On the Partial Differential Equations of Electrostatic MEMS Devices: Stationary Case

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Abstract

We analyze the nonlinear elliptic problem $\Delta u = \frac{\lambda f(x)}{(1+u)^2}$ on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. When a voltage –represented here by λ– is applied, the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices. The mathematical model lends to a nonlinear parabolic problem for the dynamic deflection of the elastic membrane which will be considered in a forthcoming paper [15]. Here, we focus on the stationary equation and on estimates for λ^* in terms of material properties of the membrane, which can be fabricated with a spatially varying dielectric permittivity profile f. Applying analytical and numerical techniques, we establish upper and lower bounds for λ^* in terms of the volume and shape of the domain, the dimension of the ambient space, as well as the permittivity profile. We analyze the first branch of stable steady-states when $\lambda < \lambda^*$, and prove that a semi-stable (extremal) solution exists at $\lambda = \lambda^*$ in dimension $1 \leq N \leq 7$, and that classical extremal solutions may not exist for dimension $N \geq 8$. More refined properties of stable steady states –such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results– are also established. The analysis of branches of unstable solutions is more elaborate and is tackled in the companion paper [13].

Key words: MEMS; pull-in voltage; power law permittivity profile; minimal solutions.

Contents

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1 Introduction

Micro-Electromechanical Systems (MEMS) are often used to combine electronics with micro-size mechanical devices in the design of various types of microscopic machinery. MEMS devices have therefore become key components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches and chemical sensors and so on (see for example [24]). The simplicity and importance of this technique have inspired numerous researchers to study mathematical models of electrostatic-elastic interactions. The mathematical analysis of these systems started in the late 1960s with the pioneering work of H. C. Nathanson and his coworkers [22] who constructed and analyzed a mass-spring model of electrostatic actuation, and offered the first theoretical explanation of pull-in instability. At roughly the same time, G. I. Taylor [28] studied the electrostatic deflection of two oppositely charged soap films, and he predicted that when the applied voltage was increased beyond a certain critical voltage, the two soap films would touch together. Since Nathanson and Taylor's seminal work, numerous investigators have analyzed and developed mathematical models of electrostatic actuation in attempts to understand further and control pull-in instability. An overview of the physical phenomena of the mathematical models associated with the rapidly developing field of MEMS technology is given in [24].

The key component of many modern MEMS is the simple idealized electrostatic device shown in Fig. 1. The upper part of this device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage V is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate, and when V is increased beyond a certain critical value V^* –known as pull-in voltage– the steady-state of the elastic membrane is lost, and proceeds to touchdown or snap through at a finite time creating the so-called pull-in instability.

A mathematical model of the physical phenomena, leading to a partial differential equation for the dimensionless dynamic deflection of the membrane, was derived and analyzed in [14] and [17]. In the damping-dominated limit, and using a narrow-gap asymptotic analysis, the dimensionless dynamic deflection $u = u(x, t)$ of the membrane on a bounded domain Ω in $\mathbb{R}^2$, is found to satisfy the following parabolic problem

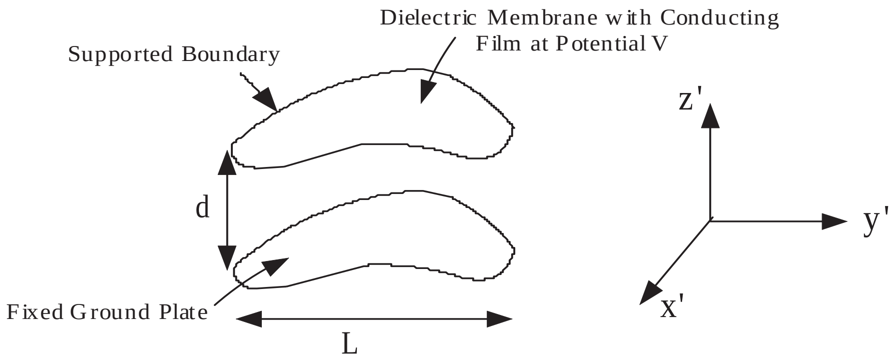

Figure 1: *The simple electrostatic MEMS device.*

$$\frac{\partial u}{\partial t} = \Delta u - \frac{\lambda f(x)}{(1+u)^2} \qquad \text{for} \quad x \in \Omega\,, \tag{1.1a}$$

$$u(x,t) = 0 \qquad \text{for} \quad x \in \partial\Omega\,, \tag{1.1b}$$

$$u(x,0) = 0 \qquad \text{for} \quad x \in \Omega\,. \tag{1.1c}$$

An outline of the derivation of (1.1) was given in Appendix A of [17]. This initial condition in (1.1c) assumes that the membrane is initially undeflected and the voltage is suddenly applied to the upper surface of the membrane at time $t = 0$. The parameter $\lambda > 0$ in (1.1a) characterizes the relative strength of the electrostatic and mechanical forces in the system, and is given in terms of the applied voltage V by

$$\lambda = \frac{\varepsilon_0 V^2 L^2}{2T_e d^3}\,, \tag{1.2}$$

where d is the undeflected gap size (see Fig. 1), L is the length scale of the membrane, T_e is the tension of the membrane, and ε_0 is the permittivity of free space in the gap between the membrane and the bottom plate. In view of relation (1.2), we shall use from now on the parameter λ and λ^* to represent the applied voltage V and pull-in voltage V^*, respectively. Referred to as the *permittivity profile*, $f(x)$ in (1.1a) is defined by the ratio

$$f(x) = \frac{\varepsilon_0}{\varepsilon_2(x)}\,, \tag{1.3}$$

where $\varepsilon_2(x)$ is the dielectric permittivity of the thin membrane.

There are several issues that must be considered in the actual design of MEMS devices. Typically one of the primary goals is to achieve the maximum possible stable deflection before touchdown occurs, which is referred to as *pull-in distance* (cf. [17] and [23]). Another consideration is to increase the stable operating range of the device by improving the pull-in voltage λ^* subject to the constraint that the range of the applied voltage is limited by the available power supply. Such improvements in the stable operating range is important for the design of certain MEMS devices such as microresonators. One way of achieving larger values of λ^*, while simultaneously increasing the pull-in distance, is to use a voltage control scheme imposed by an external circuit in which the device is placed (cf. [25]). This approach leads to a nonlocal problem for the dynamic deflection of the membrane. A different approach studied in [23] and [17] is to introduce a spatially varying dielectric permittivity $\varepsilon_2(x)$ of the membrane. The idea is to locate the region where the membrane deflection would normally be largest under a spatially uniform permittivity, and then make sure that a new dielectric permittivity $\varepsilon_2(x)$ is largest –and consequently the profile $f(x)$ smallest– in that region.

This latter approach requires the membrane to have varying dielectric properties, a framework investigated recently in [23] and [17]. In [23] J. Pelesko studied the steady-states of (1.1), when $f(x)$ is assumed to be bounded away from zero, i.e.,

$$0 < C \leq f(x) \leq 1 \quad \text{for all} \quad x \in \Omega. \tag{1.4}$$

He established in this case an upper bound for λ^*, and derived numerical results for the power-law permittivity profile, from which the larger pull-in voltage and thereby the larger pull-in distance, the existence and multiplicity of the steady-states were observed. From the strictly mathematical point of view, it turned out that –at least for $f \equiv 1$– there already exist in the literature many interesting results concerning the properties of the branch of semi-stable solutions for Dirichlet boundary value problems of the form $-\Delta u = \lambda h(u)$ where h is a regular nonlinearity (for example of the form e^u or $(1+u)^p$ for $p > 1$). See for example the seminal papers [12, 19, 20] and also [9] for a survey on the subject and an exhaustive list of related references. After a first version of this paper was circulated, X. Cabre informed us that even the case of singular nonlineariltles involved in MEMS devices had already been considered in [7] and in a more general context in [21].

Recently, Y. Guo, Z. Pan and M. Ward studied in [17] the dynamic behavior of (1.1), which is also of great practical interest. They also considered a more general class of profiles $f(x)$, where the membrane is allowed to be perfectly conducting, i.e.,

$$0 \leq f(x) \leq 1 \quad \text{for all} \quad x \in \Omega\,, \tag{1.5}$$

with $f(x) > 0$ on a subset of positive measure of Ω. By using both analytical and numerical techniques, they obtained larger pull-in voltage λ^* and larger pull-in distance for different classes of varying permittivity profiles. Besides the above practical considerations, the model turned out to be a very rich source of interesting mathematical phenomena. Numerics give lots of information and point to many conjectures, but the available arsenal of nonlinear analysis and PDE techniques can only tackle a precious few, even in the case of power-law permittivity profiles $f(x) = |x|^\alpha$.

This paper is a first in a series where we try to provide a rigorous mathematical analysis for various phenomena related to this model, that were observed numerically. Estimates on the pull-in voltage λ^* depend on the size and geometry of the domain, but also on the dimension of the ambient space and the permittivity profile f. A similar dependence occurs for the refined properties of steady states –such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results. The same complexity carries to the dynamic case where issues related to the "touchdown profile" –in finite or infinite time– or to global convergence towards a stable steady state, present many interesting mathematical challenges.

In this first paper, we shall focus on the stable and semi-stable stationary deflections of the elastic membrane, while the unstable case is considered in [13]. The dynamic case is dealt with in our forthcoming papers [15]. For convenience, we shall set $v = -u$ in such a way that our discussion will center on the following elliptic problem

$$\begin{aligned} -\Delta v &= \frac{\lambda f(x)}{(1-v)^2} && x \in \Omega; \\ 0 < v &< 1 && x \in \Omega\,; \\ v &= 0 && x \in \partial\Omega. \end{aligned} \tag*{$(S)_\lambda$}$$

Throughout the paper and unless mentioned otherwise, the permittivity profile f will be assumed to satisfy (1.5), and solutions for $(S)_\lambda$ will be taken in the classical sense.

This paper is organized as follows: In §2 we mainly show the existence of a specific pull-in voltage and study its dependence on the size and shape of the domain, as well as on the permittivity profile. These monotonicity properties will help us establish in §3 new lower and upper bound estimates on the pull-in voltage. To be more specific, we shall write $|\Omega|$ for the volume of a domain Ω in $\mathbb{R}^N$ and $P(\Omega) := \int_{\partial\Omega} ds$ for its "perimeter", with ω_N referring to the volume of the unit ball $B_1(0)$ in $\mathbb{R}^N$. We denote by μ_Ω the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and by ϕ_Ω the corresponding positive eigenfunction normalized with $\int_\Omega \phi_\Omega dx = 1$. We shall prove the following estimates, the upper ones in (1.6) below having been established in [23] and [17] respectively.

Theorem 1.1. *Assume f is a non-negative continuous function on a bounded domain Ω, then there exists a finite pull-in voltage $\lambda^* := \lambda^*(\Omega, f) > 0$ such that:*

1. *If $0 \le \lambda < \lambda^*$, there exists at least one solution for $(S)_\lambda$.*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*
3. *The following bounds on λ^* hold for any bounded domain Ω:*

$$\max\left\{\frac{8N}{27}, \frac{6N-8}{9}\right\}\frac{1}{\sup_\Omega f}\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \le \lambda^*(\Omega) \le \min\left\{\bar\lambda_1 := \frac{4\mu_\Omega}{27\inf\limits_{x\in\Omega} f(x)}, \bar\lambda_2 := \frac{\mu_\Omega}{3\int_\Omega f\phi_\Omega\, dx}\right\} \tag{1.6}$$

4. *If Ω is a strictly star-shaped domain, that is if $x \cdot \nu(x) \ge a > 0$ for all $x \in \partial\Omega$, where $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$, and if $f \equiv 1$, then*

$$\lambda^*(\Omega) \le \bar\lambda_3 = \frac{(N+2)^2 P(\Omega)}{8aN|\Omega|}. \tag{1.7}$$

   In particular, if $\Omega = B_1(0) \subset \mathbb{R}^N$ then we have the bound $\lambda^*(B_1(0)) \le \frac{(N+2)^2}{8}$.

5. *If $f(x) \equiv |x|^\alpha$ with $\alpha \ge 0$ and Ω is a ball of radius R, then we have*

$$\lambda^*(B_R, |x|^\alpha) \ge \max\{\frac{4(2+\alpha)(N+\alpha)}{27}, \frac{(2+\alpha)(3N+\alpha-4)}{9}\}R^{-(2+\alpha)}. \tag{1.8}$$

   Moreover, if $N \ge 8$ and $0 \le \alpha \le \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$, we have

$$\lambda^*(B_1, |x|^\alpha) = \frac{(2+\alpha)(3N+\alpha-4)}{9}. \tag{1.9}$$

In §3.3 we give some numerical estimates on λ^* to compare them with the analytic bounds given in Theorem 1.1 above. Note that the upper bound $\bar\lambda_1$ is relevant only when f is bounded away from 0, while the upper bound $\bar\lambda_2$ is valid for all permittivity profiles. However, the order between these two upper bounds can vary in general. For example, in the case of exponential permittivity profiles of the form $f(x) = e^{\alpha(|x|^2-1)}$ on the unit disc, one can see that $\bar\lambda_1$ is a better upper bound than $\bar\lambda_2$ for small α while the reverse holds true for larger values of α. The lower bounds in (1.6) and (1.8) can be improved in small dimensions, but they are optimal –at least for the ball– in dimension larger than 8.

We also consider issues of uniqueness and multiplicity of solutions for $(S)_\lambda$ with $0 < \lambda \le \lambda^*$. The following bifurcation diagrams show the complexity of the situation, even in the radially symmetric case. One can then see that the number of branches –and of solutions– is closely connected to the space dimension, a fact that we establish analytically in section §4, by focussing on the very first branch of solutions considered to be 'minimal" in the following way:

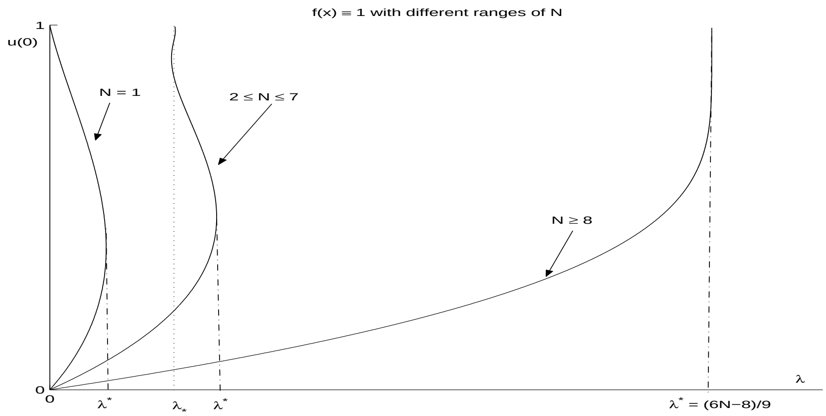

Figure 2: *Plots of $|u(0)|$ versus λ for the constant permittivity profile $f(x) \equiv 1$ defined in the unit ball $B_1(0) \subset \mathbb{R}^N$ with different ranges of N. In the case of $N \geq 8$, we have $\lambda^* = (6N-8)/9$.*

Definition 1.1. A solution $0 < u_\lambda(x) < 1$ is said to be a minimal (positive) solution of $(S)_\lambda$, if for any solution $0 < u(x) < 1$ of $(S)_\lambda$ we have $u_\lambda(x) \leq u(x)$ in Ω.

