A variational principle for nonlinear transport equations

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Abstract

We verify -after appropriate modifications- an old conjecture of Brezis-Ekeland [3] concerning the feasibility of a global and variational approach to the problems of existence and uniqueness of solutions of non-linear transport equations, which do not normally fit in an Euler-Lagrange framework. Our method is based on a concept of "anti-self duality" that seems to be inherent in many problems, including gradient flows of convex energy functionals treated in [9] and other parabolic evolution equations ([6] and ([10]).

1 Introduction

Second order boundary value problems involving self-adjoint operators have often been connected to variational principles since they often arise as Euler-Lagrange equations associated to certain energy or action functionals. In 1976, Brezis and Ekeland formulated in [3] an intriguing minimization principle which can be associated to gradient flows of convex energy functionals as well as to transport equations. However, they could not use it to establish existence of solutions for associated equations because the method required the identification of the infimum, which they could not establish.

In [9] we offered a variant of the Brezis-Ekeland principle which gave an alternate variational proof of the existence and uniqueness of gradient flows of convex energy functionals. The semi-convex case was dealt with in [8]. In this paper, we again modify the Brezis-Ekeland method to provide a complete variational proof for the existence and uniqueness of solutions of certain non-linear transport equations. In a forthcoming paper ([6], we develop a general framework for a far-reaching variational approach to many equations which do not normally fit into the standard Euler-Lagrange theory. This approach is based on the concept of *anti-self dual Lagrangians* which seems to be inherent in many important differential equations.

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2 A new Lagrangian associated to the transport equation

Let $\mathbf{a} : \Omega \to \mathbf{R^n}$ and $a_0 : \Omega \to \mathbf{R}$ be two smooth functions on a bounded domain Ω of $\mathbf{R^n}$, and consider the first order linear operator

$$Av = \mathbf{a} \cdot \nabla v = \Sigma_{i=1}^n a_i \frac{\partial v}{\partial x_i} \quad \text{and} \quad \Lambda v = \mathbf{a} \cdot \nabla v + a_0 v.$$

As in [2], we shall assume throughout that the vector field $\Sigma_{i=1}^n a_i \frac{\partial v}{\partial x_i}$ is actually the restriction of a smooth vector field $\Sigma_{i=1}^n \bar{a}_i \frac{\partial v}{\partial x_i}$ defined on an open neighborhood X of $\bar{\Omega}$ and that each $\bar{a}_i$ is a $C^{1,1}$ function on X. We also assume that the boundary of Ω is piecewise C^1, in such a way that the outer normal $\mathbf{n}$ is defined almost everywhere on $\partial\Omega$. In this case, if we denote by

$$\Sigma_- = \{x \in \partial\Omega;\, \mathbf{n}(x) \cdot \mathbf{a}(x) < 0\} \quad \text{and } \Sigma_+ = \partial\Omega \setminus \Sigma_- = \{x \in \partial\Omega;\, \mathbf{n}(x) \cdot \mathbf{a}(x) \geq 0\},$$

then a trace $u_{|\Sigma_-}$ makes sense in $L^2_{\text{loc}}(\Sigma_-)$ as soon as $u \in L^2(\Omega)$ and $\Lambda u \in L^2(\Omega)$.

Let now $\beta : \mathbf{R} \to \mathbf{R}$ be a continuous nondecreasing function convex, and let $f \in L^2(\Omega)$. We are interested in finding variationally solutions for the nonlinear transport equation:

$$\begin{cases} -\Lambda u &= \beta(u) + f \ \text{ on } \ \Omega \\ u(x) &= 0 \qquad\quad \text{ on } \ \Sigma_-. \end{cases} \tag{1}$$

under the following coercivity condition:

$$a_0(x) - \tfrac{1}{2}\text{div} a(x) \geq \alpha > 0 \text{ on } \Omega. \tag{2}$$

In [3], Brezis and Ekeland propose the following variational principle to solve (1): Let j be an antiderivative of β, and consider the functional

$$I_{\text{BE}}(u) = \int_\Omega \left[j(u) + j^*(f - \Lambda u) - fu + (a_0 - \frac{1}{2}\text{div a})u^2 \right] dx + \frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n} \cdot \mathbf{a}\, d\sigma \tag{3}$$

on the set

$$K = \{v \in L^2(\Omega);\, \Lambda v \in L^2(\Omega), j(v) \,\text{and}\, j^*(f - \Lambda v) \in L^1(\Omega) \ \text{ with } v = 0 \text{ on } \Sigma_-\}.$$

They argued that if

$$\inf_{v \in K} I_{\text{BE}}(v) = I_{\text{BE}}(\bar{v}) = 0, \tag{4}$$

Then $\bar{v}$ is a solution of (1). However, they could neither show that the infimum is attained nor that it is zero, which was an unfortunate impediment to the use of this approach for establishing existence and uniqueness results.

