On the Partial Differential Equations of Electrostatic MEMS Devices: Stationary Case

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Abstract

We analyze the nonlinear elliptic problem $\Delta u = \frac{\lambda f(x)}{(1+u)^2}$ on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. When a voltage –represented here by λ– is applied, the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices. The mathematical model lends to a nonlinear parabolic problem for the dynamic deflection of the elastic membrane which will be considered in forthcoming papers [11] and [12]. For now, we focus on the stationary equation where the challenge is to estimate λ^* in terms of material properties of the membrane, which can be fabricated with a spatially varying dielectric permittivity profile f. Applying analytical and numerical techniques, the existence of λ^* is established together with rigorous bounds. We show the existence of at least one steady-state when $\lambda < \lambda^*$ (and when $\lambda = \lambda^*$ in dimension $N < 8$) while none is possible for $\lambda > \lambda^*$. More refined properties of steady states –such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results– are shown to depend on the dimension of the ambient space and on the permittivity profile.

Key words: MEMS; pull-in voltage; power law permittivity profile; minimal solutions.

Contents

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1 Introduction

Micro-Electromechanical Systems (MEMS) are often used to combine electronics with micro-size mechanical devices in the design of various types of microscopic machinery. MEMS devices have therefore become key components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches and chemical sensors and so on (see for example [20]). The simplicity and importance of this technique have inspired numerous researchers to study mathematical models of electrostatic-elastic interactions. The mathematical analysis of these systems started in the late 1960s with the pioneering work of H. C. Nathanson and his coworkers [18] who constructed and analyzed a mass-spring model of electrostatic actuation, and offered the first theoretical explanation of pull-in instability. At roughly the same time, G. I. Taylor [24] studied the electrostatic deflection of two oppositely charged soap films, and he predicted that when the applied voltage was increased beyond a certain critical voltage, the two soap films would touch together. Since Nathanson and Taylor's seminal work, numerous investigators have analyzed and developed mathematical models of electrostatic actuation in attempts to understand further and control pull-in instability. An overview of the physical phenomena of the mathematical models associated with the rapidly developing field of MEMS technology is given in [20].

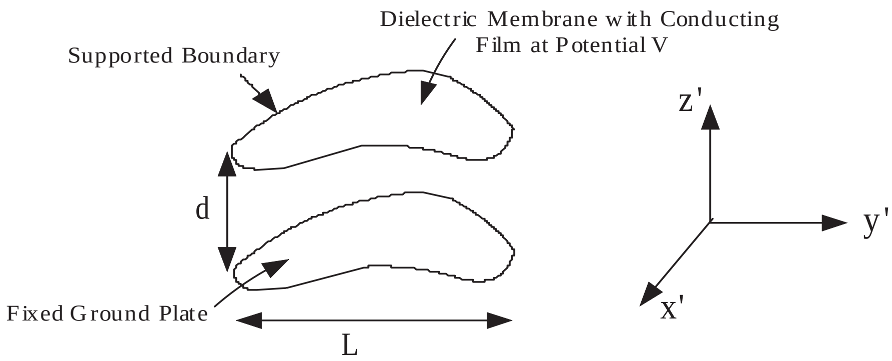

Figure 1: *The simple electrostatic MEMS device.*

The key component of many modern MEMS is the simple idealized electrostatic device shown in Fig. 1. The upper part of this device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage V is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate, and when V is increased beyond a certain critical value V^* –known as pull-in voltage– the steady-state of the elastic membrane is lost, and proceeds to touchdown or snap through at a finite time creating the so-called pull-in instability.

A mathematical model of the physical phenomena, leading to a partial differential equation for the dimensionless dynamic deflection of the membrane, was derived and analyzed in [10] and [14]. In the damping-dominated limit, and using a narrow-gap asymptotic analysis, the dimensionless dynamic deflection $u = u(x,t)$ of the membrane on a bounded domain Ω in $\mathbb{R}^2$, is found to satisfy the following parabolic problem

$$\frac{\partial u}{\partial t} = \Delta u - \frac{\lambda f(x)}{(1+u)^2} \quad \text{for} \quad x \in \Omega\,, \tag{1.1a}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega\,, \tag{1.1b}$$

$$u(x,0) = 0 \quad \text{for} \quad x \in \Omega\,. \tag{1.1c}$$

An outline of the derivation of (1.1) was given in Appendix A of [14]. This initial condition in (1.1c) assumes that the membrane is initially undeflected and the voltage is suddenly applied to the upper surface of the membrane at time $t = 0$. The parameter $\lambda > 0$ in (1.1a) characterizes the relative strength of the electrostatic and mechanical forces in the system, and is given in terms of the applied voltage V by

$$\lambda = \frac{\varepsilon_0 V^2 L^2}{2T_e d^3}\,, \tag{1.2}$$

where d is the undeflected gap size (see Fig. 1), L is the length scale of the membrane, T_e is the tension of the membrane, and ε_0 is the permittivity of free space in the gap between the membrane and the bottom plate. In view of relation (1.2), we shall use from now on the parameter λ and λ^* to represent the applied voltage V and pull-in voltage V^*, respectively. Referred to as the *permittivity profile*, $f(x)$ in (1.1a) is defined by the ratio

$$f(x) = \frac{\varepsilon_0}{\varepsilon_2(x)}\,, \tag{1.3}$$

where $\varepsilon_2(x)$ is the dielectric permittivity of the thin membrane.

There are several issues that must be considered in the actual design of MEMS devices. Typically one of the primary goals is to achieve the maximum possible stable deflection before touchdown occurs, which is referred to as *pull-in distance* (cf. [14] and [19]). Another consideration is to increase the stable operating range of the device by improving the pull-in voltage λ^* subject to the constraint that the range of the applied voltage is limited by the available power supply. Such improvements in the stable operating range is important for the design of certain MEMS devices such as microresonators. One way of achieving larger values of λ^*, while simultaneously increasing the pull-in distance, is to use a voltage control scheme imposed by an external circuit in which the device is placed (cf. [21]). This approach leads to a nonlocal problem for the dynamic deflection of the membrane. A different approach studied in [19] and [14] is to introduce a spatially varying dielectric permittivity $\varepsilon_2(x)$ of the membrane. The idea is to locate the region where the membrane deflection would normally be largest under a spatially uniform permittivity, and then make sure that a new dielectric permittivity $\varepsilon_2(x)$ is largest –and consequently the profile $f(x)$ smallest– in that region.

This latter approach requires the membrane having varying dielectric properties, a framework investigated recently in [19] and [14]. In [19] J. Pelesko studied the steady-states of (1.1), when $f(x)$ is assumed to be bounded away from zero, i.e.,

$$0 < C_0 \le f(x) \le 1 \quad \text{for all} \quad x \in \Omega. \tag{1.4}$$

He established in this case an upper bound $\bar\lambda_1$ for λ^*, and derived numerical results for the power-law permittivity profile, from which the larger pull-in voltage and thereby the larger pull-in distance, the existence and multiplicity of the steady-states were observed. Recently, Y. Guo, Z. Pan and M. Ward studied in [14] the dynamic behavior of (1.1), which is also of great practical interest. They considered a more general class of profiles $f(x)$, where the membrane is allowed to be perfectly conducting, i.e.,

$$0 \le f(x) \le 1 \quad \text{for all} \quad x \in \Omega\,, \tag{1.5}$$

with $f(x) > 0$ on a subset of positive measure of Ω. By using both analytical and numerical techniques, they obtained larger pull-in voltage λ^* and larger pull-in distance for different classes of varying permittivity profiles. They obtained new upper bounds $\bar\lambda_2$ for the pull-in voltage λ^* that correspond to the more general profiles $f(x)$ satisfying (1.5). Moreover, having estimated λ^* numerically as some saddle-node bifurcation value, they showed that λ^* is generally strictly smaller than both $\bar\lambda_1$ and $\bar\lambda_2$ (cf. Table 1 of [14]).

In this paper, we shall focus on the stationary deflection of the elastic membrane, leaving the dynamic case to our forthcoming papers [11] and [12]. For convenience, we shall set $v = -u$ in such a way that our discussion will center on the following elliptic problem

$$\begin{aligned} -\Delta v &= \frac{\lambda f(x)}{(1-v)^2} && x \in \Omega\,; \\ 0 < v &< 1 && x \in \Omega\,; \\ v &= 0 && x \in \partial\Omega. \end{aligned} \tag{S}_\lambda$$

We shall continue the investigation of optimal upper and lower bounds for the pull-in voltage, and how they relate to the permittivity profile f which will be assumed to satisfy (1.5) throughout the paper unless mentioned otherwise. We shall also discuss the issues of existence, multiplicity, and other related properties of steady-states for $(S)_\lambda$, and their remarkable dependence on space-dimension.

This paper is organized as follows: In §2 we mainly show the existence of a specific pull-in voltage and establish lower and upper bound estimates. For that, we shall write $\omega_{_N}$ for the volume of the unit ball $B_1(0)$ in $\mathbb{R}^N$, and for any bounded domain Γ in $\mathbb{R}^n$, we denote by $\mu_{_\Gamma}$ the first eigenvalue of $-\Delta$ on $H_0^1(\Gamma)$and by $\phi_{_\Gamma}$ (resp., $\psi_{_\Gamma}$) the corresponding positive eigenfunction normalized with $\int_\Gamma \phi_{_\Gamma} dx = 1$ (resp., $\sup_{x\in\Gamma} \psi_{_\Gamma} = 1$). We shall also associate to any domain Ω in $\mathbb{R}^N$ the following parameter:

$$\nu_{_\Omega} = \sup\{\mu_{_\Gamma} H(\inf_{\Omega} \psi_\Gamma);\ \Gamma \text{ domain of } \mathbb{R}^N,\ \Gamma \supset \bar\Omega\} \tag{1.6}$$

where H is the function $H(t) = \frac{t(t+1+2\sqrt{t})}{(t+1+\sqrt{t})^3}$.

We then prove the following lower estimates, the upper ones having been established in [19] and [14].

Theorem 1.1. *There exists a finite pull-in voltage $\lambda^* > 0$ such that*

1. *If $0 \le \lambda < \lambda^*$, there exists at least one solution for $(S)_\lambda$;*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*

Moreover, we have the bounds

$$\max\{\frac{\nu_{_\Omega}}{\sup\limits_{x\in\Omega} f(x)}, \frac{8N}{27 \sup\limits_{x\in\Omega} f(x)}\Big(\frac{\omega_{_N}}{|\Omega|}\Big)^{\frac{2}{N}}\} =: \underline{\lambda} \le \lambda^* \le \bar\lambda := \min\Big\{\frac{4\mu_{_\Omega}}{27 \inf\limits_{x\in\Omega} f(x)}, \frac{\mu_{_\Omega}}{3\int_\Omega f\phi_{_\Omega}\, dx}\Big\}$$

Furthermore, if $f(x) \equiv |x|^\alpha$ on Ω with $\alpha \ge 0$, then we have the more refined lower bound

$$\lambda_c(\alpha) := \frac{4(2+\alpha)(N+\alpha)}{27}\Big(\frac{\omega_{_N}}{|\Omega|}\Big)^{\frac{2+\alpha}{N}} \le \lambda^*\,. \tag{1.7}$$

In §2.3 we give some numerical estimates on λ^* to compare them with analytic bounds given in Theorem 1.1. Note that the upper bound $\bar\lambda_1 = \frac{4\mu_\Omega}{27}\Big(\inf_{x\in\Omega} f(x)\Big)^{-1}$ is relevant only when f is bounded away from 0, while the upper bound $\bar\lambda_2 = \frac{\mu_\Omega}{3}\Big(\int_\Omega f\phi_{_\Omega}\, dx\Big)^{-1}$ is valid for all permittivity profiles. In the case of a uniform permittivity profile $f \equiv 1$ on Ω, where Ω is a strictly star-shaped domain containing 0, we give a more explicit upper bound $\bar\lambda_3$ of λ^* in Proposition 2.4. In particular, we show that $\bar\lambda_3 = \frac{(N+2)^2}{8}$ is an upper bound in the case where the domain is the unit ball $\Omega = B_1(0) \subset \mathbb{R}^N$.

The issues of uniqueness and multiplicity of solutions for $(S)_\lambda$ with $0 < \lambda < \lambda^*$, and even mere existence for $(S)_{\lambda^*}$ seem to be quite interesting. We address these problems beginning in section §3 by first considering minimal (positive) solutions of $(S)_\lambda$ defined as follows.

Definition 1.1. A solution $0 < u_\lambda(x) < 1$ is said to be a minimal (positive) solution of $(S)_\lambda$, if for any solution $0 < u(x) < 1$ of $(S)_\lambda$ we have $u_\lambda(x) \le u(x)$ in Ω.