On the other hand, one can introduce for any solution u of $(S)_\lambda$, the linearized operator at u defined by $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$ and its eigenvalues $\{\mu_{k,\lambda}(u); k = 1, 2, ...\}$. The first eigenvalue is then simple and is given by:

$$\mu_{1,\lambda}(u) = \inf \left\{ \langle L_{u,\lambda}\phi, \phi\rangle_{H_0^1(\Omega)}\, ; \, \phi \in C_0^\infty(\Omega), \int_\Omega |\phi(x)|^2 dx = 1 \right\}.$$

Stable solutions (resp., *semi-stable solutions*) of $(S)_\lambda$ are those solutions u such that $\mu_{1,\lambda}(u) > 0$ (resp., $\mu_{1,\lambda}(u) \geq 0$). Our main results in this direction can be stated as follows.

Theorem 1.2. *Suppose f is a non-negative continuous function on a bounded domain Ω, and consider $\lambda^* := \lambda^*(\Omega, f)$ as defined in Theorem 1.1. Then,*

1. *For any $0 \leq \lambda < \lambda^*$, there exists a unique minimal solution u_λ of $(S)_\lambda$ such that $\mu_{1,\lambda}(u_\lambda) > 0$. Moreover for each $x \in \Omega$, the function $\lambda \to u_\lambda(x)$ is strictly increasing and differentiable on $(0, \lambda^*)$.*

2. *If $1 \leq N \leq 7$ then –by means of energy estimates– one has $\sup_{\lambda\in(0,\lambda^*)} \| u_\lambda \|_\infty < 1$ and consequently, $u^* = \lim\limits_{\lambda\uparrow\lambda^*} u_\lambda$ exists in $C^{2,\alpha}(\bar\Omega)$ with $0 < \alpha < 1$ and is a solution for $(S)_{\lambda^*}$ such that $\mu_{1,\lambda^*}(u^*) = 0$. In particular, u^* –often referred to as the extremal solution of problem $(S)_\lambda$– is unique.*

3. *On the other hand, if $N \geq 8$, $f(x) = |x|^\alpha$ with $0 \leq \alpha \leq \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$ and Ω is the unit ball, then the extremal solution is necessarily $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ and is therefore singular.*

We note that in general, the function u^* exists in any dimension, does solve $(S)_{\lambda^*}$ in a suitable weak sense and is the unique solution in an appropriate class. The above theorem says that it is however a classical solution in dimensions $1 \leq N \leq 7$, and this will allow us to start another branch of non-minimal (unstable) solutions. Indeed, we show in §5 –following ideas of Crandall-Rabinowitz [12]– that, for $1 \leq N \leq 7$, and for λ close enough to λ^*, there exists a unique second branch U_λ of solutions for $(S)_\lambda$, bifurcating from u^*, such that

$$\mu_{1,\lambda}(U_\lambda) < 0 \quad \text{while} \quad \mu_{2,\lambda}(U_\lambda) > 0. \tag{1.10}$$

In the companion paper [13], we shall provide a variational (mountain pass) characterization of these unstable solutions and more importantly, we establish –under the same dimension restriction as above– a compactness result along the second branch of unstable solutions leading to a –non zero– second bifurcation point.

Issues of uniqueness, multiplicity and other qualitative properties of the solutions for $(S)_\lambda$ are still far from being well understood, even in the radially symmetric case which we consider in §6. Some of the classical work of Joseph-Lundgren [19] and many that followed can be adapted to this situation when the permittivity profile is constant. However, the case of a power-law permittivity profile $f(x) = |x|^\alpha$ defined in a unit ball already presents a much richer situation. We present some numerical evidence for various conjectures relating to this case, some of which have been tackled in [13]. The bifurcation diagrams show four possible regimes –at least if the domain is a ball:

A. There is exactly one branch of solution for $0 < \lambda < \lambda^*$. This regime occurs when $N \geq 8$, and if $0 \leq \alpha \leq \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$. The results of this paper actually show that in this range, the first branch of solutions "disappears" at λ^* which happens to be equal to $\lambda_*(\alpha, N) = \frac{(2+\alpha)(3N+\alpha-4)}{9}$.

B. There exists an infinite number of branches of solutions. This regime occurs when

- $N = 1$ and $\alpha \geq \alpha^* := -\frac{1}{2} + \frac{1}{2}\sqrt{27/2}$.
- $2 \leq N \leq 7$ and $\alpha \geq 0$;
- $N \geq 8$ and $\alpha > \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$.

In this case, $\lambda_*(\alpha, N) < \lambda^*$ and the multiplicity becomes arbitrarily large as λ approaches –from either side– $\lambda_*(\alpha, N)$ at which there is a touchdown solution u (i.e., $\| u \|_\infty = 1$).

C. There exists a finite number of branches of solutions. We have again that $\lambda_*(\alpha, N) < \lambda^*$, but now the branch approaches the value 1 monotonically, and the number of solutions increase but remains finite as λ approaches $\lambda_*(\alpha, N)$. This regime occurs when $N = 1$ and $1 < \alpha \leq \alpha^* := -\frac{1}{2} + \frac{1}{2}\sqrt{27/2}$.

D. There exist exactly two branches of solutions for $0 < \lambda < \lambda^*$ and one solution for $\lambda = \lambda^*$. The bifurcation diagram vanishes when it returns to $\lambda = 0$. This regime occurs when $N = 1$ and $0 \leq \alpha \leq 1$.

Some of these questions have been considered in [13]. A detailed and involved analysis of compactness along the unstable branches will be discussed there, as well as some information about the second bifurcation point.

We finally mention that the above results can be extended to more general elliptic problems of the form

$$\begin{aligned} -\Delta v &= \frac{\lambda f(x)}{(1-v)^\beta}, \quad x \in \Omega\,; \\ v(x) &= 0, \qquad\quad\; x \in \partial\Omega \end{aligned} \tag{$(S)_{\lambda,\beta}$}$$

with $\beta > 0$. Here the critical dimension depends on the parameter β, but the techniques are straightforward extensions of the case $\beta = 2$ and are left to the interested reader.

2 The pull-in voltage

In this section, we establish the existence and some monotonicity properties for the pull-in voltage λ^* which is defined as:

$$\lambda^*(\Omega, f) = \sup\{\lambda > 0 \mid (S)_\lambda \text{ possesses at least one solution}\}\,. \tag{2.1}$$

In other words, λ^* is called pull-in voltage if there exist uncollapsed states for $0 < \lambda < \lambda^*$ while there are none for $\lambda > \lambda^*$. We then study how $\lambda^*(\Omega, f)$ varies with the domain Ω, the dimension and the permittivity profile f.

2.1 Existence of the pull-in voltage

For any bounded domain Γ in $\mathbb{R}^n$, we denote by μ_Γ the first eigenvalue of $-\Delta$ on $H_0^1(\Gamma)$and by ψ_Γ the corresponding positive eigenfunction normalized with $\sup_{x\in\Gamma}\psi_\Gamma = 1$. We also associate to any domain Ω in $\mathbb{R}^N$ the following parameter:

$$\nu_\Omega = \sup\{\mu_\Gamma H(\inf_\Omega \psi_\Gamma);\, \Gamma \text{ domain of } \mathbb{R}^N,\, \Gamma \supset \bar\Omega\} \tag{2.2}$$

where H is the function $H(t) = \frac{t(t+1+2\sqrt{t})}{(t+1+\sqrt{t})^3}$.

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, and assume f is a non-negative bounded function on Ω, then there exists a finite pull-in voltage $\lambda^* := \lambda^*(\Omega, f) > 0$ such that:*

1. *If $\lambda < \lambda^*$, there exists at least one solution for $(S)_\lambda$;*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*

Moreover, we have the lower bound

$$\frac{\nu_\Omega}{\sup_{x\in\Omega} f(x)} \le \lambda^*. \tag{2.3}$$

Proof: We need to show that $(S)_\lambda$ has at least one solution when $\lambda < \nu_\Omega(\sup_\Omega f(x))^{-1}$. Indeed, it is clear that $u \equiv 0$ is a sub-solution of $(S)_\lambda$ for all $\lambda > 0$. To construct a super-solution of $(S)_\lambda$, we consider a bounded domain $\Gamma \supset \bar\Omega$ with smooth boundary, and let $(\mu_\Gamma, \psi_\Gamma)$ be its first eigenpair normalized in such a way that

$$\sup_{x\in\Gamma} \psi_\Gamma(x) = 1 \text{ and } \inf_{x\in\Omega}\psi_\Gamma(x) := s_1 > 0.$$

We construct a super-solution in the form $\psi = A\psi_\Gamma$ where A is a scalar to be chosen later. First, we must have $A\psi_\Gamma \ge 0$ on $\partial\Omega$ and $0 < 1 - A\psi_\Gamma < 1$ in Ω, which requires that

$$0 < A < 1. \tag{2.4}$$

We also require

$$-\Delta\psi - \frac{\lambda f(x)}{(1-A\psi)^2} \ge 0 \quad \text{in} \quad \Omega\,, \tag{2.5}$$

which can be satisfied as long as:

$$\mu_\Gamma A\ \psi_\Gamma \ge \frac{\lambda \sup_\Omega f(x)}{(1-A\ \psi_\Gamma)^2} \quad \text{in} \quad \Omega\,, \tag{2.6}$$

or

$$\lambda \sup_\Omega f(x) < \beta(A,\Gamma) := \mu_\Gamma \inf\{g(sA);\, s \in [s_1(\Gamma), 1]\}\,, \tag{2.7}$$

where $g(s) = s(1-s)^2$. In other words, $\lambda^* \sup_\Omega f(x) \ge \sup\{\beta(A,\Gamma);\, 0 < a < 1, \Gamma \supset \bar\Omega\}$, and therefore it remains to show that

$$\nu_\Omega = \sup\{\beta(A,\Gamma);\, 0 < a < 1, \Gamma \supset \bar\Omega\}. \tag{2.8}$$

For that, we note first that

$$\inf_{s\in[s_1,1]} g(As) = \min\big\{g(As_1), g(A)\big\}.$$

We also have that $g(As_1) \le g(A)$ if and only if $A^2(s_1^3-1) - 2A(s_1^2-1) + (s_1-1) \le 0$ which happens if and only if $A^2(s_1^2+s_1+1) - 2A(s_1+1) + 1 \ge 0$ or if and only if either $A \le A_-$ or $A \ge A_+$ where

$$A_+ = \frac{s_1+1+\sqrt{s_1}}{s_1^2+1+s_1} = \frac{1}{s_1+1-\sqrt{s_1}}\,, \quad A_- = \frac{s_1+1-\sqrt{s_1}}{s_1^2+1+s_1} = \frac{1}{s_1+1+\sqrt{s_1}}$$

Since $A_- < 1 < a_+$, we get that

$$G(A) = \inf_{s\in[s_1,1]} g(As) = \begin{cases} g(As_1) & \text{if} \quad 0 \le A \le A_-\,, \\ g(A) & \text{if} \quad A_- \le A \le 1\,. \end{cases} \tag{2.9}$$

We now have that $\frac{dG}{dA} = g'(As_1)s_1 \ge 0$ for all $0 \le A \le A_-$. And since $A_- \ge \frac{1}{3}$, we have $\frac{dG}{dA} = g'(A) \le 0$ for all $A_- \le A \le 1$. It follows that

$$\begin{aligned} \sup_{0<a<1} \inf_{s\in[s_1,1]} g(As) &= \sup_{0<a<1} G(A) = G(A_-) = g(A_-) \\ &= \frac{1}{s_1+1+\sqrt{s_1}}\big(1 - \frac{1}{s_1+1+\sqrt{s_1}}\big)^2 \\ &= \frac{s_1(s_1+1+2\sqrt{s_1})}{(s_1+1+\sqrt{s_1})^3} \\ &= H(\inf_\Omega \psi_\Gamma)) \end{aligned}$$

which proves our lower estimate.

Now that we know that $\lambda^* > 0$, pick $\lambda \in (0,\lambda^*)$ and use the definition of λ^* to find a $\bar\lambda \in (\lambda,\lambda^*)$ such that $(S)_{\bar\lambda}$ has a solution $u_{\bar\lambda}$, *i.e*,

$$-\Delta u_{\bar\lambda} = \frac{\bar\lambda f(x)}{(1-u_{\bar\lambda})^2}\,, \quad x \in \Omega\,; \qquad u_{\bar\lambda} = 0\,, \quad x \in \partial\Omega\,.$$

and in particular $-\Delta u_{\bar\lambda} \ge \frac{\lambda f(x)}{(1-u_{\bar\lambda})^2}$ for $x \in \Omega$ which then implies that $u_{\bar\lambda}$ is a super-solution of $(S)_\lambda$. Since $u \equiv 0$ is a sub-solution of $(S)_\lambda$, then we can again conclude that there is a solution u_λ of $(S)_\lambda$ for every $\lambda \in (0,\lambda^*)$.

It is also easy to show that λ^* is finite, since if $(S)_\lambda$ has at least one solution $0 < u < 1$, then by integrating against the first (positive) eigenfunction ϕ_Ω, we get

$$+\infty > \mu_\Omega \ge \mu_\Omega \int_\Omega u\phi_\Omega = -\int_\Omega u\Delta\phi_\Omega = -\int_\Omega \phi_\Omega \Delta u = \lambda \int_\Omega \frac{\phi_\Omega f}{(1-u)^2}dx \ge \lambda \int_\Omega \phi_\Omega f dx \tag{2.10}$$

and therefore $\lambda^* < +\infty$. The definition of λ^* implies that there is no solution of $(S)_\lambda$ for any $\lambda > \lambda^*$. ■

2.2 Monotonicity results for the pull-in voltage

In this subsection, we give a more precise characterization of λ^*, namely as the endpoint for the branch of minimal solutions. This will allow us to establish various monotonicity properties for λ^* that will help in the estimates given in the next subsections. First we give a recursive scheme for the construction of minimal solutions.