In this note, we propose a variation of the Brezis-Ekeland functional, which will remedy the situation and which will allow us to establish variationally, existence and uniqueness of solutions for non-linear transport equations. First, we identify the appropriate underlying space. Consider first

$$H^1(\Omega) = \{u \in L^2(\Omega);\, Au \in L^2(\Omega)\}.$$

equipped with the norm $\|u\|_{H^1} = \|u\|_2 + \|Au\|_2$. As noticed in [2], that the fact that a function u is in $H^1(\Omega)$ does not necessarily guarantee that $u_{|\Sigma_-}$ is in the space

$$L^2_A(\Sigma_-) = \left\{ u \in L^2_{\mathrm{loc}}(\Sigma_-); \int_{\Sigma_-} |u(x)|^2 |\mathbf{n}(x) \cdot \mathbf{a}(x)| d\sigma < +\infty \right\}.$$

However, if $u \in H^1(\Omega)$ and $u_{\Sigma_-} \in L^2_A(\Sigma_-)$, then necessarily $u_{\Sigma_+} \in L^2_A(\Sigma_+)$. The appropriate space for our setting is therefore

$$H^1_A(\Omega) = \{u \in H^1(\Omega); u_{|\Sigma_-} \in L^2_A(\Sigma_-)\}.$$

equipped with the norm $\|u\|_{H^1_A} = \|u\|_2 + \|Au\|_2 + \|u_{|\Sigma_-}\|_{L^2_A(\Sigma_-)}$.

To define appropriate boundary spaces, we follow [4] and consider for each open subset Γ of $\partial\Omega$, the space

$$H^{1/2}_{00}(\Gamma) = \left\{ v \in L^2_A(\Gamma);\ \exists w \in H^1(\Omega),\ w = 0 \text{ on } \partial\Omega \setminus \Gamma, \text{ and } w = v \text{ on } \Gamma \right\}$$

A *trace theorem* ([11], Vol III. p. 307) or [1]) yields that the restriction mapping $u \to u_\Gamma$ is a continuous surjective map from $V = \{v \in H^1_A(\Omega); v_{|\partial\Omega\setminus\Gamma} = 0\}$ onto $H^{1/2}_{00}(\Gamma)$. It follows that there is a continuous surjection from $H^1_A(\Omega)$ onto $H^{1,0}_A(\Omega) \oplus H^{1/2}_{00}(\Sigma_-) \oplus H^{1/2}_{00}(\Sigma_+)$ via the map

$$T : H^1_A(\Omega) \to H^{1,0}_A(\Omega) \oplus H^{1/2}_{00}(\Sigma_-) \oplus H^{1/2}_{00}(\Sigma_+),$$

given by $Tu = (Ku, u_{|\Sigma_-}, u_{|\Sigma_+})$, where K is the projection from $H^1_A(\Omega)$ onto the kernel $H^{1,0}_A(\Omega)$ of $u \to (u_{|\Sigma_-}, u_{|\Sigma_+})$ in $H^1_A(\Omega)$. We denote by $H^{-1}_A(\Omega)$ the dual of $H^{1,0}_A(\Omega)$.

Now we consider the convex functional ψ on $L^2(\Omega)$ defined by:

$$\psi(u) = \int_\Omega \left\{ j(u(x)) + f(x)u(x) + \frac{1}{2}(a_0 - \frac{1}{2}\mathrm{div\,a})u^2) \right\} dx$$

and its conjugate ψ^* defined by

$$\psi^*(v) = \sup\{\int_\Omega uv\, dx - \psi(u); u \in H^1_A(\Omega)\}.$$

Let Λ_1 be the operator defined on $H^1_A(\Omega)$ by:

$$\Lambda_1(u) = \Sigma_{i=1}^n a_i \frac{\partial u}{\partial x_i} + \frac{1}{2}\mathrm{div}(\mathbf{a})u = \Lambda(u) - (a_0 - \frac{1}{2}\mathrm{div}\,\mathbf{a})u$$

in such a way that

$$\int_\Omega v\Lambda_1 u\, dx = -\int_\Omega u\Lambda_1 v\, dx + \int_{\partial\Omega} u(x)v(x)\mathbf{n}(x) \cdot \mathbf{a}(x) d\sigma. \tag{5}$$