Our main results in this direction can be stated as follows.

Theorem 1.2. *Under the above assumptions, and with λ^* as defined in Theorem 1.1, there exists for any $\lambda < \lambda^*$, a unique minimal positive classical solution $u_\lambda(x)$ of $(S)_\lambda$. It is obtained as the limit of the sequence $\{u_n(\lambda; x)\}$ constructed recursively as follows: $u_0 \equiv 0$ in Ω and for each $n \ge 1$,*

$$\begin{aligned} -\Delta u_n &= \frac{\lambda f(x)}{(1-u_{n-1})^2}\,, && x \in \Omega\,;\\ 0 \le\ u_n &< 1\,, && x \in \Omega\,;\\ u_n &= 0\,, && x \in \partial\Omega. \end{aligned} \tag{1.8}$$

Moreover, minimal solutions satisfy the following properties:

1. *For each $x \in \Omega$, the function $\lambda \to u_\lambda(x)$ is strictly increasing and differentiable on $(0, \lambda^*)$;*

2. *If* $1 \leq N < 8$, *then there exists a constant* $0 < C(N) < 1$ *such that* $\| u_\lambda \|_{C(\Omega)} \leq C(N)$ *for all* $\lambda < \lambda^*$.

We refer to Lemma 3.6 in §3.2 for a more general version of Theorem 1.2(2). Based on the results of Theorem 1.2, the existence and related properties of minimal solutions at critical voltage $\lambda = \lambda^*$ will be studied in §3.3. More precisely, we shall establish the following.

Theorem 1.3. *If* $1 \leq N < 8$ *then* $u_{\lambda^*} = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ *exists in the topology of* $C^{2,\alpha}(\bar{\Omega})$ *with* $0 < \alpha < 1$, *and* u_{λ^*} *is the unique classical solution of* $(S)_{\lambda^*}$.

§4 is devoted to the uniqueness and multiplicity of solutions which remarkably depend again on the space-dimension.

Theorem 1.4. *Under the above assumptions, with* λ^* *defined as in Theorem 1.1, we have:*

1. *If* $N > 2$, *then for any* $M > 0$ *there exists a voltage* $0 < \lambda_1^*(M) < \lambda^*$ *such that for every* $\lambda \in (0, \lambda_1^*(M))$, *there exists a unique positive solution for* $(S)_\lambda$ *–namely the minimal solution* u_λ*– that satisfies* $\int_\Omega |\frac{f}{(1-u)^3}|^{\frac{N}{2}} dx \leq M$;

2. *If* $1 \leq N < 8$ *then there exists* $0 < \lambda_2^* < \lambda^*$ *such that* $(S)_\lambda$ *has at least two solutions for* $\lambda \in (\lambda_2^*, \lambda^*)$.

A uniqueness result in the spirit of (1) also holds for dimension 1 (resp., dimension 2) with $N/2$ replaced by 1 (resp., $1+\epsilon$). However, in spite of above results, issues of uniqueness, multiplicity and other qualitative properties of the solutions for $(S)_\lambda$ are still far from being well understood. For example, we conjecture that no solution exists for $(S)_{\lambda^*}$ with $N \geq 8$ –at least when $f \equiv 1$. In §5 we shall present some numerical evidences for various conjectures relating to the case of power-law permittivity profile $f(x) = |x|^\alpha$ defined in a unit ball. It looks like there are two critical exponents $\alpha^* = -\frac{1}{2} + \frac{1}{2}\sqrt{27/2}$ and $\alpha^{**}(N) = \frac{4-6N+3\sqrt{6}(N-2)}{4}$ (which is relevant for $N \geq 8$) such that the following four regimes are possible:

1. There exist exactly two solutions for $0 < \lambda < \lambda^*$, and one solution for $\lambda = \lambda^*$. This regime occurs when $N = 1$ and $\alpha \leq 1$.

2. There exists exactly one solution for $0 < \lambda < \lambda_1^*$, exactly 2 solutions for $\lambda_1^* < \lambda < \lambda^*$ and exactly one at $\lambda = \lambda^*$. This regime occurs when $N = 1$ and $1 \leq \alpha \leq \alpha^*$.

3. There exists exactly one solution for $0 < \lambda < \lambda_1^*$, exactly two solutions for $\lambda_2^* < \lambda < \lambda^*$, while multiple solutions can be obtained for $\lambda_1^* < \lambda < \lambda_2^*$. Moreover, the multiplicity becomes arbitrarily large as λ approaches another critical value $\lambda_* \in (\lambda_1^*, \lambda_2^*)$, at which there is a touchdown (quenching) solution u characterized with $\| u \|_\infty = 1$. This regime occurs when
   - $2 \leq N \leq 7$ and $\alpha \geq 0$;
   - $N \geq 8$ and $\alpha^{**} < \alpha$.

4. There is exactly one solution if $0 < \lambda < \lambda^*$ and none for $\lambda \geq \lambda^*$. This regime occurs when $N \geq 8$, and $0 \leq \alpha \leq \alpha^{**}$.

We finally mention that the above results can be extended to more general elliptic problems of the form

$$\begin{aligned} -\Delta v &= \frac{\lambda f(x)}{(1-v)^\beta}\,, \quad x \in \Omega\,; \\ v(x) &= 0, \qquad\qquad x \in \partial\Omega \end{aligned} \tag{$(S)_{\lambda,\beta}$}$$

with $\beta > 0$. Here the critical dimension depends on the parameter β, and this is the subject of a work in progress.

2 Pull-In Voltage

In this section, we study the steady-state deflection u which satisfies $(S)_\lambda$, and we establish the existence and some estimates on the pull-in voltage λ^* for $(S)_\lambda$ defined as:

$$\lambda^* = \sup\{\lambda > 0 \mid (S)_\lambda \textit{ possesses at least one solution}\}\,. \tag{2.1}$$

In other words, λ^* is called pull-in voltage if there exist uncollapsed states for $0 < \lambda < \lambda^*$ while there are none of them for $\lambda > \lambda^*$.

Theorem 2.1. *There exists a finite pull-in voltage $\lambda^* > 0$ such that*

1. *If $\lambda < \lambda^*$, there exists at least one solution for $(S)_\lambda$;*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*

Moreover, with ν_Ω defined by (1.6), we have the lower bound

$$\nu_\Omega \Big(\sup_{x\in\Omega} f(x) \Big)^{-1} \le \lambda^*\,. \tag{2.2}$$

Proof: We need to show that $(S)_\lambda$ has at least one solution when $\lambda < \nu_\Omega(\sup_\Omega f(x))^{-1}$. Indeed, it is clear that $u \equiv 0$ is a sub-solution of $(S)_\lambda$ for all $\lambda > 0$. To construct a super-solution of $(S)_\lambda$, we consider a bounded domain $\Gamma \supset \bar\Omega$ with smooth boundary, and let $(\mu_\Gamma, \psi_\Gamma)$ be its first eigenpair normalized in such a way that

$$\sup_{x\in\Gamma} \psi_\Gamma(x) = 1 \text{ and } \inf_{x\in\Omega} \psi_\Gamma(x) := s_1 > 0.$$

We construct a super-solution in the form $\psi = A\psi_\Gamma$ where A is a scalar to be chosen later. First, we must have $A\psi_\Gamma \ge 0$ on $\partial\Omega$ and $0 < 1 - A\psi_\Gamma < 1$ in Ω, which requires that

$$0 < a < 1. \tag{2.3}$$

We also require

$$-\Delta\psi - \frac{\lambda f(x)}{(1-A\psi)^2} \ge 0 \quad \text{in} \quad \Omega\,, \tag{2.4}$$

which can be satisfied as long as:

$$\mu_\Gamma A\ \psi_\Gamma \ge \frac{\lambda \sup_\Omega f(x)}{(1 - A\ \psi_\Gamma)^2} \quad \text{in} \quad \Omega\,, \tag{2.5}$$

or

$$\lambda \sup_\Omega f(x) < \beta(A,\Gamma) := \mu_\Gamma \inf\{g(sA);\ s \in [s_1(\Gamma), 1]\}\,, \tag{2.6}$$

where $g(s) = s(1-s)^2$. In other words, $\lambda^* \sup_\Omega f(x) \ge \sup\{\beta(A,\Gamma);\ 0 < a < 1, \Gamma \supset \bar\Omega\}$, and therefore it remains to show that

$$\nu_\Omega = \sup\{\beta(A,\Gamma);\ 0 < a < 1, \Gamma \supset \bar\Omega\}. \tag{2.7}$$

For that, we note first that

$$\inf_{s\in[s_1,1]} g(As) = \min\big\{g(As_1), g(A)\big\}.$$

We also have that $g(As_1) \le g(A)$ if and only if $A^2(s_1^3-1) - 2A(s_1^2-1) + (s_1-1) \le 0$ which happens if and only if $A^2(s_1^2+s_1+1) - 2A(s_1+1) + 1 \ge 0$ or if and only if either $A \le A_-$ or $A \ge A_+$ where

$$A_+ = \frac{s_1 + 1 + \sqrt{s_1}}{s_1^2+1+s_1} = \frac{1}{s_1+1-\sqrt{s_1}}\,, \quad A_- = \frac{s_1+1-\sqrt{s_1}}{s_1^2+1+s_1} = \frac{1}{s_1+1+\sqrt{s_1}}$$

Since $A_- < 1 < a_+$, we get that

$$G(A) = \inf_{s\in[s_1,1]} g(As) = \begin{cases} g(As_1) & \text{if} \quad 0 \le A \le A_-\,, \\ g(A) & \text{if} \quad A_- \le A \le 1\,. \end{cases} \tag{2.8}$$

We now have that $\frac{dG}{dA} = g'(As_1)s_1 \geq 0$ for all $0 \leq A \leq A_-$. And since $A_- \geq \frac{1}{3}$, we have $\frac{dG}{dA} = g'(A) \leq 0$ for all $A_- \leq A \leq 1$. It follows that

$$\begin{aligned}
\sup_{0<a<1} \inf_{s\in[s_1,1]} g(As) &= \sup_{0<a<1} G(A) = G(A_-) = g(A_-) \\
&= \frac{1}{s_1+1+\sqrt{s_1}}\Big(1-\frac{1}{s_1+1+\sqrt{s_1}}\Big)^2 \\
&= \frac{s_1(s_1+1+2\sqrt{s_1})}{(s_1+1+\sqrt{s_1})^3} \\
&= H(\inf_\Omega \psi_\Gamma))
\end{aligned}$$

which proves our lower estimate.

Now that we know that $\lambda^* > 0$, pick $\lambda \in (0, \lambda^*)$ and use the definition of λ^* to find a $\bar\lambda \in (\lambda, \lambda^*)$ such that $(S)_{\bar\lambda}$ has a solution $u_{\bar\lambda}$, *i.e*,

$$-\Delta u_{\bar\lambda} = \frac{\bar\lambda f(x)}{(1-u_{\bar\lambda})^2}\,, \quad x \in \Omega\,; \qquad u_{\bar\lambda} = 0\,, \quad x \in \partial\Omega\,.$$

and in particular $-\Delta u_{\bar\lambda} \geq \frac{\lambda f(x)}{(1-u_{\bar\lambda})^2}$ for $x \in \Omega$ which then implies that $u_{\bar\lambda}$ is a super-solution of $(S)_\lambda$. Since $u \equiv 0$ is a sub-solution of $(S)_\lambda$, then we can again conclude that there is a solution u_λ of $(S)_\lambda$ for every $\lambda \in (0, \lambda^*)$.

It is also easy to show that λ^* is finite, since if $(S)_\lambda$ has at least one solution $0 < u < 1$, then by integrating against the first (positive) eigenfunction ϕ_Ω, we get

$$+\infty > \mu_\Omega \geq \mu_\Omega \int_\Omega u\phi_\Omega = -\int_\Omega u\Delta\phi_\Omega = -\int_\Omega \phi_\Omega \Delta u = \lambda \int_\Omega \frac{\phi_\Omega f}{(1-u)^2}dx \geq \lambda \int_\Omega \phi_\Omega f dx \tag{2.9}$$

and therefore $\lambda^* < +\infty$. The definition of λ^* implies that there is no solution of $(S)_\lambda$ for any $\lambda > \lambda^*$. ∎

2.1 Lower bounds for λ^*

It is desirable to seek more computationally accessible lower bounds on λ^*. For that we consider for every subset $\Gamma \subset \mathbb{R}^N$ and any function f on Γ such that $0 \leq f \leq 1$, the corresponding pull-in voltage $\lambda^*(\Gamma, f)$, that is the value λ^* defined above for the problem

$$-\Delta u = \frac{\lambda f(x)}{(1-u)^2} \qquad x \in \Gamma\,, \tag{2.10a}$$

$$0 < u < 1 \qquad x \in \Gamma\,, \tag{2.10b}$$

$$u = 0 \qquad x \in \partial\Gamma. \tag{2.10c}$$

We need the following result which can be found in [2] (Theorem 4.1).