Theorem 2.2. *Assume f is a non-negative continuous function on a bounded domain Ω in $\mathbb{R}^N$, then for any $0 < \lambda < \lambda^*(\Omega, f)$, there exists a unique minimal positive solution u_λ for $(S)_\lambda$. It is obtained as the limit* *of the sequence* $\{u_n(\lambda;x)\}$ *constructed recursively as follows:* $u_0 \equiv 0$ *in* Ω *and for each* $n \geq 1$,

$$\begin{aligned} -\Delta u_n &= \frac{\lambda f(x)}{(1-u_{n-1})^2}\,, \quad && x \in \Omega\,;\\ 0 \leq\ u_n &< 1\,, && x \in \Omega\,;\\ u_n\ &= 0\,, && x \in \partial\Omega. \end{aligned} \tag{2.11}$$

Proof: Let u be any positive solution for $(S)_\lambda$, and consider the sequence $\{u_n(\lambda;x)\}$ defined in (2.11). Clearly $u(x) > u_0 \equiv 0$ in Ω, and whenever $u(x) \geq u_{n-1}$ in Ω, then

$$\begin{aligned} -\Delta(u-u_n) = \lambda f(x)\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_{n-1})^2}\Big] \geq 0\,, \quad x \in \Omega\\ u - u_n = 0\,, \qquad x \in \partial\Omega\,. \end{aligned}$$

The maximum principle and an immediate induction yield that $1 > u(x) \geq u_n$ in Ω for all $n \geq 0$. In a similar way, the maximum principle implies that the sequence $\{u_n(\lambda;x)\}$ is monotone increasing. Therefore, $\{u_n(\lambda;x)\}$ converges uniformly to a positive solution $u_\lambda(x)$, satisfying $u(x) \geq u_\lambda(x)$ in Ω, which is a minimal positive solution of $(S)_\lambda$. It is also clear that u_λ is unique in this class of solutions. ■

Remark 2.1. Let $g(x,\xi,\Omega)$ be the Green's function of Laplace operator, with $g(x,\xi,\Omega) = 0$ on $\partial\Omega$. Then the iteration in (2.11) can be replaced by: $u_0 \equiv 0$ in Ω and for each $n \geq 1$,

$$\begin{aligned} u_n(\lambda;x) &= \lambda \int_\Omega \frac{f(\xi)g(x,\xi,\Omega)}{(1-u_{n-1}(\lambda;\xi))^2}d\xi\,, \quad && x \in \Omega\,;\\ u_n(\lambda;x) &= 0\,, && x \in \partial\Omega\,. \end{aligned} \tag{2.12}$$

The same reasoning as above yields that $\lim_{n\to\infty} u_n(\lambda;x) = u_\lambda(x)$ for all $x \in \Omega$.

The above construction of solutions yields the following monotonicity result for the pull-in voltage.

Proposition 2.3. *If* $\Omega_1 \subset \Omega_2$ *and if* f *is a non-negative continuous function on* Ω_2*, then* $\lambda^*(\Omega_1) \geq \lambda^*(\Omega_2)$ *and the corresponding minimal solutions satisfy* $u_{\Omega_1}(\lambda,x) \leq u_{\Omega_2}(\lambda,x)$ *on* Ω_1 *for every* $0 < \lambda < \lambda^*(\Omega_2)$.

Proof: Again the method of sub/super-solutions immediately yields that $\lambda^*(\Omega_1) \geq \lambda^*(\Omega_2)$. Now consider for $i = 1,2$, the sequences $\{u_n(\lambda,x,\Omega_i)\}$ on Ω_i defined by (2.12) where $g(x,\xi,\Omega_i)$ are the corresponding Green's functions on Ω_i. Since $\Omega_1 \subset \Omega_2$, we have that $g(x,\xi,\Omega_1) \leq g(x,\xi,\Omega_2)$ on Ω_1. Hence, it follows that

$$u_1(\lambda,x,\Omega_2) = \lambda \int_{\Omega_2} f(\xi)g(x,\xi,\Omega_2)d\xi \geq \lambda \int_{\Omega_1} f(\xi)g(x,\xi,\Omega_1)d\xi = u_1(\lambda,x,\Omega_1)$$

on Ω_1. By induction we conclude that $u_n(\lambda,x,\Omega_2) \geq u_n(\lambda,x,\Omega_1)$ on Ω_1 for all n. On the other hand, since $u_n(\lambda,x,\Omega_2) \leq u_{n+1}(\lambda,x,\Omega_2)$ on Ω_2 for n, we get that $u_n(\lambda,x,\Omega_1) \leq u_{\Omega_2}(\lambda,x)$ on Ω_1, and we are done. ■

We also note the following easy comparison results and we omit the details.

Corollary 2.4. *Suppose* $f_1, f_2 : \Omega \to \mathbb{R}$ *are two non-negative continuous functions such that* $f_1(x) \leq f_2(x)$ *on* Ω*, then* $\lambda^*(\Omega,f_1) \geq \lambda^*(\Omega,f_2)$ *and for* $0 < \lambda < \lambda^*(\Omega,f_2)$ *we have* $u_1(\lambda,x) \leq u_2(\lambda,x)$ *on* Ω*, where* $u_1(\lambda,x)$ *(resp.,* $u_2(\lambda,x)$*) are the unique minimal positive solution of*

$$-\Delta u = \tfrac{\lambda f_1(x)}{(1-u)^2} \text{ (resp., } -\Delta u = \tfrac{\lambda f_2(x)}{(1-u)^2}\text{) on } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

Moreover, if $f_2(x) > f_1(x)$ *on a subset of positive measure, then* $u_1(\lambda,x) < u_2(\lambda,x)$ *for all* $x \in \Omega$.

We shall also need the following result which is adapted from [2] (Theorem 4.10) where it is proved for non-singular non-linearities.

Proposition 2.5. *For any bounded domain Γ in $\mathbb{R}^N$ and any non-negative continuous function f on Γ, we have*

$$\lambda^*(\Gamma, f) \geq \lambda^*(B_R, f^*)$$

where $B_R = B_R(0)$ is the Euclidean ball in $\mathbb{R}^N$ with radius $R > 0$ and with volume $|B_R| = |\Gamma|$, and where f^* is the Schwarz symmetrization of f.

Proof: For any bounded $\Gamma \subset \mathbb{R}^N$, define its symmetrized domain $\Gamma^* = B_R$ to be the ball $\{x : |x| < R\}$ with $|\Gamma| = |B_R|$. If u is a real-valued function on Γ, we define its symmetrized function $u^* : \Gamma^* = B_R \to \mathbb{R}$ by $u^*(x) = \sup\{\mu : x \in B_R(\mu)\}$ where $B_R(\mu)$ is the symmetrization of the superlevel set $\Gamma(\mu) = \{x \in \Omega : \mu \leq u(x)\}$ (i.e., $B_R(\mu) = \Gamma(\mu)^*$). If h and g are continuous functions on Γ, then the following inequality holds (See Lemma 2.4 of [2])

$$\int_\Gamma hg dx \leq \int_{B_R} h^* g^* dx\,. \tag{2.13}$$

As in Theorem 4.10 of [2], we consider for any $\lambda \in (0, \lambda^*(B_R))$ the minimal sequence $\{u_n\}$ for (S_λ) in Γ as defined in (2.11), and let $\{v_n\}$ be the minimal sequence for the corresponding Schwarz symmetrized problem:

$$-\Delta v = \frac{\lambda f^*(x)}{(1-v)^2} \qquad x \in B_R\,, \tag{2.14a}$$

$$v = 0 \qquad x \in \partial B_R \tag{2.14b}$$

with $0 < v < 1$ on $B_R = \Gamma^*$. Since $\lambda \in (0, \lambda^*(B_R))$, we can consider the corresponding minimal solution v_λ for (2.14b). As in Theorem 2.2 we have $0 \leq v_n \leq v_\lambda < 1$ on B_R for all $n \geq 1$. We shall show that $\{u_n\}$ also satisfies $0 \leq u_n^* \leq v_\lambda < 1$ on B_R for all $n \geq 1$.

We now write $\widetilde{f}(a = \omega_N |x|^N)$ for $f^*(x)$, and $\widetilde{u}_n(a = \omega_N |x|^N)$ for $u_n^*(x)$. Applying (2.13) and the argument for (4.9) in [2], we obtain that $\widetilde{u}_0 = \widetilde{v}_0 = 0$ in $(0, R)$, and for each $n \geq 1$,

$$\frac{d\widetilde{u}_n}{da} + \frac{\lambda}{q(a)} \int_0^a \frac{\widetilde{f}}{(1-\widetilde{u}_{n-1})^2} dr \geq 0 \quad in \quad (0, R)\,, \tag{2.15}$$

and

$$\frac{d\widetilde{v}_n}{da} + \frac{\lambda}{q(a)} \int_0^a \frac{\widetilde{f}}{(1-\widetilde{v}_{n-1})^2} dr = 0 \quad in \quad (0, R) \tag{2.16}$$

with $q(a) = [N\omega_N^{1/N} a^{(N-1)/N}]^2 > 0$. We claim that for any $n \geq 1$, we have

$$\widetilde{u}_n(a) \leq \widetilde{v}_n(a) \quad a \in (0, R)\,. \tag{2.17}$$

In fact, for $n = 1$ we have $d\widetilde{u}_1/da \geq d\widetilde{v}_1/da$, and integration yields that

$$-\widetilde{u}_1(a) = \widetilde{u}_1(R) - \widetilde{u}_1(a) \geq \widetilde{v}_1(R) - \widetilde{v}_1(a) = -\widetilde{v}_1(a)\,,$$

and hence $\widetilde{u}_1(a) \leq \widetilde{v}_1(a)$ on $[0, R]$. (2.17) is now proved by induction. Suppose it holds for $n \leq k-1$, then one gets from (2.15) and (2.16) that $d\widetilde{u}_k/da \geq d\widetilde{v}_k/da$, which establishes (2.17) for all $n \geq 1$.

Therefore, the minimal sequence $\{u_n(x)\}$ on Γ is bounded by $\max_{x \in B_R} v_\lambda(x) < 1$, and again as in the proof of Theorem 2.2, there exists a minimal solution u_λ for $(S)_\lambda$ on Γ. This means $\lambda^*(\Gamma, f) \geq \lambda^*(B_R, f^*)$. ■

3 Estimates for the pull-in voltage

While the lower bound in (2.3) is useful to prove existence, it is not easy to compute. The following subsection gives more computationally accessible lower estimates for λ^*.

3.1 Lower bounds for λ^*

Proposition 3.1. *If Ω is a bounded smooth domain in $\mathbb{R}^N$ and f is a non-negative continuous function on Ω, then we have the following lower bound:*

$$\lambda^*(\Omega, f) \geq \max\{\frac{8N}{27}, \frac{6N-8}{9}\}\frac{1}{\sup_\Omega f}\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}}. \tag{3.1}$$

Moreover, if $f(x) \equiv |x|^\alpha$ with $\alpha \geq 0$ and Ω is a ball of radius R, then we have

$$\lambda^*(B_R, |x|^\alpha) \geq \max\{\frac{4(2+\alpha)(N+\alpha)}{27}, \frac{(2+\alpha)(3N+\alpha-4)}{9}\}R^{-(2+\alpha)}. \tag{3.2}$$

Finally, if $N \geq 8$ and $0 \leq \alpha \leq \alpha^{**}(N) := \frac{4-6N+3\sqrt{6}(N-2)}{4}$, we have

$$\lambda^*(B_1, |x|^\alpha) = \frac{(2+\alpha)(3N+\alpha-4)}{9}. \tag{3.3}$$

Proof: Setting $R = \Big(\frac{|\Omega|}{\omega_N}\Big)^{\frac{1}{N}}$, it suffices –in view of Proposition 2.5– and since $\sup\limits_{B_R} f^* = \sup\limits_{\Omega} f$, to show that

$$\lambda^*(B_R, f^*) \geq \max\{\frac{8N}{27R^2 \sup_\Omega f^*}, \frac{6N-8}{9R^2 \sup_\Omega f^*}\} \tag{3.4}$$

for the case where $\Omega = B_R$. In fact, the function $w(x) = \frac{1}{3}(1 - \frac{|x|^2}{R^2})$ satisfies on B_R

$$\begin{aligned} -\Delta w = \frac{2N}{3R^2} &= \frac{2N(1-\frac{1}{3})^2}{3R^2}\frac{1}{(1-\frac{1}{3})^2} \\ &\geq \frac{8N}{27R^2 \sup_\Omega f}\frac{f(x)}{[1-\frac{1}{3}(1-\frac{|x|^2}{R^2})]^2} \\ &= \frac{8N}{27R^2 \sup_\Omega f}\frac{f(x)}{(1-w)^2}\,. \end{aligned}$$

So for $\lambda \leq \frac{8N}{27R^2 \sup_\Omega f}$, w is a super-solution of $(S)_\lambda$ in B_R. Since on the other hand $w_0 \equiv 0$ is a sub-solution of $(S)_\lambda$ and $w_0 \leq w$ in B_R, then there exists a solution of $(S)_\lambda$ in B_R which proves a part of (3.4).

A similar computation applied to the function $v(x) = 1 - (\frac{|x|}{R})^{\frac{2}{3}}$ shows that v is also a supersolution as long as $\lambda \leq \frac{6N-8}{9R^2 \sup_\Omega f}$.

In order to prove (3.2), it suffices to note that $w(x) = \frac{1}{3}\big(1 - \frac{|x|^{2+\alpha}}{R^{2+\alpha}}\big)$ is a super- solution for $(S)_\lambda$ on B_R provided $\lambda \leq \frac{4(2+\alpha)(N+\alpha)}{27R^{2+\alpha}}$, and that $v(x) = 1 - (\frac{|x|}{R})^{\frac{2+\alpha}{3}}$ is a super- solution for $(S)_\lambda$ on B_R provided $\lambda \leq \frac{(2+\alpha)(3N+\alpha-4)}{9R^{2+\alpha}}$.

In order to complete the proof of Proposition 3.1, we need to establish that the function $u^*(x) = 1-|x|^{\frac{2+\alpha}{3}}$ is the extremal function as long as $N \geq 8$ and $0 \leq \alpha \leq \alpha^{**}(N) = \frac{4-6N+3\sqrt{6}(N-2)}{4}$. This will then yield that for such dimensions and these values of α, the voltage $\lambda = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ is exactly the pull-in voltage λ^*.

First, it is easy to check that u^* is a $H^1_0(\Omega)$-weak solution of $(S)_{\lambda^*}$. Since $\|u^*\|_\infty = 1$, and by the characterization of Theorem 5.1 below, we need only to prove that

$$\int_\Omega |\nabla\phi|^2 \geq \int_\Omega \frac{2\lambda|x|^\alpha}{(1-u^*)^3}\phi^2 \quad \forall\phi \in H^1_0(\Omega). \tag{3.5}$$

But Hardy's inequality gives for $N \geq 2$:

$$\int_{B_1} |\nabla\phi|^2 \geq \frac{(N-2)^2}{4}\int_{B_1}\frac{\phi^2}{|x|^2}$$

for any $\phi \in H^1_0(B_1)$, which means that (3.5) holds whenever $2\lambda^* \leq \frac{(N-2)^2}{4}$ or equivalently, if $N \geq 8$ and $0 \leq \alpha \leq \alpha^{**} = \frac{4-6N+3\sqrt{6}(N-2)}{4}$. ■

Remark 3.1. The above lower bounds can be improved at least in low dimensions. First note that if $N > \frac{12+\alpha}{5}$, then $\lambda_2 = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ is the better lower bound and it is actually sharp on the ball as soon as $N \geq 8$ and $\alpha \leq \alpha^{**}$. For lower dimensions, the best lower bounds are more complicated even when one considers supersolutions of the form $v(x) = a(1-(\frac{|x|}{R})^k)$ and optimize $\lambda(a,k,R)$ over a and k. For example, in the case where $\alpha = 0$, $N = 2$ and $R = 1$, one can see that a better lower bound can be obtained via the supersolution $v(x) = \frac{1}{2.4}(1-|x|^{1.6})$.

3.2 Upper bounds for λ^*

We note that (2.10) already yields a finite upper bound for λ^*. However, Pohozaev-type arguments can be used to establish better and more computable upper bounds. In this subsection, we establish the upper estimates claimed in Theorem 1.1. For a general domain Ω, the following upper bounds on $\lambda^*(\Omega)$ were established in [23] and [17] respectively. We sketch their easy proofs for completeness.