Consider finally the following functional on the space $H^1_A(\Omega)$

$$I_G(u) = \psi(u) + \psi^*(-\Lambda_1 u) + \frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n} \cdot \mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(x)|^2 \mathbf{n} \cdot \mathbf{a}\, d\sigma. \tag{6}$$

We shall show the following

Theorem 2.1 *Under the above conditions, there exists $\bar{u} \in H^1_A(\Omega)$ such that*

$$I_G(\bar{u}) = \inf\{I_G(u); u \in H^1_A(\Omega)\} = 0. \tag{7}$$

and $\bar{u}$ solves

$$\begin{cases} -\Lambda u &= \beta(u) + f \text{ on } \Omega \\ u(x) &= 0 \quad\quad \text{on } \Sigma_-. \end{cases} \tag{8}$$

Proof: First we show how (7) implies (8). Indeed, since for each $u \in H^1_A(\Omega)$, we have

$$\psi(u) + \psi^*(-\Lambda_1 u) + \int_\Omega u\Lambda_1 u \geq 0, \tag{9}$$

it follows that

$$\begin{aligned} I_G(u) &= \psi(u) + \psi^*(-\Lambda_1 u) + \frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma \\ &= \psi(u) + \psi^*(-\Lambda_1 u) + \int_\Omega u\Lambda_1 u \\ &\quad - \int_\Omega u\Lambda_1 u + \frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma \\ &\geq \psi(u) + \psi^*(-\Lambda_1 u) + \int_\Omega u\Lambda_1 u \\ &\quad -\frac{1}{2}\int_{\Sigma_+\cup\Sigma_-} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma + \frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma \\ &= -\int_{\Sigma_-} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma \geq 0. \end{aligned}$$

So, if $I_G(\bar{u}) = 0$, then $\bar{u} = 0$ on Σ_- since $\mathbf{n}(x)\cdot\mathbf{a}(x) < 0$ on Σ_-. Moreover, we get that

$$\psi(\bar{u}) + \psi^*(-\Lambda_1 \bar{u}) + \int_\Omega \bar{u}\Lambda_1 \bar{u} = 0,$$

from which follows that $-\Lambda_1 \bar{u} \in \partial\psi(\bar{u})$, that is

$$-\Lambda_1 \bar{u} = \beta(\bar{u}) + f + (a_0 - \frac{1}{2}\mathrm{div}\,\mathrm{a})\bar{u}$$

which means that $\bar{u}$ solves (8).

It remains to show that the infimum in (7) is zero and that it is attained. For that, we need a few lemmas

Lemma 2.2 *Let φ be any proper convex and lower semi-continuous function on $L^2(\Omega)$, and consider the following two functionals on the space $H^1_A(\Omega)$ and on the dual space $H^{-1}_A(\Omega) \oplus H^{1/2}_{00}(\Sigma_-)^* \oplus H^{1/2}_{00}(\Sigma_+)^*$ respectively:*

$$I(u) = \varphi(u) + \varphi^*(-\Lambda_1 u) + \frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma,$$

and

$$J(v, g_1, g_2) = \inf_{u\in H^1_A(\Omega)} \left\{\varphi(u) + \varphi^*(-\Lambda_1 u - K^* v) + \frac{1}{2}\int_{\Sigma_+} |u + g_1|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u - g_2|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma\right\}.$$

Then we have:

$$J^*(Tu) = I(-u) \quad \text{for each } u \in H^1_A(\Omega), \tag{10}$$

where the Legendre-Fenchel adjoint of J is taken in the duality between the space $H^{1,0}_A(\Omega) \oplus H^{1/2}_{00}(\Sigma_-) \oplus H^{1/2}_{00}(\Sigma_+)$ and its dual $H^{-1}_A(\Omega) \oplus H^{1/2}_{00}(\Sigma_-)^* \oplus H^{1/2}_{00}(\Sigma_+)^*$.