Lemma 2.2. *For any bounded domain Γ in $\mathbb{R}^N$ and any function f on Γ such that $0 \leq f \leq 1$, we have*

$$\lambda^*(\Gamma, f) \geq \lambda^*(B_R, f^*)$$

where $B_R = B_R(0)$ is the Euclidean ball in $\mathbb{R}^N$ with radius $R > 0$ and with volume $|B_R| = |\Gamma|$, and where f^* is the Schwarz symmetrization of f.

We now establish the following refined lower bounds for λ^* of $(S)_\lambda$.

Lemma 2.3. *We have the following lower bound for λ^*.*

$$\lambda^* \geq \frac{8N}{27\sup_\Omega f}\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2}{N}}. \tag{2.11}$$

Moreover, if $f(x) \equiv |x|^\alpha$ on Ω with $\alpha \geq 0$, then we have

$$\lambda^* \geq \frac{4(2+\alpha)(N+\alpha)}{27}\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{2+\alpha}{N}}. \tag{2.12}$$

Proof: Setting $R = \left(\frac{|\Omega|}{\omega_N}\right)^{\frac{1}{N}}$, it suffices –in view of Lemma 2.2– and since $\sup\limits_{B_R} f^* = \sup\limits_{\Omega} f$, to show that

$$\lambda^* \geq \frac{8N}{27R^2 \sup_\Omega f}\,, \tag{2.13}$$

for the case where $\Omega = B_R$. In fact, the function $w(x) = \frac{1}{3}(1 - \frac{|x|^2}{R^2})$ satisfies on B_R

$$\begin{aligned} -\Delta w = \frac{2N}{3R^2} &= \frac{2N(1-\frac{1}{3})^2}{3R^2}\frac{1}{(1-\frac{1}{3})^2} \\ &\geq \frac{8N}{27R^2 \sup_\Omega f}\frac{f(x)}{[1-\frac{1}{3}(1-\frac{|x|^2}{R^2})]^2} \\ &= \frac{8N}{27R^2 \sup_\Omega f}\frac{f(x)}{(1-w)^2}\,. \end{aligned}$$

So for $\lambda \leq \frac{8N}{27R^2 \sup_\Omega f}$, w is a super-solution of $(S)_\lambda$ in B_R. Since on the other hand $w_0 \equiv 0$ is a sub-solution of $(S)_\lambda$ and $w_0 \leq w$ in B_R, then there exists a solution of $(S)_\lambda$ in B_R which proves (2.13) and hence (2.11).

In order to prove (2.12), it suffices to note that $w(x) = \frac{1}{3}\big(1 - \frac{|x|^{2+\alpha}}{R^{2+\alpha}}\big)$ is a super- solution for $(S)_\lambda$ on B_R provided $\lambda \leq \frac{4(2+\alpha)(N+\alpha)}{27R^{2+\alpha}}$. This completes the proof of Lemma 2.3. ■

2.2 Upper bounds for λ^*

We note that (2.9) already yields a finite upper bound for λ^*. However, Pohozaev-type arguments such as the one used in [14] can be used to establish better and more computable upper bounds. In this subsection, we establish these estimates and hence complete the proof of Theorem 1.1.

We shall consider problem $(S)_\lambda$ in the case where $\Omega \subset \mathbb{R}^N$ is a strictly star-shaped domain containing 0, meaning that Ω satisfies the additional property that there exists a positive constant a such that

$$x \cdot \nu \geq a \int_{\partial\Omega} ds \quad \text{for all} \quad x \in \partial\Omega\,, \tag{2.14}$$

where ν is the unit outer normal to Ω at $x \in \partial\Omega$.

Proposition 2.4. *Suppose $f \equiv 1$ and that the strictly star-shaped domain $\Omega \subset \mathbb{R}^N$ satisfies* (2.14). *Then the pull-in voltage $\lambda^*(\Omega)$ satisfies:*

$$\lambda^*(\Omega) \leq \bar{\lambda}_3 = \frac{(N+2)^2}{8aN|\Omega|}\,. \tag{2.15}$$

In particular, if $\Omega = B_1(0) \subset \mathbb{R}^N$ then we have the bound

$$\lambda^*(B_1(0)) \leq \frac{(N+2)^2}{8}\,.$$

Proof: Recall the well-known Pohozaev's identity: If u is a solution of

$$\begin{aligned} \Delta u + \lambda g(u) &= 0 \qquad \text{for} \quad x \in \Omega\,, \\ u &= 0 \qquad \text{for} \quad x \in \partial\Omega\,, \end{aligned}$$

then

$$N\lambda \int_\Omega G(u)dx - \frac{N-2}{2}\lambda \int_\Omega ug(u)dx = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\big(\frac{\partial u}{\partial \nu}\big)^2 ds\,, \tag{2.16}$$

where $G(u) = \int_0^u g(s)ds$. Applying it with $g(u) = \frac{1}{(1-u)^2}$ and $G(u) = \frac{u}{1-u}$, it yields

$$\begin{aligned}
\frac{\lambda}{2} \int_\Omega \frac{u(N+2-2Nu)}{(1-u)^2} dx &= \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \Big(\frac{\partial u}{\partial \nu} \Big)^2 ds \\
&\geq \frac{a}{2} \Big(\int_{\partial\Omega} \frac{\partial u}{\partial \nu} ds \Big)^2 \\
&= \frac{a}{2} \Big(- \int_\Omega \Delta u dx \Big)^2 \\
&= \frac{a\lambda^2}{2} \Big(\int_\Omega \frac{dx}{(1-u)^2} \Big)^2 ,
\end{aligned} \tag{2.17}$$

where we have used the Divergence Theorem and Holder's inequality

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} ds \leq \Big(\int_{\partial\Omega} \big(- \frac{\partial u}{\partial \nu} \big)^2 ds \Big)^{1/2} \Big(\int_{\partial\Omega} ds \Big)^{1/2} .$$

Since

$$\begin{aligned}
\int_\Omega \frac{u(N+2-2Nu)}{(1-u)^2} dx &= \int_\Omega \Big[-2N \big(u - \frac{N+2}{4N} \big)^2 + \frac{(N+2)^2}{8N} \Big] \frac{1}{(1-u)^2} dx \\
&\leq \frac{(N+2)^2}{8N} \int_\Omega \frac{dx}{(1-u)^2} ,
\end{aligned}$$

we deduce from (2.17) that

$$\frac{(N+2)^2}{8N} \geq a\lambda \int_\Omega \frac{dx}{(1-u)^2} \geq a\lambda |\Omega| ,$$

which implies the upper bound (2.15) for λ^*.

Finally, for the special case where $\Omega = B_1(0) \subset \mathbb{R}^N$, we have $a = \frac{1}{N\omega_N}$ with $\omega_N = |B_1(0)|$, and hence the bound $\lambda^* \leq \bar{\lambda}_3 = \frac{(N+2)^2}{8}$. ■

For a general domain Ω, the following upper bounds on $\lambda^*(\Omega)$ established in [19] and [14] respectively, complete the proof of Theorem 1.1.

Proposition 2.5. *(1) If f satisfies $0 < C_0 \leq f(x) \leq 1$ on Ω, then*

$$\lambda^* \leq \bar{\lambda}_1 \equiv \frac{4\mu_\Omega}{27C_0} . \tag{2.18}$$

(2) If f satisfies $0 \leq f(x) \leq 1$ on Ω, and if $f > 0$ on a set of positive measure, then

$$\lambda_* \leq \bar{\lambda}_2 \equiv \frac{\mu_\Omega}{3} \left(\int_\Omega f \phi_\Omega \, dx \right)^{-1} . \tag{2.19}$$

Here μ_Ω and ϕ_Ω are the first eigenpair of $-\Delta$ on $H_0^1(\Omega)$ with $\int_\Omega \phi_\Omega dx = 1$.

2.3 Numerical estimates for λ^*

In the computations below we shall consider two choices for the domain Ω,

$$\Omega : [-1/2, 1/2] \quad (\text{Slab}) ; \quad \Omega : x^2 + y^2 \leq 1 \quad (\text{Unit Disk}) . \tag{2.20}$$

To compute the bounds $\bar{\lambda}_1$ and $\bar{\lambda}_2$, we must calculate the first eigenpair μ_Ω and ϕ_Ω of $-\Delta$ on Ω, normalized by $\int_\Omega \phi_\Omega \, dx = 1$, for each of these domains. A simple calculation yields that

$$\mu_\Omega = \pi^2 , \qquad \phi_\Omega = \frac{\pi}{2} \sin \left[\pi \left(x + \frac{1}{2} \right) \right] , \quad (\text{Slab}) ; \tag{2.21a}$$

$$\mu_\Omega = z_0^2 \approx 5.783 , \qquad \phi_\Omega = \frac{z_0}{J_1(z_0)} J_0(z_0 |x|) , \quad (\text{Unit Disk}) . \tag{2.21b}$$

Exponential Profiles:

Ω	α	$\underline{\lambda}$	λ^*	$\bar{\lambda}_1$	$\bar{\lambda}_2$
(Slab)	0	1.185	1.401	1.462	3.290
(Slab)	1.0	1.185	1.733	1.878	4.023
(Slab)	3.0	1.185	2.637	3.095	5.965
(Slab)	6.0	1.185	4.848	6.553	10.50
(Unit Disk)	0	0.593	0.789	0.857	1.928
(Unit Disk)	0.5	0.593	1.153	1.413	2.706
(Unit Disk)	1.0	0.593	1.661	2.329	3.746
(Unit Disk)	3.0	0.593	6.091	17.21	11.86

Table 1: *Numerical values for pull-in voltage λ^* with the bounds given in Theorem 1.1. Here the exponential permittivity profile is chosen as (2.22).*

Power-Law Profiles:

Ω	α	$\lambda_c(\alpha)$	λ^*	$\bar{\lambda}_1$	$\bar{\lambda}_2$
(Slab)	0	1.185	1.401	1.462	3.290
(Slab)	1.0	3.556	4.388	∞	9.044
(Slab)	3.0	11.851	15.189	∞	28.247
(Slab)	6.0	33.185	43.087	∞	76.608
(Unit Disk)	0	0.593	0.789	0.857	1.928
(Unit Disk)	1.0	1.333	1.775	∞	3.019
(Unit Disk)	5.0	7.259	9.676	∞	15.82
(Unit Disk)	20	71.70	95.66	∞	161.54

Table 2: *Numerical values for pull-in voltage λ^* with the bounds given in Theorem 1.1. Here the power-law permittivity profile is chosen as (2.22).*

Here J_0 and J_1 are Bessel functions of the first kind, and $z_0 \approx 2.4048$ is the first zero of $J_0(z)$. The bounds $\bar{\lambda}_1$ and $\bar{\lambda}_2$ can be evaluated by substituting (2.21) into (2.18) and (2.19). Notice that $\bar{\lambda}_2$ is, in general, determined only up to a numerical quadrature.

Using Newton's method and COLSYS [1], one can also solve the boundary value problem $(S)_\lambda$ and numerically calculate λ^* as the saddle-node point for the following two choices of the permittivity profile:

$$\text{(Slab)}: \quad f(x) = |2x|^\alpha\,, \quad \text{(power-law)}\,; \quad f(x) = e^{\alpha(x^2-1/4)} \quad \text{(exponential)}\,, \tag{2.22a}$$

$$\text{(Unit Disk)}: \quad f(x) = |x|^\alpha\,, \quad \text{(power-law)}\,; \quad f(x) = e^{\alpha(|x|^2-1)}\,, \quad \text{(exponential)}\,, \tag{2.22b}$$

where $\alpha \geq 0$. Table 1 contains numerical values for λ^* in the case of exponential profiles, while Table 2 deals with power-law profiles. What is remarkable is that $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are not comparable even when f is bounded away from 0 and that neither one of them provides the optimal value for λ^*. This leads us to conjecture that there should be a better estimate for λ^*, one involving the distribution of f in Ω, as opposed to the infimum or its average against the first eigenfunction ϕ_Ω.

3 Minimal Positive Solutions

In this section, we are concerned with *minimal positive solutions* for $(S)_\lambda$. We establish their existence, uniqueness and other related properties. We consider the case $\lambda \in (0, \lambda^*)$ in §3.1, and $\lambda = \lambda^*$ in §3.3, but first we give a recursive scheme for the construction of such solutions.