Proposition 3.2. *(1) If f is a positive bounded function on Ω and $\inf_\Omega f > 0$, then*

$$\lambda^* \leq \bar{\lambda}_1 \equiv \frac{4\mu_\Omega}{27}(\inf_\Omega f)^{-1}. \tag{3.6}$$

(2) If f is a non-negative bounded function on Ω, and if $f > 0$ on a set of positive measure, then

$$\lambda_* \leq \bar{\lambda}_2 \equiv \frac{\mu_\Omega}{3}\left(\int_\Omega f\phi_\Omega\, dx\right)^{-1}. \tag{3.7}$$

Here μ_Ω and ϕ_Ω are the first eigenpair of $-\Delta$ on $H^1_0(\Omega)$ with $\int_\Omega \phi_\Omega dx = 1$.

Proof: We multiply the equation $(S)_\lambda$ by ϕ_Ω, integrate the resulting equation over Ω, and use Green's identity to obtain

$$\int_\Omega \left(-\mu_\Omega u + \frac{\lambda f(x)}{(1-u)^2}\right)\phi_\Omega\, dx = 0. \tag{3.8}$$

Since $C := \inf_\Omega f > 0$ and $\phi_\Omega > 0$, the equality in (3.8) is impossible when

$$-\mu_\Omega u + \frac{\lambda C}{(1-u)^2} > 0, \quad \text{for all } x \in \Omega. \tag{3.9}$$

A simple calculation using (3.9) shows that (3.9) holds when $\lambda > \bar{\lambda}_1$, where $\bar{\lambda}_1$ is given in (3.6). This completes the proof of Proposition 3.2(1). ■

As shown below, the bound (3.6) on λ^* is rather good when applied to the constant permittivity profile $f(x) \equiv 1$. However, this bound is useless when the minimum of $f(x)$ on Ω is zero, and cannot be used to estimate λ^* for the power-law permittivity profile $f(x) = |x|^\alpha$ with $\alpha > 0$. Therefore, it is desirable to obtain a bound on λ^* that depends more on the global properties of f. Such a bound was established in [17] and here is a sketch of its proof.

Multiply now (S_λ) by $\phi_\Omega(1-u)^2$, and integrate the resulting equation over Ω to get

$$\int_\Omega \lambda f \phi_\Omega \, dx = \int_\Omega \phi_\Omega (1-u)^2 \Delta u \, dx \,. \tag{3.10}$$

Using the identity $\nabla \cdot (Hg) = g \nabla \cdot H + H \cdot \nabla g$ for any smooth scalar field g and vector field H, together with the Divergence theorem, we calculate

$$\int_\Omega \lambda f \phi_\Omega \, dx = \int_{\partial\Omega} (1-u)^2 \phi_\Omega \nabla u \cdot \hat{n} \, dS + \int_\Omega \nabla u \cdot \nabla \left[\phi_\Omega (1-u)^2\right] \, dx \,, \tag{3.11}$$

where $\hat{n}$ is the unit outward normal to $\partial\Omega$. Since $\phi_\Omega = 0$ on $\partial\Omega$, the first term on the right-hand side of (3.11) vanishes. By calculating the second term on the right-hand side of (3.11) we get:

$$\int_\Omega \lambda f \phi_\Omega \, dx = -\int_\Omega 2(1-u)\phi_\Omega |\nabla u|^2 \, dx + \int_\Omega (1-u)^2 \nabla u \cdot \nabla \phi_\Omega \, dx \,, \tag{3.12a}$$

$$\leq -\int_\Omega \frac{1}{3} \nabla \phi_\Omega \cdot \nabla \left[(1-u)^3\right] \, dx \,. \tag{3.12b}$$

The right-hand side of (3.12b) is evaluated explicitly by

$$\int_\Omega \lambda f \phi_\Omega \, dx \leq -\frac{1}{3} \int_{\partial\Omega} (1-u)^3 \nabla \phi_\Omega \cdot \hat{n} \, dS - \frac{\mu_\Omega}{3} \int_\Omega (1-u)^3 \phi_\Omega \, dx \,. \tag{3.13}$$

For $0 \leq u < 1$, the last term on the right-hand side of (3.13) is positive. Moreover, $u = 0$ on $\partial\Omega$ so that $\int_{\partial\Omega} \nabla \phi_\Omega \cdot \hat{n} \, dS = -\mu_\Omega$ since $\int_\Omega \phi_\Omega \, dx = 1$. Therefore, if (S_λ) has a solution, then (3.13) yields

$$\lambda \int_\Omega f \phi_\Omega \, dx \leq \frac{\mu_\Omega}{3} \,. \tag{3.14}$$

This proves that there is no solution for $\lambda > \bar{\lambda}_2$, which gives (3.7). ∎

Remark 3.2. The above estimate is not sharp, at least in dimensions $1 \leq N \leq 7$, as one can show that there exists $1 > \alpha(\Omega, N) > 0$ such that

$$\lambda \leq \frac{\mu_\Omega}{3} (1 - \alpha(\Omega, N)) \left(\int_\Omega f \phi_\Omega dx \right)^{-1} . \tag{3.15}$$

Indeed, this follows from inequality (3.13) above and Theorem 4.5 below where it will be shown that in these dimensions, there exists $0 < C(\Omega, N) < 1$ independent of λ such that $\|u_\lambda\|_\infty \leq C(\Omega, N)$ for any minimal solution u_λ. It is now easy to see that $\alpha(\Omega, N)$ can be taken to be

$$\alpha(\Omega, N) := \big(1 - C(\Omega, N)\big)^3 \int_\Omega \phi_\Omega dx.$$

We now consider problem $(S)_\lambda$ in the case where $\Omega \subset \mathbb{R}^N$ is a strictly star-shaped domain containing 0, meaning that Ω satisfies the additional property that there exists a positive constant a such that

$$x \cdot \nu \geq a > 0 \quad \text{for all} \quad x \in \partial\Omega \,, \tag{3.16}$$

where ν is the unit outer normal to Ω at $x \in \partial\Omega$.

Proposition 3.3. *Suppose $f \equiv 1$ and that the strictly star-shaped smooth domain $\Omega \subset \mathbb{R}^N$ satisfies* (3.16). *Then the pull-in voltage $\lambda^*(\Omega)$ satisfies:*

$$\lambda^*(\Omega) \le \bar{\lambda}_3 = \frac{(N+2)^2 P(\Omega)}{8aN|\Omega|}\,. \tag{3.17}$$

where $|\Omega|$ is the volume and $P(\Omega)$ is the perimeter of Ω.
In particular, if Ω is the Euclidean unit ball in $\mathbb{R}^N$, then we have the bound

$$\lambda^*(B_1(0)) \le \frac{(N+2)^2}{8}\,.$$

Proof: Recall Pohozaev's identity: If u is a solution of

$$\begin{aligned} \Delta u + \lambda g(u) = 0 \qquad &\text{for} \quad x \in \Omega\,,\\ u = 0 \qquad &\text{for} \quad x \in \partial\Omega\,, \end{aligned}$$

then

$$N\lambda \int_\Omega G(u)dx - \frac{N-2}{2}\lambda \int_\Omega ug(u)dx = \frac{1}{2}\int_{\partial\Omega} (x\cdot\nu)\big(\frac{\partial u}{\partial\nu}\big)^2 ds\,, \tag{3.18}$$

where $G(u) = \int_0^u g(s)ds$. Applying it with $g(u) = \frac{1}{(1-u)^2}$ and $G(u) = \frac{u}{1-u}$, it yields

$$\begin{aligned} \frac{\lambda}{2}\int_\Omega \frac{u(N+2-2Nu)}{(1-u)^2}dx &= \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\big(\frac{\partial u}{\partial\nu}\big)^2 ds\\ &\ge \frac{a}{2P(\Omega)}\Big(\int_{\partial\Omega}\frac{\partial u}{\partial\nu}ds\Big)^2\\ &= \frac{a}{2P(\Omega)}\Big(-\int_\Omega \Delta u dx\Big)^2\\ &= \frac{a\lambda^2}{2P(\Omega)}\Big(\int_\Omega \frac{dx}{(1-u)^2}\Big)^2, \end{aligned} \tag{3.19}$$

where we have used the Divergence Theorem and Hölder's inequality

$$\int_{\partial\Omega}\frac{\partial u}{\partial\nu}ds \le \Big(\int_{\partial\Omega}\big(-\frac{\partial u}{\partial\nu}\big)^2 ds\Big)^{1/2}\Big(\int_{\partial\Omega}ds\Big)^{1/2}.$$

Since

$$\begin{aligned} \int_\Omega \frac{u(N+2-2Nu)}{(1-u)^2}dx &= \int_\Omega \Big[-2N\big(u-\frac{N+2}{4N}\big)^2 + \frac{(N+2)^2}{8N}\Big]\frac{1}{(1-u)^2}dx\\ &\le \frac{(N+2)^2}{8N}\int_\Omega \frac{dx}{(1-u)^2}\,, \end{aligned}$$

we deduce from (3.19) that

$$\frac{(N+2)^2}{8N} \ge \frac{a\lambda}{P(\Omega)}\int_\Omega \frac{dx}{(1-u)^2} \ge \frac{a\lambda|\Omega|}{P(\Omega)}$$

which implies the upper bound (3.17) for λ^*.

Finally, for the special case where $\Omega = B_1(0) \subset \mathbb{R}^N$, we have $a = 1$ and $\frac{P(B_1(0))}{\omega_N} = N$ and hence the bound $\lambda^*(B_1(0)) \le \bar{\lambda}_3 = \frac{(N+2)^2}{8}$. ■

Exponential Profiles:

Ω	α	$\underline{\lambda}$	λ^*	$\bar{\lambda}_1$	$\bar{\lambda}_2$
(Slab)	0	1.185	1.401	1.462	3.290
(Slab)	1.0	1.185	1.733	1.878	4.023
(Slab)	3.0	1.185	2.637	3.095	5.965
(Slab)	6.0	1.185	4.848	6.553	10.50
(Unit Disk)	0	0.593	0.789	0.857	1.928
(Unit Disk)	0.5	0.593	1.153	1.413	2.706
(Unit Disk)	1.0	0.593	1.661	2.329	3.746
(Unit Disk)	3.0	0.593	6.091	17.21	11.86

Table 1: *Numerical values for pull-in voltage λ^* with the bounds given in Theorem 1.1. Here the exponential permittivity profile is chosen as (3.22).*

3.3 Numerical estimates for λ^*

In the computations below we shall consider two choices for the domain Ω,

$$\Omega : [-1/2, 1/2] \quad (\text{Slab})\,; \quad \Omega : x^2 + y^2 \leq 1 \quad (\text{Unit Disk})\,. \tag{3.20}$$

To compute the bounds $\bar{\lambda}_1$ and $\bar{\lambda}_2$, we must calculate the first eigenpair μ_Ω and ϕ_Ω of $-\Delta$ on Ω, normalized by $\int_\Omega \phi_\Omega\, dx = 1$, for each of these domains. A simple calculation yields that

$$\mu_\Omega = \pi^2\,, \qquad \phi_\Omega = \frac{\pi}{2} \sin\left[\pi\left(x + \frac{1}{2}\right)\right]\,, \quad (\text{Slab})\,; \tag{3.21a}$$

$$\mu_\Omega = z_0^2 \approx 5.783\,, \qquad \phi_\Omega = \frac{z_0}{J_1(z_0)} J_0(z_0|x|)\,, \quad (\text{Unit Disk})\,. \tag{3.21b}$$

Here J_0 and J_1 are Bessel functions of the first kind, and $z_0 \approx 2.4048$ is the first zero of $J_0(z)$. The bounds $\bar{\lambda}_1$ and $\bar{\lambda}_2$ can be evaluated by substituting (3.21) into (3.6) and (3.7). Notice that $\bar{\lambda}_2$ is, in general, determined only up to a numerical quadrature.

Using Newton's method and COLSYS [1], one can also solve the boundary value problem $(S)_\lambda$ and numerically calculate λ^* as the saddle-node point for the following two choices of the permittivity profile:

$$(\text{Slab}): \quad f(x) = |2x|^\alpha\,, \quad (\text{power-law})\,; \quad f(x) = e^{\alpha(x^2 - 1/4)} \quad (\text{exponential})\,, \tag{3.22a}$$

$$(\text{Unit Disk}): \quad f(x) = |x|^\alpha\,, \quad (\text{power-law})\,; \quad f(x) = e^{\alpha(|x|^2 - 1)}\,, \quad (\text{exponential})\,, \tag{3.22b}$$

where $\alpha \geq 0$. Table 1 contains numerical values for λ^* in the case of exponential profiles, while Table 2 deals with power-law profiles. What is remarkable is that $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are not comparable even when f is bounded away from 0 and that neither one of them provides the optimal value for λ^*. This leads us to conjecture that there should be a better estimate for λ^*, one involving the distribution of f in Ω, as opposed to the infimum or its average against the first eigenfunction ϕ_Ω.

4 The branch of minimal solutions

The branch of minimal solutions corresponds to the lowest branch in the bifurcation diagram, the one connecting the origin point $\lambda = 0$ to the first fold at $\lambda = \lambda^*$. In this section, we analyze further the

Power-Law Profiles:

Ω	α	$\lambda_c(\alpha)$	λ^*	λ_1	λ_2
(Slab)	0	1.185	1.401	1.462	3.290
(Slab)	1.0	3.556	4.388	∞	9.044
(Slab)	3.0	11.851	15.189	∞	28.247
(Slab)	6.0	33.185	43.087	∞	76.608
(Unit Disk)	0	0.593	0.789	0.857	1.928
(Unit Disk)	1.0	1.333	1.775	∞	3.019
(Unit Disk)	5.0	7.259	9.676	∞	15.82
(Unit Disk)	20	71.70	95.66	∞	161.54

Table 2: *Numerical values for pull-in voltage λ^* with the bounds given in Theorem 1.1. Here the power-law permittivity profile is chosen as (3.22).*

properties of this branch. To do so, we consider for each solution u of $(S)_\lambda$, the operator

$$L_{u,\lambda} = -\Delta - \frac{2\lambda f}{(1-u)^3} \tag{4.1}$$

associated to the linearized problem around u. We denote by $\mu_1(\lambda, u)$ the smallest eigenvalue of $L_{u,\lambda}$, that is the one corresponding to the following Dirichlet eigenvalue problem

$$-\Delta\phi - \frac{2\lambda f(x)}{(1-u)^3}\phi = \mu_1(\lambda, u)\phi\,, \quad x \in \Omega\,; \tag{4.2a}$$

$$\phi = 0 \qquad\qquad x \in \partial\Omega\,. \tag{4.2b}$$

In other words,

$$\mu_1(\lambda, u) = \inf_{\phi \in H_0^1(\Omega)} \frac{\int_\Omega \left\{|\nabla\phi|^2 - 2\lambda f(1-u)^{-3}\phi^2\right\}dx}{\int_\Omega \phi^2 dx}\,.$$

A solution u for $(S)_\lambda$ is said to be *stable* (resp., *semi-stable*) if $\mu_1(\lambda, u) > 0$ (resp., $\mu_1(\lambda, u) \geq 0$).