Proof of lemma: Write

$$\begin{aligned}
J^*(Tu) \;=\; & \sup_{w\in H^1_A(\Omega)} \sup_{v\in H^{-1}_A(\Omega)} \sup_{g_1,g_2\in H^{1/2}_{00}(\Sigma_-)\times H^{1/2}_{00}(\Sigma_+)} \{\int_\Omega vKu dx + \int_{\Sigma_-} ug_1 d\sigma + \int_{\Sigma_+} ug_2 d\sigma \\
& -\varphi(w) - \varphi^*(-\Lambda_1 w - K^* v) \\
& -\frac{1}{2}\int_{\Sigma_-} |w(x)+g_1(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma + \frac{1}{2}\int_{\Sigma_+} |w(x)-g_2(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma\}.
\end{aligned}$$

Let $h_1(x) = w(x) + g_1(x)$, $h_2(x) = w(x) - g_2(x)$, $p = -\Lambda_1 w - K^* v$, in such a way that

$$\begin{aligned}
J^*(Tu) \;=\; & \sup_{w\in H^1_A(\Omega)} \sup_{p\in H^1_A(\Omega)^*} \sup_{h_1\in H^{1/2}_{00}(\Sigma_-)} \sup_{h_2\in H^{1/2}_{00}(\Sigma_+)} \{\int_\Omega u(-p-\Lambda_1 w) dx \\
& + \int_{\Sigma_-} u(h_1 - w) \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma + \int_{\Sigma_+} u(w - h_2) \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma \\
& -\varphi(w) - \varphi^*(p) \\
& -\frac{1}{2}\int_{\Sigma_-} |h_1(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma + \frac{1}{2}\int_{\Sigma_+} |h_2(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma\}. \\
=\; & \sup_{w\in H^1_A(\Omega)} \sup_{p\in H^1_A(\Omega)^*} \sup_{h_1\in H^{1/2}_{00}(\Sigma_-)} \sup_{h_2\in H^{1/2}_{00}(\Sigma_+)} \{\int_\Omega (-up + w\Lambda_1 u) dx \\
& - \int_{\Sigma_+} uw \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma + \int_{\Sigma_-} uw \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma \\
& + \int_{\Sigma_-} u(h_1 - w) \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma + \int_{\Sigma_+} u(w - h_2) \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma \\
& -\varphi(w) - \varphi^*(p) \\
& -\frac{1}{2}\int_{\Sigma_-} |h_1(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma + \frac{1}{2}\int_{\Sigma_+} |h_2(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma\}. \\
=\; & \sup_{w\in H^1_A(\Omega)} \sup_{H^1_A(\Omega)^*} \sup_{h_1\in H^{1/2}_{00}(\Sigma_-)} \sup_{h_2\in H^{1/2}_{00}(\Sigma_+)} \{-\int_\Omega up + \int_\Omega w\Lambda_1 u dx \\
& + \int_{\Sigma_-} uh_1 \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma - \int_{\Sigma_+} uh_2 \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma \\
& -\varphi(w) - \varphi^*(p) \\
& -\frac{1}{2}\int_{\Sigma_-} |h_1(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma + \frac{1}{2}\int_{\Sigma_+} |h_2(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma\}. \\
=\; & \sup_{w\in H^1_A(\Omega)} \sup_{p\in H^1_A(\Omega)^*} \{-\int_\Omega up + \int_\Omega w\Lambda_1 u dx - \varphi(w) - \varphi^*(p)\} \\
& + \sup_{h_1\in H^{1/2}_{00}(\Sigma_-)} \{\int_{\Sigma_-} uh_1 \;\; \mathbf{n}\cdot\mathbf{a} \;\; d\sigma - \frac{1}{2}\int_{\Sigma_-} |h_1(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma\}
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{h_2\in H^{1/2}_{00}(\Sigma_+)}\{-\int_{\Sigma_+} uh_2\ \mathbf{n}\cdot\mathbf{a}\ \ d\sigma+\frac{1}{2}\int_{\Sigma_+}|h_2(x)|^2\ \ \mathbf{n}\cdot\mathbf{a}\ \ d\sigma\}\\
&= \varphi(-u)+\varphi^*(\Lambda_1 u)+\frac{1}{2}\int_{\Sigma_+}|u(x)|^2\ \ \mathbf{n}\cdot\mathbf{a}\ \ d\sigma-\frac{1}{2}\int_{\Sigma_-}u(x)^2\ \ \mathbf{n}\cdot\mathbf{a}\ \ d\sigma\\
&= I(-u).
\end{aligned}
$$

