\Gamma, f) \geq \lambda^*(B_R, f^*) \text{ and } T_\lambda(\Gamma, f) \geq T_\lambda(B_R, f^*),$$

where $B_R = B_R(0)$ is the Euclidean ball in $\mathbb{R}^N$ with radius $R > 0$ and with volume $|B_R| = |\Gamma|$, where f^* is the Schwarz symmetrization of f.

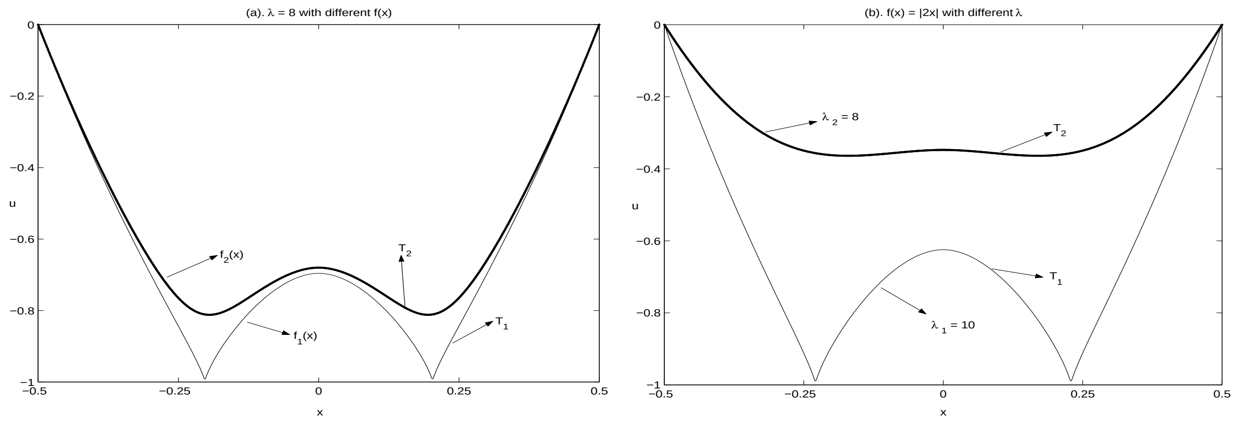

Figure 4: *Left figure: plots of u versus x with $\lambda = 8$ for different profiles $f(x)$ at the time $t = 0.185736$ in the slab domain. The finite touchdown time $T_\lambda(\Omega, f_1)$ for the case $f_1(x) = |2x|$ and $T_\lambda(\Omega, f_2)$ for the case $f_2(x)$ defined by (4.3) are 0.185736 and 0.186688, respectively. Right figure: plots of u versus x at the time $t = 0.1254864$, for $f(x) = |2x|$ with different values of λ in the slab domain. The finite touchdown time $T_{\lambda_1}(\Omega, f)$ for the case $\lambda_1 = 10$ and $T_{\lambda_2}(\Omega, f)$ for the case $\lambda_2 = 8$ are 0.1254864 and 0.185736, respectively.*

We now present numerical results comparing finite touchdown times in a slab domain.

Fig. 4(a): Dependence on the dielectric permittivity profiles f
We consider (3.1) for the cases where

$$f_1(x) = |2x| \quad \text{and} \quad f_2(x) = \begin{cases} |2x| & \text{if} \quad |x| \le \frac{1}{8}\,, \\ 1/4 + 2\sin(|x| - 1/8) & \text{otherwise.} \end{cases} \tag{4.3}$$

Using $N = 1000$ meshpoints, we plot u versus x with $\lambda = 8$ at the time $t = 0.185736$ in Fig. 4(a). The numerical results show that the finite touchdown time $T_\lambda(\Omega, f_1)$ for the case $f_1(x)$ and $T_\lambda(\Omega, f_2)$ for the case $f_2(x)$ are 0.185736 and 0.186688, respectively.

Fig. 4(b): Dependence on the applied voltage λ
Using $N = 1000$ meshpoints and the profile $f(x) = |2x|$, we plot u of (3.1) versus x with different values of λ at the time $t = 0.1254864$. The numerical results show that finite touchdown time $T_{\lambda_1}(\Omega, f)$ for applied voltage $\lambda_1 = 10$ and $T_{\lambda_2}(\Omega, f)$ for applied voltage $\lambda_2 = 8$ are 0.1254864 and 0.185736, respectively.