Theorem 3.1. *For any $\lambda < \lambda^*$ there exists a unique minimal positive solution u_λ for $(S)_\lambda$. It is obtained* *as the limit of the sequence $\{u_n(\lambda;x)\}$ constructed recursively as follows: $u_0 \equiv 0$ in Ω and for each $n \geq 1$,*

$$\begin{aligned} -\Delta u_n &= \frac{\lambda f(x)}{(1-u_{n-1})^2}\,, && x \in \Omega\,; \\ 0 \leq\, u_n &< 1\,, && x \in \Omega\,; \\ u_n &= 0\,, && x \in \partial\Omega. \end{aligned} \tag{3.1}$$

Proof: Let u be any positive solution for $(S)_\lambda$, and consider the sequence $\{u_n(\lambda;x)\}$ defined in (3.1). Clearly $u(x) > u_0 \equiv 0$ in Ω, and whenever $u(x) \geq u_{n-1}$ in Ω, then

$$\begin{aligned} -\Delta(u-u_n) &= \lambda f(x)\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_{n-1})^2}\Big] \geq 0\,, \quad x \in \Omega \\ u - u_n &= 0\,, \qquad x \in \partial\Omega\,. \end{aligned}$$

The maximum principle and an immediate induction yield that $1 > u(x) \geq u_n$ in Ω for all $n \geq 0$. In a similar way, the maximum principle implies that the sequence $\{u_n(\lambda;x)\}$ is monotone increasing. Therefore, $\{u_n(\lambda;x)\}$ converges uniformly to a positive solution $u_\lambda(x)$, satisfying $u(x) \geq u_\lambda(x)$ in Ω, which is a minimal positive solution of $(S)_\lambda$. It is also clear that $u_\lambda(x)$ is unique. ■

Remark 3.1. Let $g(x,\xi,\Omega)$ be the Green's function of Laplace operator, with $g(x,\xi,\Omega) = 0$ on $\partial\Omega$. Then the iteration in (3.1) can be replaced by: $u_0 \equiv 0$ in Ω and for each $n \geq 1$,

$$\begin{aligned} u_n(\lambda;x) &= \lambda \int_\Omega \frac{f(\xi)g(x,\xi,\Omega)}{(1-u_{n-1}(\lambda;\xi))^2}d\xi\,, && x \in \Omega\,; \\ u_n(\lambda;x) &= 0\,, && x \in \partial\Omega\,. \end{aligned} \tag{3.2}$$

The same reasoning as above yields that $\lim_{n\to\infty} u_n(\lambda;x) = u_\lambda(x)$ for all $x \in \Omega$.

The above construction of solutions yields the following monotonicity result for the pull-in voltage.

Proposition 3.2. *If $\Omega_1 \subset \Omega_2$, then $\lambda^*(\Omega_1) \geq \lambda^*(\Omega_2)$ and the corresponding minimal solutions satisfy $u_{\Omega_1}(\lambda,x) \leq u_{\Omega_2}(\lambda,x)$ on Ω_1 for every $0 < \lambda < \lambda^*(\Omega_2)$.*

Proof: Again the method of sub/super-solutions immediatly yields that $\lambda^*(\Omega_1) \geq \lambda^*(\Omega_2)$. Now consider for $i = 1,2$, the sequences $\{u_n(\lambda,x,\Omega_i)\}$ on Ω_i defined by (3.2) where $g(x,\xi,\Omega_i)$ are the corresponding Green's functions on Ω_i. Since $\Omega_1 \subset \Omega_2$, we have that $g(x,\xi,\Omega_1) \leq g(x,\xi,\Omega_2)$ on Ω_1. Hence, it follows that

$$u_1(\lambda,x,\Omega_2) = \lambda \int_{\Omega_2} f(\xi)g(x,\xi,\Omega_2)d\xi \geq \lambda \int_{\Omega_1} f(\xi)g(x,\xi,\Omega_1)d\xi = u_1(\lambda,x,\Omega_1)$$

on Ω_1. By induction we conclude that $u_n(\lambda,x,\Omega_2) \geq u_n(\lambda,x,\Omega_1)$ on Ω_1 for all n. On the other hand, since $u_n(\lambda,x,\Omega_2) \leq u_{n+1}(\lambda,x,\Omega_2)$ on Ω_2 for n, we get that $u_n(\lambda,x,\Omega_1) \leq u_{\Omega_2}(\lambda,x)$ on Ω_1, and we are done. ■

3.1 Spectral properties of minimal solutions

For a further study of minimal (positive) solutions, we now consider for each positive solution u of $(S)_\lambda$, the operator

$$L_{u,\lambda} = -\Delta - \frac{2\lambda f}{(1-u)^3} \tag{3.3}$$

associated to the linearized problem around u. We see that minimal solutions in the above sense correspond to variational solutions that are local minimizers. We denote by $\mu(\lambda,u)$ the smallest eigenvalue of $L_{u,\lambda}$, that is the one corresponding to the following Dirichlet eigenvalue problem

$$-\Delta\phi - \frac{2\lambda f(x)}{(1-u)^3}\phi = \mu(\lambda,u)\phi\,, \quad x \in \Omega\,; \tag{3.4a}$$

$$\phi = 0 \qquad x \in \partial\Omega\,. \tag{3.4b}$$

In other words,

$$\mu(\lambda, u) = \inf_{\phi \in H_0^1(\Omega)} \frac{\int_\Omega \{|\nabla\phi|^2 - 2\lambda f(1-u)^{-3}\phi^2\}dx}{\int_\Omega \phi^2 dx}.$$

Proposition 3.3. *The following hold:*

1. $\lambda^* = \sup\{\lambda;\ L_{u_\lambda,\lambda}$ *has positive first eigenvalue for minimal solution* u_λ *of* $(S)_\lambda\}$.
2. *If* $0 < \lambda < \lambda^*$ *then the smallest eigenvalue* $\mu_\lambda := \mu(\lambda, u_\lambda)$ *of* $L_{u_\lambda,\lambda}$ *–corresponding to the minimal solution* u_λ*– is positive and* $\lambda \to \mu_\lambda$ *is decreasing on* $(0, \lambda^*)$.

For Proposition 3.3, we need the following crucial lemma.

Lemma 3.4. *Suppose* u *is a positive solution of* $(S)_\lambda$*, and let* $\mu(\lambda, u)$ *be the corresponding first eigenvalue. Consider any -classical- supersolution* v *of* $(S)_\lambda$*, that is*

$$-\Delta v \geq \frac{\lambda f(x)}{(1-v)^2} \qquad x \in \Omega, \tag{3.5a}$$

$$0 \leq v(x) < 1 \qquad x \in \Omega \tag{3.5b}$$

$$v = 0 \qquad x \in \partial\Omega. \tag{3.5c}$$

If $\mu(\lambda, u) > 0$ *then* $v \geq u$ *on* Ω*, and if* $\mu(\lambda, u) = 0$ *then* $v = u$ *on* Ω.

Proof: For a given λ and $x \in \Omega$, use the fact that $f(x) \geq 0$ and that $t \to \frac{\lambda f(x)}{(1-t)^2}$ is convex on $(0, 1)$, to obtain

$$-\Delta(u + \tau(v-u)) - \frac{\lambda f(x)}{[1 - (u + \tau(v-u))]^2} \geq 0 \quad x \in \Omega, \tag{3.6}$$

for $\tau \in [0, 1]$. Note that (3.6) is an identity at $\tau = 0$, which means that the first derivative of the left side for (3.6) with respect to τ is nonnegative at $\tau = 0$, *i.e*,

$$-\Delta(v-u) - \frac{2\lambda f(x)}{(1-u)^3}(v-u) \geq 0 \qquad x \in \Omega, \tag{3.7a}$$

$$v - u = 0 \qquad x \in \partial\Omega. \tag{3.7b}$$

Thus, the maximal principle implies that if $\mu(\lambda, u) > 0$ we have $v \geq u$ on Ω, while if $\mu(u) = 0$ we have

$$-\Delta(v-u) - \frac{2\lambda f(x)}{(1-u)^3}(v-u) = 0 \qquad x \in \Omega. \tag{3.8}$$

In the latter case the second derivative of the left side for (3.6) with respect to τ is nonnegative a $\tau = 0$ again, *i.e*,

$$-\frac{6\lambda f(x)}{(1-u)^4}(v-u)^2 \geq 0 \quad x \in \Omega, \tag{3.9}$$

From (3.9) we deduce that $v \equiv u$ in $\Omega \setminus \Omega_0$, where

$$\Omega_0 = \{x \in \Omega :\ f(x) = 0 \text{ for } x \in \Omega\}. \tag{3.10}$$

On the other hand, (3.8) reduces to

$$-\Delta(v-u) = 0 \quad x \in \Omega_0,$$
$$v - u = 0 \quad x \in \partial\Omega_0,$$

which implies $v \equiv u$ on Ω_0. Hence if $\mu(\lambda, u) = 0$ then $v \equiv u$ on Ω, which completes the proof of Lemma 3.4. ■

Proof of Proposition 3.3: (1) Let

$$\lambda^{**} = \sup\{\lambda;\ L_{u_\lambda,\lambda} \text{ has positive first eigenvalue for minimal solution } u_\lambda \text{ of } (S)_\lambda\}.$$

It is clear that $\lambda^{**} \le \lambda^*$, so it suffices to prove that there is no minimal solution for $(S)_\mu$ with $\mu > \lambda^{**}$. In fact, suppose w is a minimal solution of $(S)_{\lambda^{**}+\delta}$ with $\delta > 0$, then we would have for $\lambda \le \lambda^{**}$,

$$-\Delta w = \frac{(\lambda^{**}+\delta)f(x)}{(1-w)^2} \ge \frac{\lambda f(x)}{(1-w)^2} \quad x \in \Omega\,.$$

Since the minimal solutions u_λ satisfy $-\Delta u_\lambda = \frac{\lambda f(x)}{(1-u_\lambda)^2} \quad x \in \Omega$ for all $0 < \lambda < \lambda^{**}$, it follows from Lemma 3.4 that $1 > w \ge u_\lambda$ for all $0 < \lambda < \lambda^{**}$. Consequently, $\underline{u} = \lim_{\lambda \nearrow \lambda^{**}} u_\lambda$ would exist. Now from the definition of λ^{**} and Lemma 3.4, we must have $w \equiv \underline{u}$ and $\delta = 0$ on Ω which is a contradiction, and hence $\lambda^{**} = \lambda^*$.

(2) From the first part we conclude that if $0 < \lambda < \lambda^*$ and $u = u_\lambda$, then the smallest eigenvalue of $-\Delta - \frac{2\lambda f(x)}{(1-u)^3}$ is positive. Applying the maximum principle, it is easy to show that $u_\lambda(x)$ is increasing with respect to λ (More details can be found in the proof of Theorem 1.2(1) below). That μ_λ is decreasing with respect to λ follows now easily from the variational characterization of μ_λ and the convexity of $(1-u)^{-3}$ with respect to u. ■

Remark 3.2. For the case where $f(x) > 0$ on Ω, Lemma 3 of [7] gives $\mu(1,0)$ as an upper bound for $\lambda^{**}(=\lambda^*)$. It is worth noting that our upper bound $\bar{\lambda}$ in Theorem 1.1 gives a better estimate. Indeed,if $f \equiv 1$ then $\mu(1,0) = \mu_\Omega/2$ while the estimate in Theorem 1.1 gives $\frac{4\mu_\Omega}{27}$ for an upper bound.

Proof of Theorem 1.2(1): By Theorem 3.1, it suffices to prove that for each $x \in \Omega$, the function $\lambda \to u_\lambda(x)$ is differentiable and strictly increasing on $(0,\lambda^*)$. Setting $F(\lambda, u_\lambda(x)) = -\Delta u_\lambda - \frac{\lambda f(x)}{(1-u_\lambda)^2}$, Proposition 3.3 then implies that $F_{u_\lambda}(\lambda, u_\lambda)$ on Ω is invertible for $0 < \lambda < \lambda^*$. It then follows from the Implicit Function Theorem that $u_\lambda(x)$ is differentiable with respect to λ.