Lemma 2.3 *Suppose φ is also coercive on $H^{1,0}_A(\Omega)$, that is $\lim_{\|u\|\to+\infty}\frac{\varphi(u)}{\|u\|_{H^{1,0}_A}}=+\infty$. Then, the corresponding functional J is convex and subdifferentiable at the origin on the space $H^{-1}_A(\Omega)\oplus H^{1/2}_{00}(\Sigma_-)^*\oplus H^{1/2}_{00}(\Sigma_+)^*$. Furthermore, any $p\in H^1_A$ such that $T(p)\in\partial J(0)$ satisfies*

$$I(-p)=\inf_{u\in H^1_A(\Omega)} I(u)=0$$

Proof: The convexity is standard. To prove differentiability at zero, it suffices to show that J is bounded on the balls of $H^{-1}_A(\Omega)\oplus H^{1/2}_{00}(\Sigma_-)^*\oplus H^{1/2}_{00}(\Sigma_+)^*$. Since φ is coercive on $H^{1,0}_A(\Omega)$, it follows that φ^* is bounded on the bounded sets of $H^{-1}_A(\Omega)$. It follows that

$$
\begin{aligned}
J(v,g_1,g_2) &= \inf_{u\in H^1_A(\Omega)}\{\varphi(u)+\varphi^*(-\Lambda_1 u-K^*v)\\
&\quad+\frac{1}{2}\int_{\Sigma_+}|u(x)+g_1(x)|^2\mathbf{n}(x)\cdot\mathbf{a}(x)d\sigma-\frac{1}{2}\int_{\Sigma_-}|u(x)-g_2(x)|^2\mathbf{n}(x)\cdot\mathbf{a}(x)d\sigma\\
&\le \varphi(0)+\varphi^*(-K^*v)+\frac{1}{2}\int_{\Sigma_+}|g_1(x)|^2\mathbf{n}(x)\cdot\mathbf{a}(x)d\sigma-\frac{1}{2}\int_{\Sigma_-}|g_2(x)|^2\mathbf{n}(x)\cdot\mathbf{a}(x)d\sigma.
\end{aligned}
$$

which is bounded on the ball of the form

$$\left\{(v,g_1,g_2)\in H^{-1}_A(\Omega)\oplus H^{1/2}_{00}(\Sigma_-)^*\oplus H^{1/2}_{00}(\Sigma_+)^*;\|v\|_{H^{-1}_A}\le 1,\|g_1\|_{L^2_A(\Sigma_-)}\le 1,\|g_2\|_{L^2_A(\Sigma_+)}\le 1\right\}.$$

Recall now that $\inf_{u\in H^1_A(\Omega)} I_G(u)\ge 0$. On the other hand, taking any p so that $T(p)\in\partial J(0,0,0)$ and applying Young-Fenchel duality, we obtain

$$J(0,0,0)+J^*(Tp)=0.$$

By Lemma 2.2 we have

$$-\inf_{u\in H^1_A(\Omega)} I(u)=-J(0,0,0)=J^*(Tp)=I(-p)\ge\inf_{u\in H^1_A(\Omega)} I(u).$$

In other words, $\inf_{u\in H^1_A(\Omega)} I(u)=I(-p)=0$.

End of Proof of Theorem 2.1: The only problem remaining is the fact that the convex functional ψ defined by:

$$\psi(u)=\int_\Omega\left\{j(u(x))+f(x)u(x)+\frac{1}{2}(a_0-\frac{1}{2}\mathrm{div}\,\mathrm{a})u^2)\right\}dx$$

is not necessarily coercive on $H^{1,0}_A(\Omega)$, so we consider instead for each $\epsilon>0$, the functional

$$\varphi_\epsilon(u)=\psi(u)+\frac{\epsilon}{2}\int_\Omega|\nabla u|^2dx$$

which obviously is. Setting

$$I_\epsilon(u) = \varphi_\epsilon(u) + \varphi_\epsilon^*(-\Lambda_1 u) + \frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(x)|^2 \mathbf{n}\cdot\mathbf{a}\, d\sigma. \tag{11}$$