4.1 Spectral properties of minimal solutions

We start with the following crucial lemma which shows among other things that semi-stable solutions are necessarily minimal solutions.

Lemma 4.1. *Let f be a non-negative and continuous function on a bounded domain Ω and let $\lambda^* := \lambda^*(\Omega, f)$. Suppose u is a positive solution of $(S)_\lambda$, and consider any -classical- supersolution v of $(S)_\lambda$, that is*

$$-\Delta v \;\; \geq \;\; \frac{\lambda f(x)}{(1-v)^2} \qquad x \in \Omega\,, \tag{4.3a}$$

$$0 \leq v(x) < 1 \qquad x \in \Omega \tag{4.3b}$$

$$v \;\; = \;\; 0 \qquad x \in \partial\Omega. \tag{4.3c}$$

If $\mu_1(\lambda, u) > 0$ then $v \geq u$ on Ω, and if $\mu_1(\lambda, u) = 0$ then $v = u$ on Ω.

Proof: For a given λ and $x \in \Omega$, use the fact that $f(x) \geq 0$ and that $t \to \frac{\lambda f(x)}{(1-t)^2}$ is convex on $(0, 1)$, to obtain

$$-\Delta(u + \tau(v-u)) - \frac{\lambda f(x)}{[1-(u+\tau(v-u))]^2} \geq 0 \quad x \in \Omega\,, \tag{4.4}$$

for $\tau \in [0,1]$. Note that (4.4) is an identity at $\tau = 0$, which means that the first derivative of the left side for (4.4) with respect to τ is nonnegative at $\tau = 0$, *i.e*,

$$-\Delta(v-u) - \frac{2\lambda f(x)}{(1-u)^3}(v-u) \geq 0 \qquad x \in \Omega\,, \tag{4.5a}$$

$$v - u = 0 \qquad x \in \partial\Omega\,. \tag{4.5b}$$

Thus, the maximal principle implies that if $\mu_1(\lambda, u) > 0$ we have $v \geq u$ on Ω, while if $\mu_1(\lambda, u) = 0$ we have

$$-\Delta(v-u) - \frac{2\lambda f(x)}{(1-u)^3}(v-u) = 0 \qquad x \in \Omega\,. \tag{4.6}$$

In the latter case the second derivative of the left side for (4.4) with respect to τ is nonnegative a $\tau = 0$ again, *i.e*,

$$-\frac{6\lambda f(x)}{(1-u)^4}(v-u)^2 \geq 0 \quad x \in \Omega\,, \tag{4.7}$$

From (4.7) we deduce that $v \equiv u$ in $\Omega \setminus \Omega_0$, where

$$\Omega_0 = \{x \in \Omega :\ f(x) = 0 \text{ for } x \in \Omega\}\,. \tag{4.8}$$

On the other hand, (4.6) reduces to

$$\begin{aligned} -\Delta(v-u) &= 0 \quad x \in \Omega_0\,, \\ v - u &= 0 \quad x \in \partial\Omega_0\,, \end{aligned}$$

which implies $v \equiv u$ on Ω_0. Hence if $\mu_1(\lambda, u) = 0$ then $v \equiv u$ on Ω, which completes the proof of Lemma 4.1.

Now we can prove the following

Theorem 4.2. *Let f be a non-negative continuous function on a bounded domain Ω, let $\lambda^* := \lambda^*(\Omega, f)$ and consider on $(0, \lambda^*)$, the branch $\lambda \to u_\lambda$ of minimal solutions. Then the following hold:*

1. *For each $x \in \Omega$, the function $\lambda \to u_\lambda(x)$ is differentiable and strictly increasing on $(0, \lambda^*)$.*
2. *For each $\lambda \in (0, \lambda^*)$, the minimal solution u_λ is stable and the function $\lambda \to \mu_{1,\lambda} := \mu_1(\lambda, u_\lambda)$ is decreasing on $(0, \lambda^*)$.*

Proof: First we prove that $\lambda \to u_\lambda(x)$ is non-decreasing. For that consider $\lambda_1 < \lambda_2 < \lambda^*$, their corresponding minimal positive solutions u_{λ_1} and u_{λ_2} and let u^* be a positive solution for $(S)_{\lambda_2}$. For the monotone increasing series $\{u_n(\lambda_1; x)\}$ defined in (2.11), we then have $u^* > u_0(\lambda_1; x) \equiv 0$, and if $u_{n-1}(\lambda_1; x) \leq u^*$ in Ω, then

$$\begin{aligned} -\Delta(u^* - u_n) = f(x)\Big[\frac{\lambda_2}{(1-u^*)^2} - \frac{\lambda_1}{(1-u_{n-1})^2}\Big] \geq 0\,, &\quad x \in \Omega \\ u^* - u_n = 0\,, \qquad x \in \partial\Omega\,. & \end{aligned}$$

So we have $u_n(\lambda_1; x) \leq u^*$ in Ω. Therefore, $u_{\lambda_1} = \lim_{n\to\infty} u_n(\lambda_1; x) \leq u^*$ in Ω, and in particular $u_{\lambda_1} \leq u_{\lambda_2}$ in Ω. Therefore, $\frac{du_\lambda(x)}{d\lambda} \geq 0$ for all $x \in \Omega$.

Next we show that $\lambda \to \mu_{1,\lambda}$ is decreasing. But this follows easily from the variational characterization of $\mu_{1,\lambda}$, the monotonicity of $\lambda \to u_\lambda$ as well as the monotonicity of $(1-u)^{-3}$ with respect to u.

Now we define

$$\lambda^{**} = \sup\{\lambda;\ u_\lambda \text{ is a stable solution for } (S)_\lambda\}.$$

It is clear that $\lambda^{**} \leq \lambda^*$ and to show equality, it suffices to prove that there is no minimal solution for $(S)_\mu$ with $\mu > \lambda^{**}$. For that, suppose w is a minimal solution of $(S)_{\lambda^{**}+\delta}$ with $\delta > 0$, then we would have for $\lambda \leq \lambda^{**}$,

$$-\Delta w = \frac{(\lambda^{**}+\delta)f(x)}{(1-w)^2} \geq \frac{\lambda f(x)}{(1-w)^2} \quad x \in \Omega\,.$$

Since for $0 < \lambda < \lambda^{**}$ the minimal solutions u_λ are stable, it follows from Lemma 4.1 that $1 > w \geq u_\lambda$ for all $0 < \lambda < \lambda^{**}$. Consequently, $\underline{u} = \lim_{\lambda \nearrow \lambda^{**}} u_\lambda$ exists in $C^1(\Omega)$ and it is a solution for $(S)_{\lambda^{**}}$. Now from the definition of λ^{**}, we necessarily have $\mu_{1,\lambda^{**}} = 0$, hence by again applying Lemma 4.1 we obtain that $w \equiv \underline{u}$ and $\delta = 0$ on Ω which is a contradiction, and hence $\lambda^{**} = \lambda^*$.

It remains to show that for each $x \in \Omega$, the function $\lambda \to u_\lambda(x)$ is differentiable and strictly increasing on $(0, \lambda^*)$. But since each u_λ is stable, then by setting $F(\lambda, u_\lambda) := -\Delta - \frac{\lambda f}{(1-u_\lambda)^2}$, we get that $F_{u_\lambda}(\lambda, u_\lambda)$ is invertible for $0 < \lambda < \lambda^*$. It then follows from the Implicit Function Theorem that $u_\lambda(x)$ is differentiable with respect to λ.

Finally, by differentiating $(S)_\lambda$ with respect to λ, and since $\lambda \to u_\lambda(x)$ is non-decreasing, we get

$$\begin{aligned} -\Delta \frac{du_\lambda}{d\lambda} - \frac{2\lambda f(x)}{(1-u_\lambda)^3}\frac{du_\lambda}{d\lambda} = \frac{f(x)}{(1-u_\lambda)^2} &\geq 0\,, \quad x \in \Omega \\ \frac{du_\lambda}{d\lambda} &\geq 0\,, \quad x \in \partial\Omega\,. \end{aligned}$$

Applying the strong maximum principle, we conclude that $\frac{du_\lambda}{d\lambda} > 0$ on Ω for all $0 < \lambda < \lambda^*$. ∎

Remark 4.1. Lemma 3 of [12]) yields $\mu_1(1,0)$ as an upper bound for λ^{**} – at least in the case where $\inf_\Omega f > 0$ on Ω. Since $\lambda^{**}(= \lambda^*)$, this gives another upper bound for λ^* is our setting. It is worth noting that the upper bound in Theorem 1.1 gives a better estimate, since in the case where $f \equiv 1$, we have $\mu(1,0) = \mu_\Omega/2$ while the estimate in Theorem 1.1 gives $\frac{4\mu_\Omega}{27}$ for an upper bound.

4.2 Energy estimates and regularity

We start with the following easy observation.

Lemma 4.3. *Let f be a non-negative continuous function on a bounded domain Ω in $\mathbb{R}^N$. Then,*

1. *Any positive (weak) solution u in $H_0^1(\Omega)$ of $(S)_\lambda$ satisfies $\int_\Omega \frac{f}{(1-u)^2}dx < \infty$.*
2. *If $\inf_\Omega f > 0$ and $N \geq 3$, then any solution u such that $f/(1-u)^2 \in L^{3N/4}$ is a classical solution.*

Proof: (1) Since $u \in H_0^1(\Omega)$ is a positive solution of $(S)_\lambda$, we have

$$\int_\Omega \frac{f}{(1-u)^2} - \int_\Omega \frac{f}{1-u} = \int_\Omega \frac{uf}{(1-u)^2} = \int_\Omega |\nabla u|^2 < C\,,$$

which implies that

$$\int_\Omega \frac{f}{(1-u)^2} \leq C + \int_\Omega \frac{f}{1-u} \leq C + \int_\Omega [C\varepsilon \frac{f}{(1-u)^2} + \frac{C}{\varepsilon} f] \leq C + C\varepsilon \int_\Omega \frac{f}{(1-u)^2}$$

with $\varepsilon > 0$. Therefore, by choosing $\varepsilon > 0$ small enough, we conclude that $\int_\Omega \frac{f}{(1-u)^2} < \infty$.

(2) Suppose u is a weak solution such that $\frac{f(x)}{(1-u)^3} \in L^p(\Omega)$ which means that $\frac{f(x)}{(1-u)^2} \in L^{3p/2}(\Omega)$. By Sobolev's Theorem we can already deduce that $u \in C^{0,\alpha}$ with $\alpha = 2 - \frac{2N}{3p}$. To get more regularity, it suffices to show that $u < 1$ on Ω, but then if not we consider $x_0 \in \bar{\Omega}$ such that $u(x_0) = \|u\|_{C(\bar{\Omega})} = 1$, then we have

$$|1 - u(x)| = |u(x_0) - u(x)| \leq C|x_0 - x|^\alpha \quad on \quad \bar{\Omega}\,.$$

This inequality shows that if $p \geq \frac{N}{2}$ then we have

$$\infty > \int_\Omega \big(\frac{f(x)}{(1-u)^3}\big)^p dx \geq C' \int_\Omega |x - x_0|^{-3p\alpha} dx = C' \int_\Omega |x - x_0|^{-N} dx = \infty\,,$$

a contradiction, which implies that we must have $\|u\|_{C(\bar{\Omega})} < 1$. ■

Note that the above argument cannot be applied to the case where $f(x) \geq 0$ vanishes on Ω, and therefore we have to use the iterative scheme outlined in the next theorem.

Theorem 4.4. *For any bounded domain $\Omega \subset \mathbb{R}^N$ and any constant $C > 0$ there exists $0 < K(C,N) < 1$ such that a positive weak solution u of $(S)_\lambda$ $(0 < \lambda < \lambda^*)$ is a classical solution and $\| u \|_{C(\Omega)} \leq K(C,N)$ provided one of the following conditions holds:*

1. *$N = 1$ and $\|\frac{f}{(1-u)^3}\|_{L^1(\Omega)} \leq C$.*
2. *$N = 2$ and $\|\frac{f}{(1-u)^3}\|_{L^{1+\epsilon}(\Omega)} \leq C$ for some $\epsilon > 0$.*
3. *$N > 2$ and $\|\frac{f}{(1-u)^3}\|_{L^{N/2}(\Omega)} \leq C$.*

Proof: We prove this lemma by considering the following three cases separately:

(1) If $N = 1$, then for any $I > 0$ we write using the Sobolev inequality with constant $K(1) > 0$,

$$\begin{aligned}
K(1) \| (1-u)^{-1} - 1 \|^2_{L^\infty} &\leq \int_\Omega \big|\nabla[(1-u)^{-1} - 1]\big|^2 \\
&= \frac{1}{3} \int_\Omega \nabla u \cdot \nabla\big[(1-u)^{-3} - 1\big] \\
&= \frac{\lambda}{3} \int_\Omega f(1-u)^{-2}[(1-u)^{-3} - 1] \\
&\leq CI + C \int_{\{(1-u)^{-3} \geq I\}} f(1-u)^{-5} \\
&\leq CI + C \int_{\{(1-u)^{-3} \geq I\}} 8f(1-u)^{-2} \\
&\quad + C \int_{\{(1-u)^{-3} \geq I\}} f\big[(1-u)^{-3} + 2(1-u)^{-2} + 4(1-u)^{-1}\big]\big[(1-u)^{-1} - 1\big]^2 \\
&\leq CI + C + C \| (1-u)^{-1} - 1 \|^2_{L^\infty(\{(1-u)^{-3} \geq I\})} \int_{\{(1-u)^{-3} \geq I\}} \frac{f}{(1-u)^3} \\
&\leq CI + C + C\varepsilon(I) \| (1-u)^{-1} - 1 \|^2_{L^\infty}
\end{aligned} \tag{4.9}$$

with $\varepsilon(I) = \displaystyle\int_{\{(1-u)^{-3} \geq I\}} \frac{f}{(1-u)^3}$. From the assumption $f/(1-u)^3 \in L^1(\Omega)$, we have $\varepsilon(I) \to 0$ as $I \to \infty$. We now choose I such that $\varepsilon(I) \leq \frac{K(1)}{2C}$, so that the above estimates imply that $\| (1-u)^{-1} - 1 \|_{L^\infty} < K(C)$.

Standard regularity theory for elliptic problems now imply that $1/(1-u) \in C^{2,\alpha}(\Omega)$. Therefore, u is classical and there exists a constant $K(C,N)$ which can be taken strictly less than 1 such that $\| u \|_{C(\Omega)} \leq K(C,N) < 1$.

(2) The case when $N=2$ is similar as one can use that H_0^1 embeds in L^p for any $p < +\infty$.

(3) The case when $N > 2$ is more elaborate and we first show that $(1-u)^{-1} \in L^q(\Omega)$ for all $q \in (1,\infty)$. Since $u \in H_0^1(\Omega)$ is a solution of $(S)_\lambda$, we already have $\int_\Omega \frac{f}{(1-u)^2} < C$. Now we proceed by iteration to show that if $\int_\Omega \frac{f}{(1-u)^{2+2\theta}} < C$ for some $\theta \geq 0$, then $\int_\Omega \frac{1}{(1-u)^{2^*(1+\theta)}} < C$.