4.2 Explicit bounds on finite touchdown times

We now establish claims 1), 3) and 4) in Theorem 1.3 of the introduction.

Proposition 4.3. Suppose f is a non-negative continuous function on a bounded domain Ω, then,

1. For $\lambda > 0$, we have $T_\lambda(\Omega, f) \ge T_* := \frac{1}{3\lambda \sup_{x\in\Omega} f(x)}$.

2. If $\inf_\Omega f > 0$, and if $\lambda > \bar{\lambda}_1 := \frac{4\mu_\Omega}{27 \inf_{x\in\Omega} f(x)}$, then

$$T_\lambda(\Omega, f) \le T_{1,\lambda}(\Omega, f) := \int_0^1 \big[\frac{\lambda \inf_{x\in\Omega} f(x)}{(1-s)^2} - \mu_\Omega s\big]^{-1} ds. \tag{4.4}$$

3. If $f > 0$ on a set of positive measure, and if $\lambda > \bar{\lambda}_2 := \frac{\mu_\Omega}{3\int_\Omega f\phi_\Omega\, dx}$, then

$$T_\lambda(\Omega, f) \leq T_{2,\lambda}(\Omega, f) := -\frac{1}{\mu_\Omega} \log\big[1 - \frac{\mu_\Omega}{3\lambda}\big(\int_\Omega f\phi_\Omega\, dx\big)^{-1}\big]. \tag{4.5}$$

Proof: 1) Consider the initial value problem:

$$\begin{aligned} \frac{d\eta(t)}{dt} &= \frac{\lambda M}{(1-\eta(t))^2}\,, \\ \eta(0) &= 0\,, \end{aligned} \tag{4.6}$$

where $M = \sup_{x\in\Omega} f(x)$. From (4.6) one has $\frac{1}{\lambda M}\int_0^{\eta(t)}(1-s)^2 ds = t$. If T_* is the time where $\lim_{t\to T_*}\eta(t) = 1$, then we have $T_* = \frac{1}{\lambda M}\int_0^1(1-s)^2 ds = \frac{1}{3\lambda M}$. Obviously, $\eta(t)$ is now a super-function of $u(x,t)$ near touchdown, and thus we have $T \geq T_* = \frac{1}{3\lambda M} = \frac{1}{3\lambda \sup_{x\in\Omega} f(x)}$. which completes the proof of 1).

The following analytic upper bounds of finite touchdown time T were established in Theorem 3.1 and 3.2 of [13]. We sketch their easy proofs for completeness.

2) Multiplying (1.1a) by ϕ_Ω, the first normalized eigenfunction of $-\Delta$, and integrating over the domain, we obtain

$$\frac{d}{dt}\int_\Omega \phi_\Omega u\, dx = \int_\Omega \phi_\Omega \Delta u\, dx + \int_\Omega \frac{\lambda\phi_\Omega f(x)}{(1-u)^2}\, dx\,. \tag{4.7}$$

Using Green's theorem, together with the lower bound C_0 of f, we get

$$\frac{d}{dt}\int_\Omega \phi_\Omega u\, dx \geq -\mu_\Omega \int_\Omega \phi_\Omega u\, dx + \lambda C_0 \int_\Omega \frac{\phi_\Omega}{(1-u)^2}\, dx\,. \tag{4.8}$$

Next, we define an energy-like variable $E(t)$ by $E(t) = \int_\Omega \phi_\Omega u\, dx$ so that

$$E(t) = \int_\Omega \phi_\Omega u\, dx \leq \sup_\Omega u \int_\Omega \phi_\Omega\, dx = \sup_\Omega u\,. \tag{4.9}$$

Moreover, $E(0) = 0$ since $u = 0$ at $t = 0$. Then, using Jensen's inequality on the second term on the right-hand side of (4.8), we obtain

$$\frac{dE}{dt} + \mu_\Omega E \geq \frac{\lambda C_0}{(1-E)^2}\,, \qquad E(0) = 0\,. \tag{4.10}$$

We then compare $E(t)$ with the solution $F(t)$ of

$$\frac{dF}{dt} + \mu_\Omega F = \frac{\lambda C_0}{(1-F)^2}\,, \qquad F(0) = 0\,. \tag{4.11}$$

Standard comparison principles yield that $E(t) \geq F(t)$ on their domains of existence. Therefore,

$$\sup_\Omega u \geq E(t) \geq F(t)\,. \tag{4.12}$$

Next, we separate variables in (4.11) to determine t in terms of F, and it is easy to see that the touchdown time $\bar{T}_1$ for F is given by

$$\bar{T}_1 \equiv \int_0^1 \big[\frac{\lambda C_0}{(1-s)^2} - \mu_\Omega s\big]^{-1} ds\,. \tag{4.13}$$