The above lemma now applies and we get $p_\epsilon \in H^1_A(\Omega)$ such that

$$\inf_{u\in H^1_A(\Omega)} I_\epsilon(u) = I_\epsilon(-p_\epsilon) = 0,$$

As in the beginning of the proof of Theorem 2.1, this means that $u_\epsilon = -p_\epsilon$ belongs to $\mathrm{Dom}(\partial\varphi_\epsilon)$ and satisfies $-\Lambda_1 u_\epsilon \in \partial\varphi_\epsilon(u_\epsilon)$, which implies

$$-\Lambda_1 u_\epsilon = \beta(u_\epsilon) + f + (a_0 - \frac{1}{2}\mathrm{div}\,\mathrm{a})u_\epsilon - \epsilon\Delta u_\epsilon.$$

In other words, we have for each $\epsilon > 0$,

$$\left\{\begin{array}{rcll} \epsilon\Delta u_\epsilon - a_0 u_\epsilon - \Sigma_{i=1}^n a_i \frac{\partial u_\epsilon}{\partial x_i} & = & \beta(u_\epsilon) + f & \text{on} \quad \Omega \\ u_\epsilon & = & 0 & \text{on} \quad \Sigma_-, \\ \frac{\partial u_\epsilon}{\partial n} & = & 0 & \text{on} \quad \partial\Omega\setminus\Sigma_-. \end{array}\right. \tag{12}$$

It is now standard to show that, as $\epsilon \to 0$, u_ϵ converges in $L^2(\Omega)$ to a solution u of (8). For details, see Bardos [2].

3 More general transport equations

The above method applies to a more general transport equation of the following form

$$\left\{\begin{array}{lcll} -\Lambda u & = & \beta(u) + Bu + f & \text{on} \quad \Omega \\ u(x) & = & u_0(x) & \text{on} \quad \Sigma_-. \end{array}\right. \tag{13}$$

where $B : H^1_A(\Omega) \to (H^1_A(\Omega))^*$ is a positive bounded linear operator, $f \in L^2(\Omega)$ and $u_0 \in L^2_A(\Sigma_-)$.
Indeed, one first decomposes B into a symmetric and an anti-symmetric part, B_s and B_a, by simply writing $B_s = \frac{1}{2}(B + B^*)$ and $B_a = \frac{1}{2}(B - B^*)$.
Now consider the convex functional defined by:

$$\psi(u) = \frac{1}{2}\langle Bu, u\rangle + \int_\Omega (j(u(x)) + f(x)u(x) + \frac{1}{2}(a_0 - \frac{1}{2}\mathrm{div}\,\mathbf{a})u^2)dx$$

and its conjugate ψ^*. Let again Λ_1 be the operator

$$\Lambda_1(u) = \Sigma_{i=1}^n a_i \frac{\partial u}{\partial x_i} + \frac{1}{2}\mathrm{div}(\mathbf{a})u = \Lambda(u) - (a_0 - \frac{1}{2}\mathrm{div}\,\mathbf{a})u.$$

The functional on the space $H^1_A(\Omega)$ is now

$$\begin{array}{rcl} \tilde{I}(u) & = & \psi(u) + \psi^*(-\Lambda_1 u - B_a u) \\ & & +\frac{1}{2}\int_{\Sigma_+} |u(x)|^2 \mathbf{n}(x)\cdot\mathbf{a}(x)\, d\sigma - \frac{1}{2}\int_{\Sigma_-} |u(x)|^2 \mathbf{n}(x)\cdot\mathbf{a}(x)\, d\sigma \\ & & +2\int_{\Sigma_-} u(x)u_0(x)\mathbf{n}(x)\cdot\mathbf{a}(x)\, d\sigma - \int_{\Sigma_-} |u_0(x)|^2 \mathbf{n}(x)\cdot\mathbf{a}(x)\, d\sigma. \end{array}$$

It us left to the reader to show the following

Theorem 3.1 *There exists $\bar{u} \in H^1_A(\Omega)$ such that*

$$\tilde{I}(\bar{u}) = \inf\{\tilde{I}(u); u \in H^1_A(\Omega)\} = 0. \tag{14}$$

and $\bar{u}$ solves equation (13).

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