Indeed, for any constant $\theta \geq 0$ and $\ell > 0$ we choose a test function $\phi = [(1-u)^{-3}-1]\min\{(1-u)^{-2\theta}, \ell^2\}$. By applying this test function to both sides of $(S)_\lambda$, we have

$$\begin{aligned}
&\lambda \int_\Omega f(1-u)^{-2}[(1-u)^{-3}-1]\min\{(1-u)^{-2\theta},\ell^2\} = \int_\Omega \nabla u \cdot \nabla\big[\big((1-u)^{-3}-1\big)\min\{(1-u)^{-2\theta},\ell^2\}\big] \\
&= 3\int_\Omega |\nabla u|^2(1-u)^{-4}\min\{(1-u)^{-2\theta},\ell^2\} + 2\theta\int_{\{(1-u)^{-\theta}\leq \ell\}} |\nabla u|^2(1-u)^{-2\theta-1}[(1-u)^{-3}-1]\,.
\end{aligned} \tag{4.10}$$

We now suppose $\int_\Omega \frac{f}{(1-u)^{2+2\theta}} < C$. We then obtain from (4.10) and the fact that $\frac{1}{(1-u)^5} \leq C_I \frac{1}{(1-u)^3}(\frac{1}{1-u}-1)^2$ whenever $(1-u)^{-3} \geq I > 1$ that:

$$\begin{aligned}
&\int_\Omega \big|\nabla[\big((1-u)^{-1}-1\big)\min\{(1-u)^{-\theta},\ell\}]\big|^2 \\
&\leq 2\int_\Omega |\nabla u|^2(1-u)^{-4}\min\{(1-u)^{-2\theta},\ell^2\} + 2\theta^2\int_{\{(1-u)^{-\theta}\leq\ell\}} |\nabla u|^2(1-u)^{-2\theta-2}\big[(1-u)^{-1}-1\big]^2 \\
&= 2\int_\Omega |\nabla u|^2(1-u)^{-4}\min\{(1-u)^{-2\theta},\ell^2\} \\
&\quad +2\theta^2\int_{\{(1-u)^{-\theta}\leq\ell\}} |\nabla u|^2(1-u)^{-2\theta-1}\big[(1-u)^{-3}-1+1+(1-u)^{-1}-2(1-u)^{-2}\big] \\
&\leq C\lambda\int_\Omega f(1-u)^{-2}[(1-u)^{-3}-1]\min\{(1-u)^{-2\theta},\ell^2\} \\
&\leq C\lambda\int_\Omega f(1-u)^{-5}\min\{(1-u)^{-2\theta},\ell^2\} \\
&\leq CI + C\int_{\{(1-u)^{-3}\geq I\}} f(1-u)^{-5}\min\{(1-u)^{-2\theta},\ell^2\} \\
&\leq CI + C\int_{\{(1-u)^{-3}\geq I\}} f(1-u)^{-3}\big[(1-u)^{-1}-1\big]^2\min\{(1-u)^{-2\theta},\ell^2\} \\
&\leq CI + C\Big[\int_{\{(1-u)^{-3}\geq I\}}\Big(\frac{f}{(1-u)^3}\Big)^{\frac{N}{2}}\Big]^{\frac{2}{N}} \\
&\quad \times\Big[\int_{\{(1-u)^{-3}\geq I\}}\big(\big[(1-u)^{-1}-1\big]\min\{(1-u)^{-\theta},\ell\}\big)^{\frac{2N}{N-2}}\Big]^{\frac{N-2}{N}} \\
&\leq CI + C\varepsilon(I)\int_\Omega \big|\nabla[\big((1-u)^{-1}-1\big)\min\{(1-u)^{-\theta},\ell\}]\big|^2
\end{aligned} \tag{4.11}$$

with

$$\varepsilon(I) = \Big[\int_{\{(1-u)^{-3} \geq I\}} \Big(\frac{f}{(1-u)^3} \Big)^{\frac{N}{2}} \Big]^{\frac{2}{N}} .$$

From the assumption $f/(1-u)^3 \in L^{\frac{N}{2}}(\Omega)$ we have $\varepsilon(I) \to 0$ as $I \to \infty$. We now choose I such that $\varepsilon(I) = \frac{1}{2C}$, and the above estimates imply that

$$\int_{\{(1-u)^{-\theta} \leq \ell\}} \big|\nabla[(1-u)^{-\theta-1} - (1-u)^{-\theta}]\big|^2 \leq CI \,,$$

where the bound is uniform with respect to ℓ. This estimate leads to

$$\begin{aligned}
\tfrac{1}{(\theta+1)^2} \textstyle\int_{\{(1-u)^{-\theta} \leq \ell\}} \big|\nabla[(1-u)^{-\theta-1}]\big|^2 &= \int_{\{(1-u)^{-\theta} \leq \ell\}} (1-u)^{-2\theta-4} \big|\nabla u\big|^2 \\
&\leq CI + C \int_{\{(1-u)^{-\theta} \leq \ell\}} (1-u)^{-2\theta-3} \big|\nabla u\big|^2 \\
&\leq CI + \int_{\{(1-u)^{-\theta} \leq \ell\}} \big[C\varepsilon(1-u)^{-2\theta-4} + C/\varepsilon\big] \big|\nabla u\big|^2 \\
&\leq CI + C\varepsilon \int_{\{(1-u)^{-\theta} \leq \ell\}} (1-u)^{-2\theta-4} \big|\nabla u\big|^2
\end{aligned}$$

with $\varepsilon > 0$. This means that for $\varepsilon > 0$ sufficiently small

$$\int_{\{(1-u)^{-\theta} \leq \ell\}} \big|\nabla (1-u)^{-\theta-1}\big|^2 = \int_{\{(1-u)^{-\theta} \leq \ell\}} (\theta+1)^2 (1-u)^{-2\theta-4} \big|\nabla u\big|^2 < C \,.$$

So we can let $\ell \to \infty$ and we get that $(1-u)^{-\theta-1} \in H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, which means that $\int_\Omega \frac{1}{(1-u)^{2^*(1+\theta)}} < C$.

By iterating the above argument for $\theta_i + 1 = \frac{N}{N-2}(\theta_{i-1}+1)$ for $i \geq 1$ and starting with $\theta_0 = 0$, we find that $1/(1-u) \in L^q(\Omega)$ for all $q \in (1, \infty)$.

Standard regularity theory for elliptic problems applies again to give that $1/(1-u) \in C^{2,\alpha}(\Omega)$. Therefore, u is a classical solution and there exists a constant $0 < K(C,N) < 1$ such that $\| \, u \, \|_{C(\Omega)} \leq K(C,N) < 1$. This completes the proof of Theorem 4.4. ∎

Theorem 4.5. *For any dimension $1 \leq N \leq 7$, there exists a constant $0 < C(N) < 1$ independent of λ such that for any $0 < \lambda < \lambda^*$, the minimal solution u_λ satisfies $\| \, u_\lambda \, \|_{C(\Omega)} \leq C(N)$.*

Consequently, $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ exists in the topology of $C^{2,\alpha}(\bar\Omega)$ with $0 < \alpha < 1$. It is the unique classical solution for $(S)_{\lambda^*}$ and satisfies $\mu_{1,\lambda^*}(u^*) = 0$.

The theorem, which gives Theorem 1.2(2), will follow from the following uniform energy estimate on the minimal solutions u_λ.

Proposition 4.6. *There exists a constant $C(p) > 0$ such that for each $\lambda \in (0, \lambda^*)$, the minimal solution u_λ satisfies $\| \frac{f}{(1-u_\lambda)^3} \|_{L^p(\Omega)} \leq C(p)$ as long as $p < 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}$.*

Proof: Since minimal solutions are stable, we have:

$$\lambda \int_\Omega \frac{2f(x)}{(1-u_\lambda)^3} w^2 dx \leq -\int_\Omega w \Delta w dx = \int_\Omega |\nabla w|^2 dx \,, \tag{4.12}$$

for all $0<\lambda<\lambda^*$ and nonnegative $w\in H_0^1(\bar\Omega)$. Setting

$$w=(1-u_\lambda)^i-1>0\,,\ where\quad -2-\sqrt{6}<i<0\,, \tag{4.13}$$

then (4.12) becomes

$$i^2\int_\Omega (1-u_\lambda)^{2i-2}|\nabla u_\lambda|^2dx\geq \lambda\int_\Omega \frac{2[1-(1-u_\lambda)^i]^2f(x)}{(1-u_\lambda)^3}dx\,. \tag{4.14}$$

On the other hand, multiplying $(S)_\lambda$ by $\frac{i^2}{1-2i}[(1-u_\lambda)^{2i-1}-1]$ and applying integration by parts yield that

$$i^2\int_\Omega (1-u_\lambda)^{2i-2}|\nabla u_\lambda|^2dx=\lambda\frac{i^2}{2i-1}\int_\Omega \frac{[1-(1-u_\lambda)^{2i-1}]f(x)}{(1-u_\lambda)^2}dx\,. \tag{4.15}$$

And hence (4.14) and (4.15) reduce to

$$\begin{aligned}&\frac{\lambda i^2}{2i-1}\int_\Omega\frac{f(x)}{(1-u_\lambda)^2}dx-2\lambda\int_\Omega\frac{f(x)}{(1-u_\lambda)^3}dx+4\lambda\int_\Omega\frac{f(x)}{(1-u_\lambda)^{3-i}}dx\\&\geq \lambda(2+\frac{i^2}{2i-1})\int_\Omega\frac{f(x)}{(1-u_\lambda)^{3-2i}}dx\,.\end{aligned} \tag{4.16}$$

From the choice of i in (4.13) we have $2+\frac{i^2}{2i-1}>0$. So (4.16) implies that

$$\begin{aligned}\int_\Omega\frac{f(x)}{(1-u_\lambda)^{3-2i}}dx&\leq C\int_\Omega\frac{f(x)}{(1-u_\lambda)^{3-i}}dx\\&\leq C\Big(\int_\Omega\Big|\frac{f^{\frac{3-i}{3-2i}}}{(1-u_\lambda)^{3-i}}\Big|^{\frac{3-2i}{3-i}}dx\Big)^{\frac{3-i}{3-2i}}\cdot\Big(\int_\Omega\Big|f^{\frac{-i}{3-2i}}\Big|^{\frac{3-2i}{-i}}dx\Big)^{\frac{-i}{3-2i}}\\&\leq C\Big(\int_\Omega\frac{f(x)}{(1-u_\lambda)^{3-2i}}dx\Big)^{\frac{3-i}{3-2i}}\,,\end{aligned} \tag{4.17}$$

where Holder's inequality is applied. From the above we deduce that

$$\int_\Omega\frac{f(x)}{(1-u_\lambda)^{3-2i}}dx\leq C\,. \tag{4.18}$$

Further we have

$$\begin{aligned}\int_\Omega\Big|\frac{f(x)}{(1-u_\lambda)^3}\Big|^{\frac{3-2i}{3}}dx&=\int_\Omega f^{\frac{-2i}{3}}\cdot\frac{f}{(1-u_\lambda)^{3-2i}}dx\\&\leq C\int_\Omega\frac{f}{(1-u_\lambda)^{3-2i}}dx\leq C\,.\end{aligned} \tag{4.19}$$

Therefore, we get that

$$\|\ \frac{f(x)}{(1-u_\lambda)^3}\ \|_{L^p}\leq C\,, \tag{4.20a}$$

where –in view of (4.13)–

$$p=\frac{3-2i}{3}\leq 1+\frac{4}{3}+2\sqrt{\frac{2}{3}}\,. \tag{4.20b}$$

■

Proof of Theorem 4.5: The existence of u^* as a classical solution follows from Proposition 4.6 and Theorem 4.4, where $p = 1$ when the dimension $N = 1$, p can be taken to be $1 + \frac{4}{3}$ when $N = 2$. For $N > 2$, the reasoning applies as long as $\frac{N}{2} < 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}$ which happens when $N \leq 7$.

Since $\mu_{1,\lambda} > 0$ on the minimal branch for any $\lambda < \lambda^*$, we have the limit $\mu_{1,\lambda^*} \geq 0$. If now $\mu_{1,\lambda^*} > 0$ the Implicit Function Theorem could be applied to the operator $L_{u_{\lambda^*},\lambda^*}$, and would allow the continuation of the minimal branch $\lambda \mapsto u_\lambda$ of classical solutions beyond λ^*, which is a contradiction and hence $\mu_{1,\lambda^*} = 0$. The uniqueness in the class of classical solutions then follows from Lemma 4.1.

5 Uniqueness and Multiplicity of Solutions

The purpose of this section is to discuss uniqueness and multiplicity of solutions for $(S)_\lambda$.

5.1 Uniqueness of the solution at $\lambda = \lambda^*$

We first note that in view of the monotonicity in λ and the uniform boundedness of the first branch of solutions, the extremal function defined by $u^*(x) = \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$ always exists, and can always be considered as a solution for $(S)_{\lambda^*}$ in a generalized sense. Now if there exists $0 < C < 1$ such that $\| u_\lambda \|_{C(\bar\Omega)} \leq C$ for each $\lambda < \lambda^*$ –just like in the case where $1 \leq N < 8$– then we have seen in Theorem 4.5 that u^* is unique among the classical solutions. In the sequel, we tackle the important case when u^* is a weak solution (i.e., in $H_0^1(\Omega)$) of $(S)_{\lambda^*}$ but with the possibility that $\|u^*\|_\infty = 1$.