Note that $\bar{T}_1$ is finite whenever the integral in (4.13) converges, and a simple calculation shows that this occurs whenever $\lambda > \bar{\lambda}_1 \equiv \frac{4\mu_\Omega}{27C_0}$. Moreover, if $\bar{T}_1$ is finite, then (4.12) implies that the touchdown time T of (1.1) must also be finite. It follows that when $\lambda > \bar{\lambda}_1 = \frac{4\mu_\Omega}{27C_0}$, we have

$$T_\lambda(\Omega, f) \leq \bar{T}_1 \equiv \int_0^1 \big[\frac{\lambda C_0}{(1-s)^2} - \mu_\Omega s\big]^{-1} ds\,. \tag{4.14}$$

3) Multiply now (1.1a) by $\phi_\Omega(1-u)^2$, and integrate the resulting equation over Ω to get

$$\frac{d}{dt}\int_\Omega \frac{\phi_\Omega}{3}(1-u)^3\,dx = -\int_\Omega \phi_\Omega(1-u)^2\Delta u\,dx - \int_\Omega \lambda f\phi_\Omega\,dx\,. \tag{4.15}$$

We calculate the first term on the right-hand side of (4.15) to get

$$\frac{d}{dt}\int_\Omega \frac{\phi_\Omega}{3}(1-u)^3\,dx = \int_\Omega \nabla u\cdot\nabla\left[\phi_\Omega(1-u)^2\right]\,dx + \int_{\partial\Omega}(1-u)^2\phi_\Omega\nabla u\cdot\hat{n}\,dS - \int_\Omega \lambda f\phi_\Omega\,dx \tag{4.16a}$$

$$= -\int_\Omega 2(1-u)\phi_\Omega|\nabla u|^2\,dx - \int_\Omega \frac{1}{3}\nabla\phi_\Omega\cdot\nabla\left[(1-u)^3\right]\,dx - \int_\Omega \lambda f\phi_\Omega\,dx \tag{4.16b}$$

$$\leq -\frac{1}{3}\int_{\partial\Omega}\nabla\phi_\Omega\cdot\hat{n}\,dS - \frac{\mu_\Omega}{3}\int_\Omega (1-u)^3\phi_\Omega\,dx - \int_\Omega \lambda f\phi_\Omega\,dx\,, \tag{4.16c}$$

where $\hat{n}$ is the unit outward normal to $\partial\Omega$. Since $\int_{\partial\Omega}\nabla\phi_\Omega\cdot\hat{n}\,dS = -\mu_\Omega$, we further estimate from (4.16c) that

$$\frac{dE}{dt} + \mu_\Omega E \leq R\,, \qquad R \equiv \frac{\mu_\Omega}{3} - \lambda\int_\Omega f\phi_\Omega\,dx\,, \tag{4.17}$$

where $E(t)$ is defined by

$$E(t) \equiv \frac{1}{3}\int_\Omega \phi_\Omega(1-u)^3\,dx\,, \qquad \text{with} \quad E(0) = \frac{1}{3}\,. \tag{4.18}$$

Next, we compare $E(t)$ with the solution $F(t)$ of

$$\frac{dF}{dt} + \mu_\Omega F = R\,, \qquad F(0) = \frac{1}{3}\,. \tag{4.19}$$

Again, comparison principles and the definition of E yield

$$\frac{1}{3}\inf_\Omega (1-u)^3 \leq E(t) \leq F(t)\,. \tag{4.20}$$

For $\lambda > \bar{\lambda}_2$ we have that $R<0$ in (4.17) and (4.19), and for $R<0$, we have necessarily that $F=0$ at some finite time $\bar{T}_2$ which, in view of (4.20), implies that $E=0$ at finite time. Thus, u must touchdown at some finite time $T<\bar{T}_2$. By estimating $\bar{T}_2$ explicitly we get that

$$T_\lambda(\Omega,f) \leq \bar{T}_2 \equiv -\frac{1}{\mu_\Omega}\log\left[1 - \frac{\mu_\Omega}{3\lambda}\Big(\int_\Omega f\phi_\Omega\,dx\Big)^{-1}\right]. \tag{4.21}$$

■

If now $\inf_{x\in\Omega} f(x) > 0$, we can establish the following upper estimate for the touchdown time for any $\lambda > \lambda^*$, which is claim 2) in Theorem 1.3 of the introduction.