Consider now for $\lambda_1 < \lambda_2 < \lambda^*$, their corresponding minimal positive solutions u_{λ_1} and u_{λ_2} and let u^* be a positive solution for $(S)_{\lambda_2}$. For the monotone increasing series $\{u_n(\lambda_1;x)\}$ defined in (3.1), we then have $u^* > u_0(\lambda_1;x) \equiv 0$, and if $u_{n-1}(\lambda_1;x) \le u^*$ in Ω, then

$$\begin{aligned}
-\Delta(u^* - u_n) &= f(x)\big[\frac{\lambda_2}{(1-u^*)^2} - \frac{\lambda_1}{(1-u_{n-1})^2}\big] \ge 0\,, \quad x \in \Omega \\
u^* - u_n &= 0\,, \qquad x \in \partial\Omega\,.
\end{aligned}$$

So we have $u_n(\lambda_1;x) \le u^*$ in Ω. Therefore, $u_{\lambda_1} = \lim_{n\to\infty} u_n(\lambda_1;x) \le u^*$ in Ω, and in particular $u_{\lambda_1} \le u_{\lambda_2}$ in Ω. Therefore, $\frac{du_\lambda(x)}{d\lambda} \ge 0$ for all $x \in \Omega$.

Finally, by differentiating $(S)_\lambda$ with respect to λ we get

$$\begin{aligned}
-\Delta\frac{du_\lambda}{d\lambda} - \frac{2\lambda f(x)}{(1-u_\lambda)^3}\frac{du_\lambda}{d\lambda} &= \frac{f(x)}{(1-u_\lambda)^2} \ge 0\,, \quad x \in \Omega \\
\frac{du_\lambda}{d\lambda} &\ge 0\,, \qquad x \in \partial\Omega\,.
\end{aligned}$$

Applying the strong maximum principle, we conclude that $\frac{du_\lambda}{d\lambda} > 0$ on Ω for all $0 < \lambda < \lambda^*$. ■

3.2 Energy estimates and regularity

We start with the following easy observation.

Lemma 3.5. *Any positive (weak) solution u in $H_0^1(\Omega)$ of $(S)_\lambda$ satisfies $\int_\Omega \frac{f}{(1-u)^2}dx < \infty$.*

Proof: Since $u \in H_0^1(\Omega)$ is a positive solution of $(S)_\lambda$, we have

$$\int_\Omega \frac{f}{(1-u)^2} - \int_\Omega \frac{f}{1-u} = \int_\Omega \frac{uf}{(1-u)^2} = \int_\Omega |\nabla u|^2 < C\,,$$

which implies that

$$\int_\Omega \frac{f}{(1-u)^2} \le C + \int_\Omega \frac{f}{1-u} \le C + \int_\Omega \big[C\varepsilon\frac{f}{(1-u)^2} + \frac{C}{\varepsilon}f\big] \le C + C\varepsilon\int_\Omega \frac{f}{(1-u)^2}$$

with $\varepsilon > 0$. Therefore, by choosing $\varepsilon > 0$ small enough, we conclude that $\int_\Omega \frac{f}{(1-u)^2} < \infty$. ■

That $f/(1-u) \in L^2(\Omega)$ is unfortunately not sufficient to obtain regularity results for the solutions. However, we now show that the situation is much better if $f/(1-u)$ has better integrability properties.

Theorem 3.6. *For any bounded domain $\Omega \subset \mathbb{R}^N$ and any constant $C > 0$ there exists $0 < K(C,N) < 1$ such that a positive weak solution u of $(S)_\lambda$ $(0 < \lambda < \lambda^*)$ is a classical solution and $\| u \|_{C(\Omega)} \le K(C,N)$ provided one of the following conditions holds:*

1. *$N = 1$ and $\|\frac{f}{(1-u)^3}\|_{L^1(\Omega)} \le C$.*
2. *$N = 2$ and $\|\frac{f}{(1-u)^3}\|_{L^{1+\epsilon}(\Omega)} \le C$ for some $\epsilon > 0$.*
3. *$N > 2$ and $\|\frac{f}{(1-u)^3}\|_{L^{N/2}(\Omega)} \le C$.*

Proof: We prove this lemma by considering the following three cases separately:

(1) If $N = 1$, then for any $I > 0$ we write using the Sobolev inequality with constant $K(1) > 0$,

$$\begin{aligned} K(1) \parallel (1-u)^{-1} - 1 \parallel^2_{L^\infty} &\le \int_\Omega \left|\nabla[(1-u)^{-1} - 1]\right|^2 \\ &= \frac{1}{3}\int_\Omega \nabla u \cdot \nabla\big[(1-u)^{-3} - 1\big] \\ &= \frac{\lambda}{3}\int_\Omega f(1-u)^{-2}[(1-u)^{-3} - 1] \\ &\le CI + C\int_{\{(1-u)^{-3}\ge I\}} f(1-u)^{-5} \\ &\le CI + C\int_{\{(1-u)^{-3}\ge I\}} 8f(1-u)^{-2} \\ &\quad + C\int_{\{(1-u)^{-3}\ge I\}} f\big[(1-u)^{-3} + 2(1-u)^{-2} + 4(1-u)^{-1}\big]\big[(1-u)^{-1} - 1\big]^2 \\ &\le CI + C + C \parallel (1-u)^{-1} - 1 \parallel^2_{L^\infty(\{(1-u)^{-3}\ge I\})} \int_{\{(1-u)^{-3}\ge I\}} \frac{f}{(1-u)^3} \\ &\le CI + C + C\varepsilon(I) \parallel (1-u)^{-1} - 1 \parallel^2_{L^\infty} \end{aligned} \tag{3.11}$$

with $\varepsilon(I) = \displaystyle\int_{\{(1-u)^{-3}\ge I\}} \frac{f}{(1-u)^3}$. From the assumption $f/(1-u)^3 \in L^1(\Omega)$, we have $\varepsilon(I) \to 0$ as $I \to \infty$. We now choose I such that $\varepsilon(I) \le \frac{K(1)}{2C}$, so that the above estimates imply that $\parallel (1-u)^{-1} - 1 \parallel_{L^\infty} < K(C)$. Standard regularity theory for elliptic problems now imply that $1/(1-u) \in C^{2,\alpha}(\Omega)$. Therefore, u is classical and there exists a constant $K(C,N)$ which can be taken strictly less than 1 such that $\| u \|_{C(\Omega)} \le K(C,N) < 1$.

(2) The case when $N = 2$ is similar as one can use that H_0^1 embeds in L^p for any $p < +\infty$.

(3) The case when $N > 2$ is more elaborate and we first show that $(1-u)^{-1} \in L^q(\Omega)$ for all $q \in (1,\infty)$. Since $u \in H_0^1(\Omega)$ is a solution of $(S)_\lambda$, we already have $\int_\Omega \frac{f}{(1-u)^2} < C$. Now we proceed by iteration to show that if $\int_\Omega \frac{f}{(1-u)^{2+2\theta}} < C$ for some $\theta \ge 0$, then $\int_\Omega \frac{1}{(1-u)^{2^*(1+\theta)}} < C$.

Indeed, for any constant $\theta \ge 0$ and $\ell > 0$ we choose a test function $\phi = [(1-u)^{-3} - 1]\min\{(1-u)^{-2\theta}, \ell^2\}$. By applying this test function to both sides of $(S)_\lambda$, we have

$$\begin{aligned} &\lambda\int_\Omega f(1-u)^{-2}[(1-u)^{-3} - 1]\min\{(1-u)^{-2\theta}, \ell^2\} = \int_\Omega \nabla u \cdot \nabla\big[\big((1-u)^{-3} - 1\big)\min\{(1-u)^{-2\theta}, \ell^2\}\big] \\ &= 3\int_\Omega |\nabla u|^2(1-u)^{-4}\min\{(1-u)^{-2\theta}, \ell^2\} + 2\theta\int_{\{(1-u)^{-\theta}\le\ell\}} |\nabla u|^2(1-u)^{-2\theta-1}[(1-u)^{-3} - 1]\,. \end{aligned} \tag{3.12}$$

We now suppose $\int_\Omega \frac{f}{(1-u)^{2+2\theta}} < C$. We then obtain from (3.12) and the fact that $\frac{1}{(1-u)^5} \leq C_I \frac{1}{(1-u)^3}(\frac{1}{1-u}-1)^2$ for $(1-u)^{-3} \geq I > 1$ that

$$
\begin{aligned}
&\int_\Omega \big|\nabla[\big((1-u)^{-1}-1\big)\min\{(1-u)^{-\theta},\ell\}]\big|^2 \\
&\leq 2\int_\Omega |\nabla u|^2(1-u)^{-4}\min\{(1-u)^{-2\theta},\ell^2\} + 2\theta^2\int_{\{(1-u)^{-\theta}\leq\ell\}} |\nabla u|^2(1-u)^{-2\theta-2}\big[(1-u)^{-1}-1\big]^2 \\
&= 2\int_\Omega |\nabla u|^2(1-u)^{-4}\min\{(1-u)^{-2\theta},\ell^2\} \\
&\quad +2\theta^2\int_{\{(1-u)^{-\theta}\leq\ell\}} |\nabla u|^2(1-u)^{-2\theta-1}\big[(1-u)^{-3}-1+1+(1-u)^{-1}-2(1-u)^{-2}\big] \\
&\leq C\lambda\int_\Omega f(1-u)^{-2}[(1-u)^{-3}-1]\min\{(1-u)^{-2\theta},\ell^2\} \\
&\leq C\lambda\int_\Omega f(1-u)^{-5}\min\{(1-u)^{-2\theta},\ell^2\} \qquad (3.13)\\
&\leq CI + C\int_{\{(1-u)^{-3}\geq I\}} f(1-u)^{-5}\min\{(1-u)^{-2\theta},\ell^2\} \\
&\leq CI + C\int_{\{(1-u)^{-3}\geq I\}} f(1-u)^{-3}\big[(1-u)^{-1}-1\big]^2\min\{(1-u)^{-2\theta},\ell^2\} \\
&\leq CI + C\Big[\int_{\{(1-u)^{-3}\geq I\}} \Big(\frac{f}{(1-u)^3}\Big)^{\frac{N}{2}}\Big]^{\frac{2}{N}} \\
&\quad\times\Big[\int_{\{(1-u)^{-3}\geq I\}} \big(\big[(1-u)^{-1}-1\big]\min\{(1-u)^{-\theta},\ell\}\big)^{\frac{2N}{N-2}}\Big]^{\frac{N-2}{N}} \\
&\leq CI + C\varepsilon(I)\int_\Omega \big|\nabla[\big((1-u)^{-1}-1\big)\min\{(1-u)^{-\theta},\ell\}]\big|^2
\end{aligned}
$$

with

$$\varepsilon(I) = \Big[\int_{\{(1-u)^{-3}\geq I\}} \Big(\frac{f}{(1-u)^3}\Big)^{\frac{N}{2}}\Big]^{\frac{2}{N}}.$$

From the assumption $f/(1-u)^3 \in L^{\frac{N}{2}}(\Omega)$ we have $\varepsilon(I) \to 0$ as $I \to \infty$. We now choose I such that $\varepsilon(I) = \frac{1}{2C}$, and the above estimates imply that

$$\int_{\{(1-u)^{-\theta}\leq\ell\}} \big|\nabla[(1-u)^{-\theta-1}-(1-u)^{-\theta}]\big|^2 \leq CI\,,$$

where the bound is uniform with respect to ℓ. This estimate leads to

$$
\begin{aligned}
\tfrac{1}{(\theta+1)^2}\textstyle\int_{\{(1-u)^{-\theta}\leq\ell\}} \big|\nabla[(1-u)^{-\theta-1}]\big|^2 &= \int_{\{(1-u)^{-\theta}\leq\ell\}} (1-u)^{-2\theta-4}\big|\nabla u\big|^2 \\
&\leq CI + C\int_{\{(1-u)^{-\theta}\leq\ell\}} (1-u)^{-2\theta-3}\big|\nabla u\big|^2 \\
&\leq CI + \int_{\{(1-u)^{-\theta}\leq\ell\}} \big[C\varepsilon(1-u)^{-2\theta-4}+C/\varepsilon\big]\big|\nabla u\big|^2 \\
&\leq CI + C\varepsilon\int_{\{(1-u)^{-\theta}\leq\ell\}} (1-u)^{-2\theta-4}\big|\nabla u\big|^2
\end{aligned}
$$

with $\varepsilon > 0$. This means that for $\varepsilon > 0$ sufficiently small

$$\int_{\{(1-u)^{-\theta}\leq\ell\}} \big|\nabla(1-u)^{-\theta-1}\big|^2 = \int_{\{(1-u)^{-\theta}\leq\ell\}} (\theta+1)^2(1-u)^{-2\theta-4}\big|\nabla u\big|^2 < C\,.$$

So we can let $\ell \to \infty$ and we get that $(1-u)^{-\theta-1} \in H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, which means that $\int_\Omega \frac{1}{(1-u)^{2^*(1+\theta)}} < C$.