We shall borrow ideas from [4, 8], where the authors deal with the case of regular nonlinearities. However, unlike these papers where solutions are considered in a very weak sense, we consider here a more focussed and much simpler situation. We establish the following useful characterization of the extremal solution:

Theorem 5.1. *Assume f is a nonnegative function in $C(\bar\Omega)$. For $\lambda > 0$, consider $u \in H_0^1(\Omega)$ to be a weak solution of $(S)_\lambda$ (in the $H_0^1(\Omega)$-sense) such that $\| u \|_{L^\infty(\Omega)} = 1$. Then the following assertions are equivalent:*

1. *$\mu_{1,\lambda} \geq 0$, that is u satisfies*

$$\int_\Omega |\nabla\phi|^2 \geq \int_\Omega \frac{2\lambda f(x)}{(1-u)^3}\phi^2 \quad \forall \phi \in H_0^1(\Omega)\,, \tag{5.1}$$

2. *$\lambda = \lambda^*$ and $u = u^*$.*

Here and in the sequel, u will be called a $H_0^1(\Omega)$-weak solution of $(S)_\lambda$ if $0 \leq u \leq 1$ a.e. while u solves $(S)_\lambda$ in the weak sense of $H_0^1(\Omega)$. We need the following uniqueness result:

Proposition 5.2. *Let $f \in C(\bar\Omega)$ be a nonnegative function. Let u_1, u_2 be two $H_0^1(\Omega)$-weak solutions of $(S)_\lambda$ so that $\mu_{1,\lambda}(u_i) \geq 0$ for $i = 1, 2$. Then $u_1 = u_2$ a.e. in Ω.*

Proof: For any $\theta \in [0,1]$ and $\phi \in H_0^1(\Omega)$, $\phi \geq 0$, we have that:

$$\begin{aligned} I_{\theta,\phi} : &= \int_\Omega \nabla\big(\theta u_1 + (1-\theta)u_2\big)\nabla\phi - \int_\Omega \frac{\lambda f(x)}{\big(1-\theta u_1 - (1-\theta)u_2\big)^2}\phi \\ &= \lambda \int_\Omega f(x)\Big(\frac{\theta}{(1-u_1)^2} + \frac{1-\theta}{(1-u_2)^2} - \frac{1}{\big(1-\theta u_1 - (1-\theta)u_2\big)^2}\Big)\phi \geq 0 \end{aligned}$$

due to the convexity of $1/(1-u)$ with respect to u. Since $I_{0,\phi} = I_{1,\phi} = 0$, the derivative of $I_{\theta,\phi}$ at $\theta = 0, 1$ provides:

$$\int_\Omega \nabla(u_1 - u_2)\nabla\phi - \int_\Omega \frac{2\lambda f(x)}{(1-u_2)^3}(u_1 - u_2)\phi \geq 0$$
$$\int_\Omega \nabla(u_1 - u_2)\nabla\phi - \int_\Omega \frac{2\lambda f(x)}{(1-u_1)^3}(u_1 - u_2)\phi \leq 0$$

for any $\phi \in H_0^1(\Omega)$, $\phi \geq 0$. Testing the first inequality on $\phi = (u_1 - u_2)^-$ and the second one on $(u_1 - u_2)^+$ we get that:

$$\int_\Omega \Big[|\nabla(u_1 - u_2)^-|^2 - \frac{2\lambda f(x)}{(1-u_2)^3}\big((u_1-u_2)^-\big)^2\Big] \leq 0$$
$$\int_\Omega \Big[|\nabla(u_1 - u_2)^+|^2 - \frac{2\lambda f(x)}{(1-u_1)^3}\big((u_1-u_2)^+\big)^2\Big] \leq 0\,.$$

Since $\mu_{1,\lambda}(u_1) \geq 0$, we have:
1) either $\mu_{1,\lambda}(u_1) > 0$ and then $u_1 \leq u_2$ a.e.;
2) or $\mu_{1,\lambda}(u_1) = 0$ which then gives:

$$\int_\Omega \nabla(u_1 - u_2)\nabla\bar\phi - \int_\Omega \frac{2\lambda f(x)}{(1-u_1)^3}(u_1 - u_2)\bar\phi = 0 \tag{5.2}$$

where $\bar\phi = (u_1 - u_2)^+$. Since $I_{\theta,\bar\phi} \geq 0$ for any $\theta \in [0,1]$ and $I_{1,\bar\phi} = \partial_\theta I_{1,\bar\phi} = 0$, we get that:

$$\partial^2_{\theta\theta} I_{1,\bar\phi} = -\int_\Omega \frac{6\lambda f(x)}{(1-u_1)^4}\big((u_1-u_2)^+\big)^3 \geq 0\,.$$

Let $Z_0 = \{x \in \Omega : \ f(x) = 0\}$. Clearly, $(u_1 - u_2)^+ = 0$ a.e. in $\Omega \setminus Z_0$ and, by (5.2) we get:

$$\int_\Omega |\nabla(u_1 - u_2)^+|^2 = 0.$$

Hence, $u_1 \leq u_2$ a.e. in Ω. The same argument applies to prove the reversed inequality: $u_2 \leq u_1$ a.e. in Ω. Therefore, $u_1 = u_2$ a.e. in Ω and the proof is complete. ∎

Since $\| u_\lambda \| < 1$ for any $\lambda \in (0, \lambda^*)$, we need –in order to prove Theorem 5.1– only to show that $(S)_\lambda$ does not have any $H_0^1(\Omega)$-weak solution for $\lambda > \lambda^*$. By the definition of λ^*, this is already true for classical solutions. We shall now extend this property to the class of weak solutions by means of the following result:

Proposition 5.3. *If w is a $H_0^1(\Omega)$-weak solution of $(S)_\lambda$, then for any $\varepsilon \in (0,1)$ there exists a classic solution w_ε of $(S)_{\lambda(1-\varepsilon)}$.*

Proof: First we prove that for any $\psi \in C^2([0,1])$ concave function so that $\psi(0) = 0$, we have that

$$\int_\Omega \nabla\psi(w)\nabla\varphi \geq \int_\Omega \frac{\lambda f}{(1-w)^2}\dot\psi(w)\varphi \tag{5.3}$$

for any $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$. Indeed, by concavity of ψ we get:

$$\begin{aligned}\int_\Omega \nabla\psi(w)\nabla\varphi &= \int_\Omega \dot\psi(w)\nabla w\nabla\varphi = \int_\Omega \nabla w \nabla\left(\dot\psi(w)\varphi\right) - \int_\Omega \ddot\psi(w)\varphi|\nabla w|^2 \\ &\geq \int_\Omega \frac{\lambda f(x)}{(1-w)^2}\dot\psi(w)\varphi\end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. By density, we get (5.3).

Let $\varepsilon \in (0,1)$. Define

$$\psi_\varepsilon(w) := 1 - \big(\varepsilon + (1-\varepsilon)(1-w)^3\big)^{\frac{1}{3}}\,, \quad 0 \leq w \leq 1\,.$$

Since $\psi_\varepsilon \in C^2([0,1])$ is a concave function, $\psi_\varepsilon(0) = 0$ and

$$\dot{\psi}_\varepsilon(w) = (1-\varepsilon)\frac{g\left(\psi_\varepsilon(w)\right)}{g(w)}\,, \quad g(s) := (1-s)^{-2},$$

by (5.3) we obtain that for any $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$:

$$\int_\Omega \nabla\psi_\varepsilon(w)\nabla\varphi \geq \int_\Omega \frac{\lambda f(x)}{(1-w)^2}\dot{\psi}_\varepsilon(w)\varphi = \lambda(1-\varepsilon)\int_\Omega f(x)g\big(\psi_\varepsilon(w)\big)\varphi = \int_\Omega \frac{\lambda(1-\varepsilon)f(x)}{(1-\psi_\varepsilon(w))^2}\varphi\,.$$

Hence, $\psi_\varepsilon(w)$ is a $H_0^1(\Omega)$-weak supersolution of $(S)_{\lambda(1-\varepsilon)}$ so that $0 \leq \psi_\varepsilon(w) \leq 1 - \varepsilon^{\frac{1}{3}} < 1$. Since 0 is a subsolution for any $\lambda > 0$, we get the existence of a $H_0^1(\Omega)$-weak solution w_ε of $(S)_{\lambda(1-\varepsilon)}$ so that $0 \leq w_\varepsilon \leq 1 - \varepsilon^{\frac{1}{3}}$. By standard elliptic regularity theory, w_ε is a classical solution of $(S)_{\lambda(1-\varepsilon)}$. ■

5.2 Uniqueness of certain solutions for small voltage

In the following we focus on the uniqueness when λ is small enough. We first define non-minimal solutions for $(S)_\lambda$ as follows:

Definition 5.1. A solution $0 \leq u < 1$ is said to be a non-minimal positive solution of $(S)_\lambda$, if there exists another positive solution v of $(S)_\lambda$ and a point $x \in \Omega$ such that $u(x) > v(x)$.

Lemma 5.4. *Suppose u is a non-minimal solution of $(S)_\lambda$ with $\lambda \in (0,\lambda^*)$. Then $\mu_1(\lambda,u) < 0$ and the function $w = u - u_\lambda$ is in the negative space of $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$.*

Proof: For a fixed $\lambda \in (0,\lambda^*)$, let u_λ be the minimal solution of $(S)_\lambda$. We have $w = u - u_\lambda \geq 0$ in Ω, and

$$-\Delta w - \frac{\lambda(2-u-u_\lambda)f}{(1-u)^2(1-u_\lambda)^2}w = 0 \quad \text{in } \Omega\,.$$

Hence the strong maximum principle yields that $u_\lambda < u$ in Ω.

Let $\Omega_0 = \{x \in \Omega : f(x) = 0\}$ and $\Omega/\Omega_0 = \{x \in \Omega : f(x) > 0\}$. Direct calculations give that

$$-\Delta(u-u_\lambda) - \frac{2\lambda f}{(1-u)^3}(u-u_\lambda) = \lambda f\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_\lambda)^2} - \frac{2}{(1-u)^3}(u-u_\lambda)\Big] = \begin{cases} 0\,, & x \in \Omega_0\,; \\ < 0\,, & x \in \Omega/\Omega_0\,. \end{cases} \tag{5.4}$$

From this we get

$$\langle L_{u,\lambda}w, w\rangle = \lambda\int_{\Omega/\Omega_0} f\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_\lambda)^2} - \frac{2}{(1-u)^3}(u-u_\lambda)\Big](u-u_\lambda) < 0\,. \tag{5.5}$$

■

Now we are able to prove the following uniqueness result.

Theorem 5.5. *For every $M > 0$ there exists $0 < \lambda_1^*(M) < \lambda^*$ such that for $\lambda \in (0,\lambda_1^*(M))$ the equation $(S)_\lambda$ has a unique solution v satisfying:*

1. $\|\frac{f}{(1-v)^3}\|_1 \le M$ *as long as the dimension* $N = 1$.

2. $\|\frac{f}{(1-v)^3}\|_{1+\epsilon} \le M$ *and* $N = 2$.

3. $\|\frac{f}{(1-v)^3}\|_{N/2} \le M$ *and* $N > 2$.

Proof: For any fixed $\lambda \in (0, \lambda^*)$, let u_λ be the minimal solution of $(S)_\lambda$ and suppose $(S)_\lambda$ has a non-minimal solution u. The preceding lemma then gives

$$\int_\Omega |\nabla(u - u_\lambda)|^2 dx < \int_\Omega \frac{2\lambda(u - u_\lambda)^2 f(x)}{(1-u)^3} dx\,.$$

This implies in the case where $N > 2$ that

$$\begin{aligned} C(N)\Big(\int_\Omega (u - u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} &< \lambda \int_\Omega \frac{2f(x)}{(1-u)^3}(u - u_\lambda)^2 dx \\ &\le 2\lambda\Big(\int_\Omega \Big|\frac{f}{(1-u)^3}\Big|^{\frac{N}{2}}\Big)^{\frac{2}{N}} \Big(\int_\Omega (u - u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} \\ &\le 2\lambda M^{\frac{2}{N}} \Big(\int_\Omega (u - u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} \end{aligned}$$

which is a contradiction if $\lambda < \frac{C(N)}{2M^{\frac{2}{N}}}$ unless $u \equiv u_\lambda$. If $N = 1$, then we write

$$C(1)\|(u - u_\lambda)\|_\infty^2 < \lambda \int_\Omega \frac{2f(x)}{(1-u)^3}(u - u_\lambda)^2 dx \le 2\lambda \|(u - u_\lambda)\|_\infty^2 \int_\Omega \frac{f}{(1-u)^3} dx$$

and the proof follows. A similar proof holds for dimension $N = 2$. ∎

Remark 5.1. The above gives uniqueness for small λ among all solutions that either stay away from 1 or those that approach it slowly. We do not know whether if λ is small enough, any positive solution v of $(S)_\lambda$ satisfy $\int_\Omega (1-v)^{-\frac{3N}{2}} dx \le M$ for some uniform bound M independent of λ. Numerical computations do show that we may have uniqueness for small λ –at least for radially symmetric solutions– as long as $N \ge 2$.

5.3 Second solutions around the bifurcation point

Our next result is quite standard.

Lemma 5.6. *Suppose there exists* $0 < C < 1$ *such that* $\| u_\lambda \|_{C(\bar\Omega)} \le C$ *for each* $\lambda < \lambda^*$. *Then there exists* $\delta > 0$ *such that the solutions of* $(S)_\lambda$ *near* $(\lambda^*, u_{\lambda^*})$ *form a curve* $\rho(s) = \{(\bar\lambda(s), v(s)) : |s| < \delta\}$, *and the pair* $(\bar\lambda(s), v(s))$ *satisfies:*

$$\bar\lambda(0) = \lambda^*,\ \bar\lambda'(0) = 0,\ \bar\lambda''(0) < 0\,, \quad \text{and} \quad v(0) = u_{\lambda^*},\ v'(0)(x) > 0 \text{ in } \Omega\,. \tag{5.6}$$

In particular, if $1 \le N \le 7$ *then for* λ *close enough to* λ^*, *there exists a unique second branch* U_λ *of solutions for* $(S)_\lambda$, *bifurcating from* u^*, *such that*

$$\mu_{1,\lambda}(U_\lambda) < 0 \quad \text{while} \quad \mu_{2,\lambda}(U_\lambda) > 0. \tag{5.7}$$

Proof: The proof is similar to a related result of Crandall and Rabinowitz cf. [11] [12], so we will be brief. Firstly, the assumed upper bound on u_λ in C^1 and standard regularity theory, show that if $f \in C(\bar{\Omega})$ then $\| u_\lambda \|_{C^{2,\alpha}(\bar{\Omega})} \leq C$ for some $0 < \alpha < 1$ (while if $f \in L^\infty$, then $\| u_\lambda \|_{C^{1,\alpha}(\bar{\Omega})} \leq C$). It follows that $\{(\lambda, u_\lambda)\}$ is precompact in the space $\mathbb{R} \times C^{2,\alpha}$, and hence we have a limiting point $(\lambda^*, u_{\lambda^*})$ as desired. Since $\frac{\lambda^* f(x)}{(1-u_{\lambda^*})^2}$ is nonnegative, Theorem 3.2 of [11] characterizes the solution set of $(S)_\lambda$ near $(\lambda^*, u_{\lambda^*})$: $\bar{\lambda}(0) = \lambda^*$, $\bar{\lambda}'(0) = 0$, $v(0) = u_{\lambda^*}$ and $v'(0) > 0$ in Ω. Finally, the same computation as in Theorem 4.8 in [11] gives that $\bar{\lambda}''(0) < 0$. ■

Remark 5.2. A version of these results will be established variationally in the companion paper [13]. Indeed, we shall give there a variational characterization for both the stable and unstable solutions u_λ, U_λ in the following sense: For $1 \leq N \leq 7$, there exists $\delta > 0$ such that for any $\lambda \in (\lambda^* - \delta, \lambda^*)$, the minimal solution u_λ is a local minimum for some regularized energy functional $J_{\varepsilon,\lambda}$ on the space $H_0^1(\Omega)$, while the second solution U_λ is a mountain pass for the functional $J_{\varepsilon,\lambda}$.

6 Radially symmetric case and power-law permittivity profiles

In this section, we discuss issues of uniqueness and multiplicity of solutions for $(S)_\lambda$ when Ω is a symmetric domain and when f is a radially symmetric permittivity profile. Here, one can again define the corresponding pull-in voltage $\lambda_r^*(\Omega, f)$ requiring the solutions to be radially symmetric, that is:

$$\lambda_r^*(\Omega, f) = \sup\{\lambda;\ (S)_\lambda \text{ has a radially symmetric solution}\}.$$

Proposition 6.1. *Let Ω be a symmetric domain and let f be a non-negative bounded radially symmetric permittivity profile on Ω, then the minimal solutions of $(S)_\lambda$ are necessarily radially symmetric and consequently $\lambda_r^*(\Omega, f) = \lambda^*(\Omega, f)$.*

Moreover, if Ω is a ball, then any radial solution of $(S)_\lambda$ attains its maximum at 0.