Remark 4.2. It follows from the above that if $\lambda > \max\left\{\bar{\lambda}_1\,,\bar{\lambda}_2\right\}$, then

$$T_\lambda(\Omega,f) \leq \min\left\{T_{0,\lambda}, T_{1,\lambda}, T_{2,\lambda}\right\}. \tag{4.22}$$

where $T_{0,\lambda}$ is given by Theorem 2.2. We note that the three estimates on the touchdown times are not comparable. Indeed, it is clear that $T_{0,\lambda}$ is the better estimate when $\lambda^* < \lambda < \min\left\{\bar{\lambda}_1\,,\bar{\lambda}_2\right\}$ since $T_{1,\lambda}$ and $T_{2,\lambda}$ are not finite. On the other hand, our numerical simulations show that $T_{0,\lambda}$ can be much worse than the others, for $\lambda > \max\left\{\bar{\lambda}_1\,,\bar{\lambda}_2\right\}$.

Here are now some numerical estimates for touchdown times for several choices of the domain Ω given by (3.2) and the exponential profile $f(x)$ satisfying

$$\text{(Slab)}: \quad f(x) = e^{\alpha(x^2-1/4)} \quad \text{(exponential)}, \tag{4.23a}$$

$$\text{(Unit Disk)}: \quad f(x) = e^{\alpha(|x|^2-1)}, \quad \text{(exponential)}, \tag{4.23b}$$

where $\alpha \geq 0$. In order to choose proper applied voltage λ satisfying $\lambda > \lambda^*$, we first compute the bounds $\bar{\lambda}_1$ and $\bar{\lambda}_2$. This requires to calculate the smallest eigenpair μ_Ω and ϕ_Ω of $-\Delta$, normalized by $\int_\Omega \phi_\Omega \, dx = 1$, for either of the domains. A simple calculation yields that

$$\mu_\Omega = \pi^2, \qquad \phi_\Omega = \frac{\pi}{2} \sin\left[\pi\left(x + \frac{1}{2}\right)\right], \quad \text{(Slab)}, \tag{4.24a}$$

$$\mu_\Omega = z_0^2 \approx 5.783, \qquad \phi_\Omega = \frac{z_0}{J_1(z_0)} J_0(z_0|x|), \quad \text{(Unit Disk)}. \tag{4.24b}$$

Here J_0 and J_1 are Bessel functions of the first kind, and $z_0 \approx 2.4048$ is the first zero of $J_0(z)$. The bounds $\bar{\lambda}_1$ and $\bar{\lambda}_2$ can be evaluated by substituting (4.24) into (1.2). Notice that $\bar{\lambda}_2$ is, in general, determined only up to a numerical quadrature.

[htb]

Ω	α	$\underline{\lambda}$	λ^*	$\bar{\lambda}_1$	$\bar{\lambda}_2$
(Slab)	0	1.185	1.401	1.462	3.290
(Slab)	1.0	1.185	1.733	1.878	4.023
(Slab)	3.0	1.185	2.637	3.095	5.965
(Slab)	6.0	1.185	4.848	6.553	10.50
(Unit Disk)	0	0.593	0.789	0.857	1.928
(Unit Disk)	0.5	0.593	1.153	1.413	2.706
(Unit Disk)	1.0	0.593	1.661	2.329	3.746
(Unit Disk)	3.0	0.593	6.091	17.21	11.86

Table 1: *Numerical values for pull-in voltage λ^* with the bounds $\underline{\lambda}$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ given in Theorem A. Here the exponential permittivity profile is chosen as (4.23).*

[htb]

Ω	α	T_*	T	$T_{0,\lambda}$	$T_{1,\lambda}$	$T_{2,\lambda}$
(Slab)	0	1/60	0.01668	0.2555	0.0175	0.01825
(Slab)	1.0	1/60	0.02096	$\leq$ 0.3383	0.0229	0.02275
(Slab)	3.0	1/60	0.03239	$\leq$ 0.6121	0.0395	0.03588
(Slab)	6.0	1/60	0.06312	$\leq$ 1.7033	0.0973	0.07544
(Unit Disk)	0	1/60	0.01667	0.2420	0.0172	0.01745
(Unit Disk)	0.5	1/60	0.02241	$\leq$ 0.4103	0.0289	0.02507
(Unit Disk)	1.0	1/60	0.02927	$\leq$ 0.7123	0.0492	0.03579
(Unit Disk)	3.0	1/60	0.09563	$\leq$ 8.9847	1.1614	0.15544