By iterating the above argument for $\theta_i + 1 = \frac{N}{N-2}(\theta_{i-1}+1)$ for $i \geq 1$ and starting with $\theta_0 = 0$, we find that $1/(1-u) \in L^q(\Omega)$ for all $q \in (1,\infty)$.

Standard regularity theory for elliptic problems applies again to give that $1/(1-u) \in C^{2,\alpha}(\Omega)$. Therefore, u is a classical solution and there exists a constant $0 < K(C,N) < 1$ such that $\| u \|_{C(\Omega)} \leq K(C,N) < 1$. This completes the proof of Theorem 3.6. ■

Theorem 3.7. *For any dimension* $1 \leq N < 8$, *there exists a constant* $0 < C(N) < 1$ *independent of* λ *such that for any* $0 < \lambda < \lambda^*$, *the minimal solution* u_λ *satisfies* $\| u_\lambda \|_{C(\Omega)} \leq C(N)$.

The theorem, which gives Theorem 1.2(2), will follow from the following uniform energy estimate on the minimal solutions u_λ.

Lemma 3.8. *There exists a constant* $C(p) > 0$ *such that for each* $\lambda \in (0,\lambda^*)$, *the minimal solution* u_λ *satisfies* $\|\frac{f}{(1-u_\lambda)^3}\|_{L^p(\Omega)} \leq C(p)$ *as long as* $p < 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}$.

Proof: Proposition 3.3 implies that

$$\lambda \int_\Omega \frac{2f(x)}{(1-u_\lambda)^3} w^2 dx \leq -\int_\Omega w\Delta w dx = \int_\Omega |\nabla w|^2 dx\,, \tag{3.14}$$

for all $0 < \lambda < \lambda^*$ and nonnegative $w \in H_0^1(\bar{\Omega})$. Setting

$$w = (1-u_\lambda)^i - 1 > 0\,,\ \textit{where} \quad -2-\sqrt{6} < i < 0\,, \tag{3.15}$$

then (3.14) becomes

$$i^2 \int_\Omega (1-u_\lambda)^{2i-2} |\nabla u_\lambda|^2 dx \geq \lambda \int_\Omega \frac{2[1-(1-u_\lambda)^i]^2 f(x)}{(1-u_\lambda)^3} dx\,. \tag{3.16}$$

On the other hand, multiplying $(S)_\lambda$ by $\frac{i^2}{1-2i}[(1-u_\lambda)^{2i-1} - 1]$ and applying integration by parts yield that

$$i^2 \int_\Omega (1-u_\lambda)^{2i-2} |\nabla u_\lambda|^2 dx = \lambda \frac{i^2}{2i-1} \int_\Omega \frac{[1-(1-u_\lambda)^{2i-1}]f(x)}{(1-u_\lambda)^2} dx\,. \tag{3.17}$$

And hence (3.16) and (3.17) reduce to

$$\begin{aligned} &\frac{\lambda i^2}{2i-1} \int_\Omega \frac{f(x)}{(1-u_\lambda)^2} dx - 2\lambda \int_\Omega \frac{f(x)}{(1-u_\lambda)^3} dx + 4\lambda \int_\Omega \frac{f(x)}{(1-u_\lambda)^{3-i}} dx \\ &\geq \lambda(2 + \frac{i^2}{2i-1}) \int_\Omega \frac{f(x)}{(1-u_\lambda)^{3-2i}} dx\,. \end{aligned} \tag{3.18}$$

From the choice of i in (3.15) we have $2 + \frac{i^2}{2i-1} > 0$. So (3.18) implies that

$$\begin{aligned} \int_\Omega \frac{f(x)}{(1-u_\lambda)^{3-2i}} dx &\leq C \int_\Omega \frac{f(x)}{(1-u_\lambda)^{3-i}} dx \\ &\leq C\Big(\int_\Omega \Big|\frac{f^{\frac{3-i}{3-2i}}}{(1-u_\lambda)^{3-i}}\Big|^{\frac{3-2i}{3-i}} dx\Big)^{\frac{3-i}{3-2i}} \cdot \Big(\int_\Omega \Big|f^{\frac{-i}{3-2i}}\Big|^{\frac{3-2i}{-i}} dx\Big)^{\frac{-i}{3-2i}} \\ &\leq C\Big(\int_\Omega \frac{f(x)}{(1-u_\lambda)^{3-2i}} dx\Big)^{\frac{3-i}{3-2i}}\,, \end{aligned} \tag{3.19}$$

where Holder's inequality is applied. From the above we deduce that

$$\int_\Omega \frac{f(x)}{(1-u_\lambda)^{3-2i}} dx \leq C\,. \tag{3.20}$$

Further we have

$$\int_\Omega \left|\frac{f(x)}{(1-u_\lambda)^3}\right|^{\frac{3-2i}{3}} dx = \int_\Omega f^{\frac{-2i}{3}} \cdot \frac{f}{(1-u_\lambda)^{3-2i}} dx \\ \leq C \int_\Omega \frac{f}{(1-u_\lambda)^{3-2i}} dx \leq C\,. \tag{3.21}$$

Therefore, we get that

$$\| \frac{f(x)}{(1-u_\lambda)^3} \|_{L^p} \leq C\,, \tag{3.22a}$$

where –in view of (3.15)–

$$p = \frac{3-2i}{3} \leq 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}\,. \tag{3.22b}$$

■

Proof of Theorem 3.7: This follows from Lemma 3.8 and Theorem 3.6, where $p = 1$ when the dimension $N = 1$, p can be taken to be $1+\frac{4}{3}$ when $N = 2$. For $N > 2$, the reasoning applies as long as $\frac{N}{2} < 1+\frac{4}{3}+2\sqrt{\frac{2}{3}}$ which happens when $N < 8$. ■

Finally, we note the following easy comparison results and we omit the details.

Corollary 3.9. *Suppose $f_1, f_2 : \Omega \to (0,1]$ satisfy $f_1(x) \leq f_2(x)$ on Ω, then $\lambda^*(\Omega, f_1) \geq \lambda^*(\Omega, f_2)$ and for $0 < \lambda < \lambda^*(\Omega, f_1)$ we have $u_1(\lambda, x) \leq u_2(\lambda, x)$ on Ω, where $u_1(\lambda, x)$ (resp., $u_2(\lambda, x)$) are the unique minimal positive solution of*

$$-\Delta u = \tfrac{\lambda f_1(x)}{(1-u)^2} \ \textit{(resp., } -\Delta u = \tfrac{\lambda f_2(x)}{(1-u)^2}\textit{) on } \Omega \textit{ and } u = 0 \textit{ on } \partial\Omega.$$

Moreover, if $f_1(x) > f_2(x)$ on a subset of positive measure, then $u_1(\lambda, x) < u_2(\lambda, x)$ for all $x \in \Omega$.

We note that if one considers the cases of power-law or exponential profiles for $(S)_\lambda$ defined in a ball, then the minimal positive solution corresponds to the lowest branch in the bifurcation diagram, the one connecting the origin point $\lambda = 0$ to the first fold at $\lambda = \lambda^*$, see section §5.

3.3 Existence of solutions at $\lambda = \lambda^*$

In this subsection, we study the existence of positive solutions at the critical voltage $\lambda = \lambda^*$. We first deal with the existence of minimal solutions u_{λ^*} for $(S)_{\lambda^*}$.

Lemma 3.10. *Suppose there exists $0 < C < 1$ such that $\| u_\lambda \|_{C(\bar{\Omega})} \leq C$ for each $\lambda < \lambda^*$. Then $u_{\lambda^*} = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ exists in the $C^{2,\alpha}(\bar{\Omega})$ topology for some $0 < \alpha < 1$. Moreover, there exists $\delta > 0$ such that the solutions of $(S)_\lambda$ near $(\lambda^*, u_{\lambda^*})$ form a curve $\rho(s) = \{(\bar{\lambda}(s), v(s)) : |s| < \delta\}$, and the pair $(\bar{\lambda}(s), v(s))$ satisfies:*

$$\bar{\lambda}(0) = \lambda^*,\ \bar{\lambda}'(0) = 0,\ \bar{\lambda}''(0) < 0\,, \quad \text{and} \quad v(0) = u_{\lambda^*}\,,\ v'(0)(x) > 0 \text{ in } \Omega\,. \tag{3.23}$$

Proof: The proof is similar to a related result of Crandall and Rabinowitz cf. [6] [7], so we will be brief. Firstly, the assumed upper bound on u_λ in C^1 and standard regularity theory, show that if $f \in C(\bar{\Omega})$ then $\| u_\lambda \|_{C^{2,\alpha}(\bar{\Omega})} \leq C$ for some $0 < \alpha < 1$ (while if $f \in L^\infty$, then $\| u_\lambda \|_{C^{1,\alpha}(\bar{\Omega})} \leq C$). It follows that $\{(\lambda, u_\lambda)\}$ is precompact in the space $\mathbb{R} \times C^{2,\alpha}$, and hence we have a limiting point $(\lambda^*, u_{\lambda^*})$ as desired. Since $\frac{\lambda^* f(x)}{(1-u_{\lambda^*})^2}$ is nonnegative, Theorem 3.2 of [6] characterizes the solution set of $(S)_\lambda$ near $(\lambda^*, u_{\lambda^*})$: $\bar{\lambda}(0) = \lambda^*$, $\bar{\lambda}'(0) = 0$, $v(0) = u_{\lambda^*}$ and $v'(0) > 0$ in Ω. Finally, the same computation as in Theorem 4.8 in [6] gives that $\bar{\lambda}''(0) < 0$. ■

Remark 3.3. Lemma 3.10 implies that if the minimal solution $u_\lambda(x)$ satisfies $\| u_\lambda \|_{C(\bar{\Omega})} \leq C < 1$ (which occurs when $N < 8$), then there exists two distinct solutions for $(S)_\lambda$ for λ in a deleted left neighborhood of λ^*. A version of this result will be established variationally in an upcoming paper.

The following theorem gives the uniqueness of (classic) solutions for $(S)_{\lambda^*}$.

Theorem 3.11. *Suppose there exists $0 < C < 1$ such that $\| u_\lambda \|_{C(\bar{\Omega})} \le C$ for each $\lambda < \lambda^*$. Then the minimal solution $u_{\lambda^*} = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ obtained above satisfies the following properties:*

1. *The smallest eigenvalue $\mu(\lambda)$ at $\lambda = \lambda^*$ of the linearized operator $L_{u_\lambda, \lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u_\lambda)^3}$ on Ω is zero.*

2. *u_{λ^*} is the unique solution of $(S)_{\lambda^*}$.*

Proof: (1) Applying Proposition 3.3(2) we see that $\mu(\lambda) > 0$ on the minimal branch for any $\lambda < \lambda^*$, hence the limit $\mu(\lambda^*) \ge 0$. If now $\mu(\lambda^*) > 0$ the Implicit Function Theorem could be applied to the operator $L_{u_{\lambda^*}, \lambda^*}$, and would allow the continuation of the minimal branch $\lambda \mapsto u_\lambda$ of classical solutions beyond λ^*, which is a contradiction and hence $\mu(\lambda^*) = 0$.

(2) Suppose now u is any solution such that $u \ge u_{\lambda^*}$. Since $\mu(\lambda^*) = 0$, let ϕ be any positive eigenfunction in the kernel of $L_{u_{\lambda^*}, \lambda^*}$ and write,

$$-\phi\Delta(u - u_{\lambda^*}) = \lambda^* f(x)\big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_{\lambda^*})^2}\big]\phi\,,$$

which yields that

$$-\int_\Omega (u - u_{\lambda^*})\Delta\phi = \lambda^* \int_\Omega f(x)\big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_{\lambda^*})^2}\big]\phi\,.$$

On the other hand, since $-\Delta\phi = \frac{2\lambda^* f(x)}{(1-u_{\lambda^*})^3}\phi$, we have

$$\lambda^* \int_\Omega f(x)\big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_{\lambda^*})^2} - \frac{2}{(1-u_{\lambda^*})^3}(u - u_{\lambda^*})\big]\phi = 0\,.$$

Since the integrand is nonnegative it follows that

$$\frac{1}{(1-u)^2} = \frac{1}{(1-u_{\lambda^*})^2} + \frac{2}{(1-u_{\lambda^*})^3}(u - u_{\lambda^*}) \quad \textit{a.e. in } \Omega\,. \tag{3.24}$$

If now $\| u \|_{L^\infty} \le C < 1$, then u is a classical solution as in Theorem 3.6, and we conclude that $u \equiv u_{\lambda^*}$ on Ω. ■

Now Theorem 1.3 is a direct result of Theorem 1.2(2) (or Theorem 3.7), Lemma 3.10 and Theorem 3.11.