Proof:: It is clear that $\lambda_r^*(\Omega, f) \leq \lambda^*(\Omega, f)$ and the reverse will be proved if we establish that every minimal solution of $(S)_\lambda$ with $0 < \lambda < \lambda^*(\Omega, f)$ is radially symmetric. But this is a straightforward application of the recursive scheme defined in Theorem 2.2 which gives a radially symmetric function at each step and therefore the resulting limiting function –which is the minimal solution– is radially symmetric.

For any radially symmetric $u(r)$ of $(S)_\lambda$ defined in the ball of radius R, we have $u_r(0) = 0$ and

$$-u_{rr} - \frac{N-1}{r} u_r = \frac{\lambda f}{(1-u)^2} \quad in \quad (0, R)\,,$$

Multiplying by r^{N-1}, we get that $-\frac{d(r^{N-1}u_r)}{dr} = \frac{\lambda f r^{N-1}}{(1-u)^2} \geq 0$, and therefore $u_r < 0$ in $(0, R)$ since $u_r(0) = 0$. This shows that $u(r)$ attains its maximum at 0. ■

The bifurcation diagrams shown in the introduction actually reflect the radially symmetric situation, and our emphasis in this section is on whether there is a better chance to analyze mathematically the higher branches of solutions in this case. Now some of the classical work of Joseph-Lundgren [19] and many that followed can be adapted to this situation when the permittivity profile is constant. However, the case of a power-law permittivity profile $f(x) = |x|^\alpha$ defined in a unit ball already presents a much richer situation. We now present various analytical and numerical evidence for various conjectures relating to this case, some of which are established rigorously in [13].

Power-law permittivity profiles

Consider the domain Ω to be a unit ball $B_1(0) \subset \mathbb{R}^N$ ($N \geq 1$) and let $f(x) = |x|^\alpha$ ($\alpha \geq 0$). We analyze in this case the branches of radially symmetric solutions of $(S)_\lambda$ for $\lambda \in (0, \lambda^*]$. In this case, $(S)_\lambda$ reduces to

$$\begin{aligned} &-u_{rr} - \frac{N-1}{r} u_r = \frac{\lambda r^\alpha}{(1-u)^2}\,, \quad 0 < r \leq 1, \\ &u'(0) = 0\,, \quad u(1) = 0\,. \end{aligned} \tag{6.1}$$

Here $r = |x|$ and $0 < u = u(r) < 1$ for $0 < r < 1$.

Looking first for a solution of the form

$$u(r) = 1 - \beta w(P) \quad \text{with} \quad P = \gamma r\,,$$

where γ, $\beta > 0$, equation (6.1) implies that

$$\gamma^2 \beta \big(w'' + \frac{N-1}{P} w'\big) = \frac{\lambda P^\alpha}{\beta^2 \gamma^\alpha} \frac{1}{w^2}\,.$$

We set $w(0) = 1$ and $\lambda = \gamma^{2+\alpha}\beta^3$. This yields the following initial value problem

$$\begin{aligned} &w'' + \frac{N-1}{P} w' = \frac{P^\alpha}{w^2}\,, \quad P > 0\,, \\ &w'(0) = 0\,, \quad w(0) = 1\,. \end{aligned} \tag{6.2}$$

Since $u(1) = 0$ we have $\beta = 1/w(\gamma)$. Therefore, we conclude that

$$\begin{cases} u(0) = 1 - \dfrac{1}{w(\gamma)}\,, \\[2mm] \lambda = \dfrac{\gamma^{2+\alpha}}{w^3(\gamma)}\,, \end{cases} \tag{6.3}$$

where $w(\gamma)$ is a solution of (6.2).

As was done in [23], one can numerically integrate the initial value problem (6.2) and use the results to compute the complete bifurcation diagram for (6.1). We show such a computation of $u(0)$ versus λ defined in (6.3) for the slab domain ($N = 1$) in Fig. 3. In this case, one observes from the numerical results that when $N = 1$, and $0 \leq \alpha \leq 1$, there exist exactly two solutions for $(S)_\lambda$ whenever $\lambda \in (0, \lambda^*)$. On the other hand, the situation becomes more complex for $\alpha > 1$ as $u(0) \to 1$. This leads to the question of determining the asymptotic behavior of $w(P)$ as $P \to \infty$. Towards this end, we proceed it as follows.

Setting $\eta = logP$ and $w(P) = P^B V(\eta) > 0$ for some positive constant B, we obtain from (6.2) that

$$P^{B-2}V'' + (2B + N - 2)P^{B-2}V' + B(B + N - 2)P^{B-2}V = \frac{P^{\alpha-2B}}{V^2}\,. \tag{6.4}$$

Choosing $B - 2 = \alpha - 2B$ so that $B = (2 + \alpha)/3$, we get that

$$V'' + \frac{3N + 2\alpha - 2}{3} V' + \frac{(2+\alpha)(3N + \alpha - 4)}{9} V = \frac{1}{V^2}\,. \tag{6.5}$$

We can already identify from this equation the following regime:

Case 1: Assume that

$$N = 1 \text{ and } 0 \leq \alpha \leq 1. \tag{6.6}$$

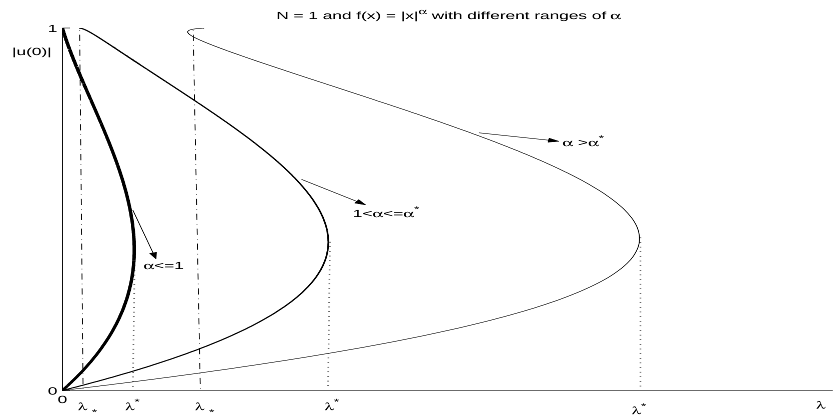

Figure 3: *Plots of* $|u(0)|$ *versus* λ *for the power-law permittivity profile* $f(x) = |x|^\alpha$ $(\alpha \geq 0)$ *defined in the slab domain* ($N = 1$). *The numerical experiments point to a constant* $\alpha^* > 1$ *(analytically given in* (6.10)*) such that the bifurcation diagrams are greatly different for different ranges of* α: $0 \leq \alpha \leq 1$, $1 < \alpha \leq \alpha^*$ *and* $\alpha > \alpha^*$.

In this case, there is no equilibrium point for equation (6.5), which means that the bifurcation diagram vanishes at $\lambda = 0$, from which one infer that in this case, there exist exactly two solutions for $\lambda \in (0, \lambda^*)$ and just one for $\lambda = \lambda^*$.

Case 2: N and α satisfy either one of the following conditions:

$$N = 1 \text{ and } \alpha > 1, \tag{6.7a}$$

$$N \geq 2. \tag{6.7b}$$

There exists then an equilibrium point V_e of (6.5) which must be positive and satisfies

$$V_e^3 = \frac{9}{(2+\alpha)(3N+\alpha-4)} > 0\,. \tag{6.8}$$

Linearizing around this equilibrium point by writing

$$V = V_e + Ce^{\sigma\eta}\,, \quad 0 < C << 1,$$

we obtain that

$$\sigma^2 + \frac{3N+2\alpha-2}{3}\sigma + \frac{(2+\alpha)(3N+\alpha-4)}{3} = 0\,.$$

This reduces to

$$\sigma_\pm = -\frac{3N+2\alpha-2}{6} \pm \frac{\sqrt{\triangle}}{6}\,, \tag{6.9a}$$

with

$$\triangle = -8\alpha^2 - (24N-16)\alpha + (9N^2 - 84N + 100)\,. \tag{6.9b}$$

We note that $\sigma_\pm < 0$ whenever $\triangle \geq 0$. Define now

$$\alpha^* = -\frac{1}{2} + \frac{1}{2}\sqrt{27/2}\,, \quad \alpha^{**} = \frac{4 - 6N + 3\sqrt{6}(N-2)}{4} \quad (N \geq 8)\,. \tag{6.10}$$

Next, we discuss on N and α by considering the sign of $\triangle$.

Case 2.A: N and α satisfy either one of the followings:

$$N = 1 \quad with \quad 1 < \alpha \leq \alpha^* \,; \tag{6.11a}$$

$$N \geq 8 \quad with \quad 0 \leq \alpha \leq \alpha^{**} \,. \tag{6.11b}$$

In this case, we have $\triangle \geq 0$ and

$$V \sim \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + C_1 e^{-\frac{3N+2\alpha-2-\sqrt{\triangle}}{6}\eta} + \cdots \,, \quad \text{as} \quad \eta \to +\infty \,.$$

Further, we conclude that

$$w \sim P^{\frac{2+\alpha}{3}} \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + C_1 P^{-\frac{N-2}{2}+\frac{\sqrt{\triangle}}{6}} + \cdots \,, \quad \text{as} \quad P \to +\infty \,.$$

In both cases, the branch approaches monotonically the value 1 as $\eta \to +\infty$. Moreover, since $\lambda = \gamma^{2+\alpha}/w^3(\gamma)$, we have

$$\lambda \sim \lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9} \quad as \quad \gamma \to \infty \,. \tag{6.12}$$

which is another important critical threshold for the voltage.

In the case $(6.11a)$ illustrated by Fig. 3, we have $\lambda_* < \lambda^*$ and the number of solutions increase but remains finite as λ approaches λ_*.

On the other hand, in the case of $(6.11b)$ illustrated by Fig. 4, we have $\lambda_* = \lambda^*$ and there seems to be only one branch of solutions.

Case 2.B: N and α satisfy any one of the following three:

$$N = 1 \quad with \quad \alpha > \alpha^* \,; \tag{6.13a}$$

$$2 \leq N \leq 7 \quad with \quad \alpha \geq 0 \,; \tag{6.13b}$$

$$N \geq 8 \quad with \quad \alpha > \alpha^{**} \,. \tag{6.13c}$$

In this case, we have $\triangle < 0$ and

$$V \sim \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + C_1 e^{-\frac{3N+2\alpha-2}{6}\eta} cos\big(\frac{\sqrt{-\triangle}}{6}\eta + C_2\big) + \cdots \,, \quad \text{as} \quad \eta \to +\infty \,.$$

We also have

$$w \sim P^{\frac{2+\alpha}{3}} \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + C_1 P^{-\frac{N-2}{2}} cos\big(\frac{\sqrt{-\triangle}}{6} log P + C_2\big) + \cdots \quad \text{as} \quad P \to +\infty \tag{6.14}$$

and from the fact that $\lambda = \gamma^{2+\alpha}/w^3(\gamma)$, we get again that

$$\lambda \sim \lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9} \quad \text{as} \quad \gamma \to \infty \,.$$

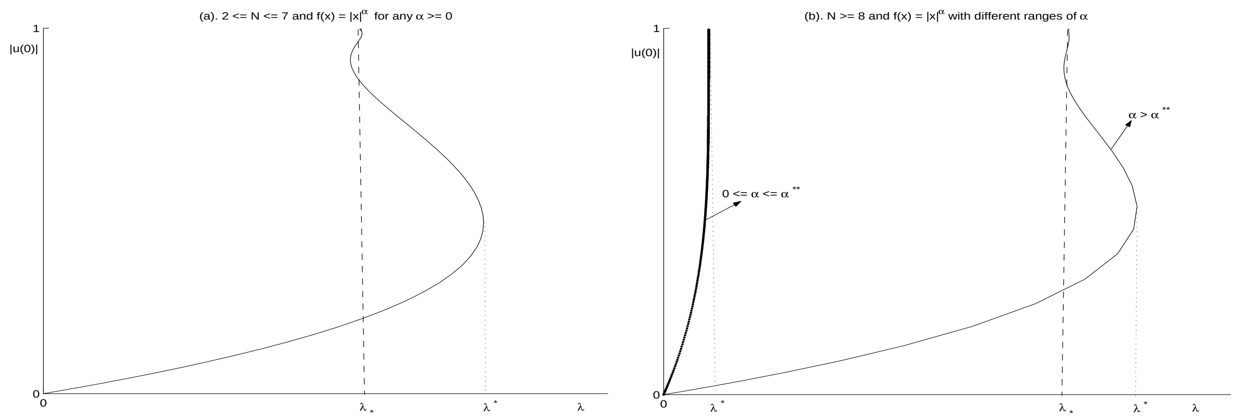

Figure 4: *Left figure: Plots of* $|u(0)|$ *versus* λ *for the power-law permittivity profile* $f(x) = |x|^\alpha$ $(\alpha \geq 0)$ *defined in the unit ball* $B_1(0) \subset \mathbb{R}^N$ *with* $2 \leq N \leq 7$*. In this case,* $|u(0)|$ *oscillates around the value* λ_* *defined in* (6.12) *and there exists a unique solution for* $(S)_{\lambda^*}$*. Right figure: Plots of* $|u(0)|$ *versus* λ *for the power-law permittivity profile* $f(x) = |x|^\alpha$ $(\alpha \geq 0)$ *defined in the unit ball* $B_1(0) \subset \mathbb{R}^N$ *with* $N \geq 8$*. The characters of the bifurcation diagrams depend on different ranges of* α*: when* $0 \leq \alpha \leq \alpha^{**}$*, there exists a unique solution for* $(S)_\lambda$ *with* $\lambda \in (0, \lambda^*)$ *and there does not exist any solution for* $(S)_{\lambda^*}$*; when* $\alpha > \alpha^{**}$*,* $|u(0)|$ *oscillates around the value* λ_* *defined in* (6.12) *and there exists a unique solution for* $(S)_{\lambda^*}$*.*

Note the oscillatory behavior of $w(P)$ in (6.14) for large P, which means that $|u(0)|$ is expected to oscillate around the value $\lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ as $P \to \infty$. The diagrams below point to the existence of a sequence $\{\lambda_i\}$ satisfying

$$\lambda_0 = 0\,, \quad \lambda_{2k} \nearrow \lambda_* \quad \text{as} \quad k \to \infty\,;$$

$$\lambda_1 = \lambda^*\,, \quad \lambda_{2k-1} \searrow \lambda_* \quad \text{as} \quad k \to \infty$$

and such that exactly $2k+1$ solutions for $(S)_\lambda$ exist when $\lambda \in (\lambda_{2k}, \lambda_{2k+2})$, while there are exactly $2k$ solutions when $\lambda \in (\lambda_{2k+1}, \lambda_{2k-1})$. Furthermore, $(S)_\lambda$ has infinitely multiple solutions at $\lambda = \lambda_*$.

The three cases (6.13a), (6.13b) and (6.13c) considered here for N and α, are illustrated by the diagrams in Fig. 3, Fig. 4(a) and Fig. 4(b), respectively.

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