Table 2: *Numerical values for finite touchdown time T with the bounds T_*, $T_{0,\lambda}$, $T_{1,\lambda}$ and $T_{2,\lambda}$ given in Proposition 3.3. Here applied voltage $\lambda = 20$ and the exponential permittivity profile is chosen as (4.23).*

In Table 1 we give numerical results for the saddle-node value λ^* with the bounds $\underline{\lambda}$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ given in Theorem A, for the exponential permittivity profile chosen as (4.23). These numerical results and Fig. 5 in

[htb]

Ω	$T(\lambda = 5)$	$T(\lambda = 10)$	$T(\lambda = 15)$	$(\lambda = 20)$
(Slab)	0.07495	0.03403	0.02239	0.01668
(Unit Disk)	0.06699	0.03342	0.02235	0.01667

Table 3: *Numerical values for finite touchdown time T at different applied voltages $\lambda = 5$, 10, 15 and 20, respectively. Here the constant permittivity profile $f(x) \equiv 1$ is chosen.*

[13] show that the pull-in voltage λ^* is seen to increase with α. Therefore, by increasing α, or equivalently by increasing the spatial extent where $f(x) \ll 1$, one can increase the stable operating range of the MEMS capacitor. From Table 1 we also observe that the bound $\bar{\lambda}_1$ for λ^* is better than $\bar{\lambda}_2$ just for small values of α. For $\alpha \gg 1$, we use Laplace's method on the integral defining $\bar{\lambda}_2$, to obtain for this exponential permittivity profile that

$$\bar{\lambda}_1 \sim \frac{4b_1}{27} e^{c_1 \alpha}\,, \qquad \bar{\lambda}_2 \sim c_2 \alpha^2\,. \tag{4.25}$$

Here $b_1 = \pi^2$, $c_1 = 1/4$, $c_2 = 1/3$ for the slab domain, and $b_1 = z_0^2$, $c_1 = 1$, $c_2 = 4/3$ for the unit disk, where z_0 is the first zero of $J_0(z)$. Therefore, for $\alpha \gg 1$ the bound $\bar{\lambda}_2$ is better than $\bar{\lambda}_1$.

Following the numerical results of Table 1, we can compute in Table 2 the values of finite touchdown time T at $\lambda = 20$, with the bounds T_*, $T_{0,\lambda}$, $T_{1,\lambda}$ and $T_{2,\lambda}$ given in Theorem 2.2 and Proposition 3.3. Using the meshpoints $N = 800$ we compute finite touchdown time T with error less than 0.00001. The numerical results in Table 2 show that the bounds $T_{1,\lambda}$ and $T_{2,\lambda}$ for T are much better than $T_{0,\lambda}$. Further the bound $T_{1,\lambda}$ is better than $T_{2,\lambda}$ for smaller values of α, and however the bound $T_{2,\lambda}$ is better than $T_{1,\lambda}$ for larger values of α. In fact, for $\alpha \gg 1$ and λ large enough we can deduce from (4.25) that

$$T_{1,\lambda} \sim \frac{1}{3\lambda} e^{d_1 \alpha}\,, \qquad T_{2,\lambda} \sim \frac{d_2}{\lambda} \alpha^2\,.$$

Here $d_1 = 1/4$, $d_2 = 1/3\pi^2$ for the slab domain, and $d_1 = 1$, $d_2 = 4/3z_0^2$ for the unit disk, where z_0 is the first zero of $J_0(z) = 0$. Therefore, for $\alpha \gg 1$ and fixed λ large enough, the bound $T_{2,\lambda}$ is better than $T_{1,\lambda}$. Table 2 also shows that for fixed applied voltage λ, the touchdown time is seen to increase once α is increased or equivalently the spatial extent where $f(x) \ll 1$ is increased. However, Theorem 3.2 tells us that for fixed permittivity profile f, by increasing the applied voltage λ within the available power supply, the touchdown time can be decreased and consequently the operating speed of MEMS devices can be improved. In Table 3 we give numerical values for finite touchdown time T with error less than 0.00001, at different applied voltages $\lambda = 5$, 10, 15 and 20, respectively. Here the constant permittivity profile $f(x) \equiv 1$ is chosen and the meshpoints $N = 800$ again.

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