4 Uniqueness and Multiplicity of Solutions

The purpose of this section is to discuss uniqueness and multiplicity of solutions for $(S)_\lambda$. Note that Lemma 3.10 gives that for some $0 < \lambda_2^* < \lambda^*$, there exists at least two solutions for $(S)_\lambda$ with $\lambda \in (\lambda_2^*, \lambda^*)$, which is Theorem 1.4(2). In the following we shall focus on the uniqueness when λ is small enough. We first define non-minimal solutions for $(S)_\lambda$ as follows:

Definition 4.1. A solution $0 \le u < 1$ is said to be a non-minimal positive solution of $(S)_\lambda$, if there exists another positive solution v of $(S)_\lambda$ and a point $x \in \Omega$ such that $u(x) > v(x)$.

Lemma 4.1. *Suppose u is a non-minimal solution of $(S)_\lambda$ with $\lambda \in (0, \lambda^*)$. Then the smallest eigenvalue $\mu(\lambda)$ of the linearized operator $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$ on Ω must be negative.*

Proof: For any fixed $\lambda \in (0, \lambda^*)$, let u_λ be the minimal solution of $(S)_\lambda$. Clearly we have $w = u - u_\lambda \ge 0$ in Ω, and

$$-\Delta w - \frac{\lambda(2 - u - u_\lambda)f}{(1-u)^2(1-u_\lambda)^2}w = 0 \quad \text{in } \Omega\,.$$

Hence we deduce from the strong maximum principle that $u_\lambda < u$ in Ω.

Let $\Omega_0 = \{x \in \Omega : f(x) = 0\}$ and $\Omega/\Omega_0 = \{x \in \Omega : f(x) > 0\}$. Direct calculations give that

$$-\Delta(u-u_\lambda) - \frac{2\lambda f}{(1-u)^3}(u-u_\lambda) = \lambda f\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_\lambda)^2} - \frac{2}{(1-u)^3}(u-u_\lambda)\Big] = \begin{cases} 0\,, & x \in \Omega_0\,; \\ < 0\,, & x \in \Omega/\Omega_0\,. \end{cases} \tag{4.1}$$

From this we get

$$\lambda \int_{\Omega/\Omega_0} f\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_\lambda)^2} - \frac{2}{(1-u)^3}(u-u_\lambda)\Big](u-u_\lambda) < 0\,. \tag{4.2}$$

Now suppose that $\mu(\lambda) \geq 0$. Then for each $\phi \in H_0^1(\Omega)$ we have

$$\langle L_{u,\lambda}\phi, \phi\rangle = \int_\Omega (|\nabla\phi|^2 - \frac{2\lambda f(x)}{(1-u)^3}\phi^2) \geq 0\,. \tag{4.3}$$

Putting $\phi = u - u_\lambda$ in (4.3), we get from the left equality of (4.1) that

$$\begin{aligned} &\lambda \int_{\Omega/\Omega_0} f\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_\lambda)^2} - \frac{2}{(1-u)^3}(u-u_\lambda)\Big](u-u_\lambda) \\ &= \lambda \int_\Omega f\Big[\frac{1}{(1-u)^2} - \frac{1}{(1-u_\lambda)^2} - \frac{2}{(1-u)^3}(u-u_\lambda)\Big](u-u_\lambda) \geq 0 \end{aligned}$$

which contradicts (4.2), and we are done. ■

Remark 4.1. Proposition 3.3 and Lemma 4.1 give that for $0 < \lambda < \lambda^*$, the smallest eigenvalue of $L_{u,\lambda}$ is necessarily negative if u is a non-minimal solution of $(S)_\lambda$, while the smallest eigenvalue of $L_{u,\lambda}$ is positive if $u = u_\lambda$ is a minimal solution of $(S)_\lambda$. For parabolic problems of the type (1.1), it is well-known that the spectrum of the linearized operator about any steady-state solution determines the stability of solutions for (1.1). Therefore, $u_\lambda(x)$ is the unique stable steady-state of (1.1). In our upcoming paper [11], we shall prove that the dynamic solution of (1.1) with $\lambda < \lambda^*$ (and $\lambda = \lambda^*$ for $N < 8$) will globally converge to its unique minimal solution $u_\lambda(x)$.

Now we are able to prove the following uniqueness result, which completes the proof of Theorem 1.4.

Theorem 4.2. *For every $M > 0$ there exists $0 < \lambda_1^*(M) < \lambda^*$ such that for $\lambda \in (0, \lambda_1^*(M))$ the equation $(S)_\lambda$ has a unique solution v satisfying:*

1. $\|\frac{f}{(1-v)^3}\|_1 \leq M$ *as long as the dimension* $N = 1$.
2. $\|\frac{f}{(1-v)^3}\|_{1+\epsilon} \leq M$ *and* $N = 2$.
3. $\|\frac{f}{(1-v)^3}\|_{N/2} \leq M$ *and* $N > 2$.

Proof: For any fixed $\lambda \in (0, \lambda^*)$, let u_λ be the minimal solution of $(S)_\lambda$ and suppose $(S)_\lambda$ has a non-minimal solution u. Lemma 4.2 then gives

$$\int_\Omega |\nabla(u-u_\lambda)|^2 dx < \int_\Omega \frac{2\lambda(u-u_\lambda)^2 f(x)}{(1-u)^3} dx\,.$$

This implies in the case where $N > 2$ that

$$\begin{aligned} C(N)\Big(\int_\Omega (u-u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} &< \lambda \int_\Omega \frac{2f(x)}{(1-u)^3}(u-u_\lambda)^2 dx \\ &\leq 2\lambda\Big(\int_\Omega \Big|\frac{f}{(1-u)^3}\Big|^{\frac{N}{2}}\Big)^{\frac{2}{N}} \Big(\int_\Omega (u-u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} \\ &\leq 2\lambda M^{\frac{2}{N}} \Big(\int_\Omega (u-u_\lambda)^{\frac{2N}{N-2}} dx\Big)^{\frac{N-2}{N}} \end{aligned}$$

which is a contradiction if $\lambda < \frac{C(N)}{2M^{\frac{2}{N}}}$ unless $u \equiv u_\lambda$. If $N=1$, then we write

$$C(1)\|(u-u_\lambda)\|_\infty^2 < \lambda \int_\Omega \frac{2f(x)}{(1-u)^3}(u-u_\lambda)^2 dx \le 2\lambda \|(u-u_\lambda)\|_\infty^2 \int_\Omega \frac{f}{(1-u)^3} dx$$

and the proof follows. A similar proof holds for dimension $N=2$. ■

Remark 4.2. The above gives uniqueness for small λ among all solutions that either stay away from 1 or those that approach it slowly. We do not know whether if λ is small enough, any positive solution v of $(S)_\lambda$ satisfy $\int_\Omega (1-v)^{-\frac{3N}{2}} dx \le M$ for some uniform bound M independent of λ. Numerical computations do show that we may have uniqueness for small λ –at least for radially symmetric solutions– as long as $N \ge 2$.

5 Steady-State: Case of Power-Law Profile

The issues of uniqueness and multiplicity of solutions for $(S)_\lambda$ with $0 < \lambda < \lambda^*$, and even mere existence for $(S)_{\lambda^*}$ with $N \ge 8$ seem to be quite challenging problems. In this section, we discuss these problems in the case where f has a power-law permittivity profile, *i.e*, $f(x) = |x|^\alpha$ ($\alpha \ge 0$). We shall also consider the domain Ω to be a unit ball $B_1(0) \subset \mathbb{R}^N$ ($N \ge 1$) and $\lambda \in (0, \lambda^*]$. In this special case, the solutions of $(S)_\lambda$ must be radially symmetric, and $(S)_\lambda$ is then reduced to the following problem

$$\begin{aligned} &-u_{rr} - \frac{N-1}{r} u_r = \frac{\lambda r^\alpha}{(1-u)^2}\,, \quad 0 < r \le 1, \\ &u'(0) = 0\,, \quad u(1) = 0\,. \end{aligned} \tag{5.1}$$

Here $r = |x|$ and $0 < u = u(r) < 1$ for $0 < r < 1$.

Looking first for a solution of the form

$$u(r) = 1 - \beta w(P) \quad \text{with} \quad P = \gamma r\,,$$

where γ, $\beta > 0$, equation (5.1) implies that

$$\gamma^2 \beta \big(w'' + \frac{N-1}{P} w'\big) = \frac{\lambda P^\alpha}{\beta^2 \gamma^\alpha} \frac{1}{w^2}\,.$$

We set $w(0) = 1$ and $\lambda = \gamma^{2+\alpha}\beta^3$. This yields the following initial value problem

$$\begin{aligned} &w'' + \frac{N-1}{P} w' = \frac{P^\alpha}{w^2}\,, \quad P > 0\,, \\ &w'(0) = 0\,, \quad w(0) = 1\,. \end{aligned} \tag{5.2}$$

Since $u(1) = 0$ we have $\beta = 1/w(\gamma)$. Therefore, we conclude that

$$\begin{cases} u(0) = 1 - \dfrac{1}{w(\gamma)}\,, \\ \lambda = \dfrac{\gamma^{2+\alpha}}{w^3(\gamma)}\,, \end{cases} \tag{5.3}$$

where $w(\gamma)$ is a solution of (5.2).

As done in [19], one can numerically integrate the initial value problem (5.2) and use the results to compute the complete bifurcation diagram for (5.1). We show such a computation of $u(0)$ versus λ defined in (5.3) for the slab domain ($N = 1$) in Fig. 2. In this case, one observes from the numerical results that when $N = 1$,

- There exists a unique solution for $(S)_{\lambda^*}$;
- For $0 \le \alpha \le 1$, there exist exactly two solutions for $(S)_\lambda$ whenever $\lambda \in (0, \lambda^*)$;

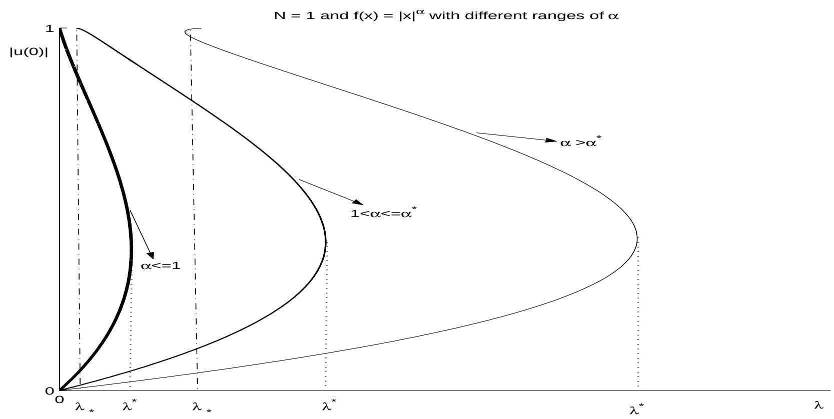

Figure 2: *Plots of* $|u(0)|$ *versus* λ *for the power-law permittivity profile* $f(x) = |x|^\alpha$ $(\alpha \geq 0)$ *defined in the slab domain (*$N = 1$*). The numerical experiments show that it seems to exist a constant* $\alpha^* > 1$ *(analytically given in* (5.8)*), such that the bifurcation diagrams are greatly different for different ranges of* α*:* $0 \leq \alpha \leq 1$*,* $1 < \alpha \leq \alpha^*$ *and* $\alpha > \alpha^*$*.*

- For $\alpha > 1$, it is however difficult to see in any other case the bifurcation diagram as $u(0) \to 1$.

This leads us to the question of determining the asymptotic behavior of $w(P)$ as $P \to \infty$. Towards this end, we proceed it as follows.

Set $\eta = logP$, $w(P) = P^B V(\eta) > 0$ for some positive constant B. Then we obtain from (5.2) that

$$P^{B-2}V'' + (2B + N - 2)P^{B-2}V' + B(B + N - 2)P^{B-2}V = \frac{P^{\alpha-2B}}{V^2}\,. \tag{5.4}$$

Choosing $B - 2 = \alpha - 2B$ so that $B = (2 + \alpha)/3$, we get that

$$V'' + \frac{3N + 2\alpha - 2}{3}V' + \frac{(2 + \alpha)(3N + \alpha - 4)}{9}V = \frac{1}{V^2}\,. \tag{5.5}$$

We notice from (5.5) that only for the case where $N \geq 2$ or $N = 1$ with $\alpha > 1$, that the equilibrium point V_e of (5.5) must be positive and satisfies

$$V_e^3 = \frac{9}{(2 + \alpha)(3N + \alpha - 4)} > 0\,. \tag{5.6}$$

When $N = 1$, this is consistent with the numerical observation of Fig. 2. Linearizing around this equilibrium point by writing

$$V = V_e + Ce^{\sigma\eta}\,, \quad 0 < C << 1,$$

we obtain that

$$\sigma^2 + \frac{3N + 2\alpha - 2}{3}\sigma + \frac{(2 + \alpha)(3N + \alpha - 4)}{3} = 0\,.$$

This reduces to

$$\sigma_\pm = -\frac{3N + 2\alpha - 2}{6} \pm \frac{\sqrt{\triangle}}{6}\,, \tag{5.7a}$$

with

$$\triangle = -8\alpha^2 - (24N - 16)\alpha + (9N^2 - 84N + 100)\,. \tag{5.7b}$$

We note that $\sigma_\pm < 0$ whenever $\triangle \geq 0$. Define now

$$\alpha^* = -\frac{1}{2} + \frac{1}{2}\sqrt{27/2}\,, \quad \alpha^{**} = \frac{4 - 6N + 3\sqrt{6}(N - 2)}{4} \quad (N \geq 8)\,. \tag{5.8}$$

Next, we discuss on N and α by considering the sign of $\triangle$:

Case (1). N and α satisfy either one of the followings:

$$N = 1 \quad with \quad 1 < \alpha \le \alpha^* \,; \tag{5.9a}$$

$$N \ge 8 \quad with \quad 0 \le \alpha \le \alpha^{**} \,. \tag{5.9b}$$

In this case, we have $\triangle \ge 0$ and

$$V \sim \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + C_1 e^{-\frac{3N+2\alpha-2-\sqrt{\triangle}}{6}\eta} + \cdots \,, \quad \text{as} \quad \eta \to +\infty \,.$$

Further, we conclude that

$$w \sim P^{\frac{2+\alpha}{3}} \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + C_1 P^{-\frac{N-2}{2}+\frac{\sqrt{\triangle}}{6}} + \cdots \,, \quad \text{as} \quad P \to +\infty \,.$$

Since $\lambda = \gamma^{2+\alpha}/w^3(\gamma)$, we have

$$\lambda \sim \lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9} \quad as \quad \gamma \to \infty \,. \tag{5.10}$$

For this case, we compute the numerical results for (5.9a) in Fig. 2 and for (5.9b) in Fig. 3(b), respectively.

Case (2). N and α satisfy any one of the following three:

$$N = 1 \quad with \quad \alpha > \alpha^* \,; \tag{5.11a}$$

$$2 \le N \le 7 \quad with \quad \alpha \ge 0 \,; \tag{5.11b}$$

$$N \ge 8 \quad with \quad \alpha > \alpha^{**} \,. \tag{5.11c}$$

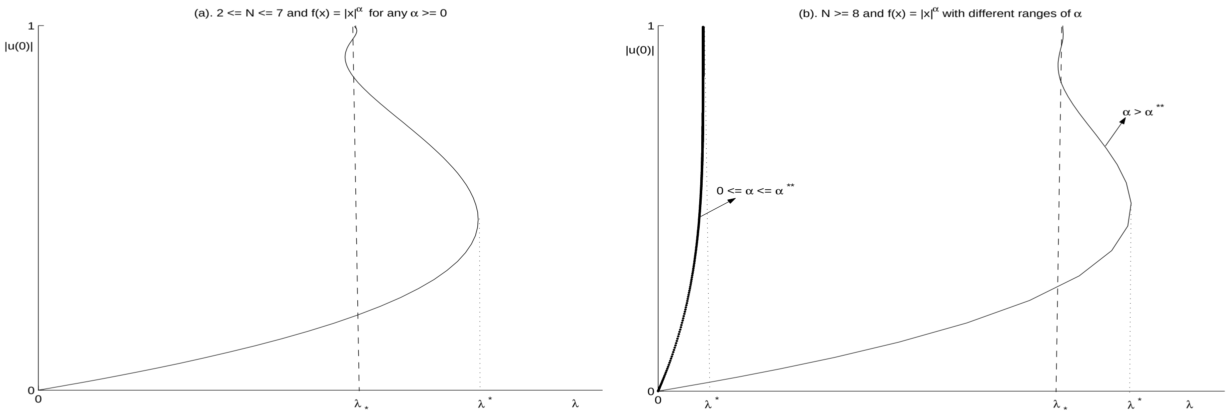

Figure 3: *Left figure: Plots of $|u(0)|$ versus λ for the power-law permittivity profile $f(x) = |x|^\alpha$ ($\alpha \ge 0$) defined in the unit ball $B_1(0) \subset \mathbb{R}^N$ with $2 \le N \le 7$. In this case, $|u(0)|$ oscillates around the value λ_* defined in* (5.10) *and there exists a unique solution for $(S)_{\lambda^*}$. Right figure: Plots of $|u(0)|$ versus λ for the power-law permittivity profile $f(x) = |x|^\alpha$ ($\alpha \ge 0$) defined in the unit ball $B_1(0) \subset \mathbb{R}^N$ with $N \ge 8$. The characters of the bifurcation diagrams depend on different ranges of α: when $0 \le \alpha \le \alpha^{**}$, there exists a unique solution for $(S)_\lambda$ with $\lambda \in (0, \lambda^*)$ and there does not exist any solution for $(S)_{\lambda^*}$; when $\alpha > \alpha^{**}$, $|u(0)|$ oscillates around the value λ_* defined in* (5.10) *and there exists a unique solution for $(S)_{\lambda^*}$.*

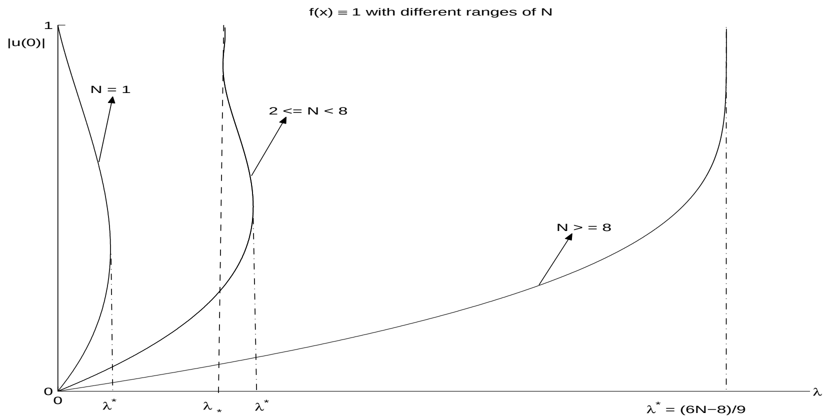

Figure 4: *Plots of $|u(0)|$ versus λ for the constant permittivity profile $f(x) \equiv 1$ defined in the unit ball $B_1(0) \subset \mathbb{R}^N$ with different ranges of N. In the case of $N \geq 8$, we have $\lambda^* = (6N-8)/9$.*

Number of solutions for the case $f(x) = |x|^\alpha$ and $N = 1$

	$\lambda = \lambda_1 (< \lambda^*)$	$\lambda < \lambda^*$	$\lambda = \lambda^*$
$0 \leq \alpha \leq 1$	——	2	1
$1 < \alpha \leq \alpha^*$	1	≥ 1	1
$\alpha > \alpha^*$	∞	≥ 1	1

Table 3: *Number of solutions to* (1.1) *which is defined in a unit ball $B_1(0)$ with $N = 1$, where $\lambda_1 = \frac{(2+\alpha)(\alpha-1)}{9} < \lambda^*$, $\alpha^* = -\frac{1}{2} + \frac{1}{2}\sqrt{27/2}$ and $f(x) = |x|^\alpha$ is chosen to be power-law permittivity profile.*

In this case, we have $\triangle < 0$ and

$$V \sim \Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + C_1 e^{-\frac{3N+2\alpha-2}{6}\eta} cos\big(\frac{\sqrt{-\triangle}}{6}\eta + C_2\big) + \cdots\,, \quad \text{as} \quad \eta \to +\infty\,.$$

Further, we conclude that

$$w \sim P^{\frac{2+\alpha}{3}}\Big(\frac{9}{(2+\alpha)(3N+\alpha-4)}\Big)^{\frac{1}{3}} + C_1 P^{-\frac{N-2}{2}} cos\big(\frac{\sqrt{-\triangle}}{6} logP + C_2\big) + \cdots\,, \quad \text{as} \quad P \to +\infty\,. \tag{5.12}$$

And we also obtain from $\lambda = \gamma^{2+\alpha}/w^3(\gamma)$ that

$$\lambda \sim \lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9} \quad \text{as} \quad \gamma \to \infty\,.$$

When N and α separately satisfy (5.11a), (5.11b) and (5.11c), the typical diagrams are computed in Fig. 2, Fig. 3(a) and Fig. 3(b), respectively.

Combining the numerical results of Fig. 2 and Fig. 3, we plot the bifurcation diagram of the constant permittivity profile defined in the unit ball with different ranges of N. The result of such a computation is

Number of solutions for the case $f(x) = |x|^\alpha$ and $2 \leq N \leq 7$

	$\lambda = \lambda_* (< \lambda^*)$	$\lambda < \lambda^*$	$\lambda = \lambda^*$
$\alpha \geq 0$	∞	≥ 1	1

Table 4: *Number of solutions to* (1.1) *which is defined in a unit ball $B_1(0)$ with $2 \leq N \leq 7$, where $\lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9} < \lambda^*$ and $f(x) = |x|^\alpha$ is chosen to be power-law permittivity profile.*

Number of solutions for the case $f(x) = |x|^\alpha$ and $N \geq 8$

	$\lambda = \lambda_*(< \lambda^*)$	$\lambda < \lambda^*$	$\lambda = \lambda_*(= \lambda^*)$	$\lambda = \lambda^*$
$0 \leq \alpha \leq \alpha^{**}$	—	1	0	0
$\alpha > \alpha^{**}$	∞	≥ 1	—	1

Table 5: *Number of solutions to* (1.1) *which is defined in a unit ball* $B_1(0)$ *with* $N \geq 8$*, where* $\lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ *(*$= \lambda^*$ *for* $0 \leq \alpha \leq \alpha^{**}$*),* $\alpha^{**} = \frac{4-6N+3\sqrt{6}(N-2)}{4}$ *and* $f(x) = |x|^\alpha$ *is chosen to be power-law permittivity profile.*

Number of solutions for the case $f(x) \equiv 1$

	$\lambda = \lambda_*(< \lambda^*)$	$\lambda < \lambda^*$	$\lambda = \lambda_*(= \lambda^*)$	$\lambda = \lambda^*$
$N = 1$	—	2	—	1
$2 \leq N \leq 7$	∞	≥ 1	—	1
$N \geq 8$	—	1	0	0

Table 6: *Number of solutions to* (1.1) *which is defined in a unit ball* $B_1(0)$*, where* $\lambda_* = \frac{6N-8}{9}$ *and* $f(x) \equiv 1$ *is chosen to be constant permittivity profile. We note that* $\lambda_* = \lambda^* = (6N-8)/9$ *for* $N \geq 8$.

shown in Fig. 4, from which one can observe the uniqueness, (infinite) multiplicity of the solutions for $(S)_\lambda$ with $\lambda \in (0, \lambda^*)$ and different ranges of N: $N = 1$, $2 \leq N \leq 7$ and $N \geq 8$. We note that $\lambda^* = (6N-8)/9$ when $N \geq 8$.

Applying above numerical results, in Table 3∼5 we give the number of solutions for (5.1) depending on N and α. In Table 6, we give the number of solutions for (5.1) with constant profile $f(x) \equiv 1$, which shows that $N = 8$ is the critical dimension for (5.1).

Remark 5.1. Under the assumptions of (5.11), since $w(P)$ in (5.12) is oscillatory for $P >> 1$, we expect that $|u(0)|$ oscillates around the value $\lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ as $P \to \infty$. In particular, this implies that for this case, $(S)_\lambda$ has infinitely multiple solutions for $(S)_{\lambda_*}$. We compute the numerical results of this case in Fig. 2 and Fig. 3, from which we can observe the uniqueness, (infinite) multiplicity of the solutions for $(S)_\lambda$: when N and α satisfy any one of the three cases in (5.11), then there exists a series of $\{\lambda_i\}$ satisfying

$$\lambda_0 = 0\,, \quad \lambda_{2k} \nearrow \lambda_* \quad \text{as} \quad k \to \infty\,;$$

$$\lambda_1 = \lambda^*\,, \quad \lambda_{2k-1} \searrow \lambda_* \quad \text{as} \quad k \to \infty$$

such that: there exist exactly $2k+1$ solutions for $(S)_\lambda$ with $\lambda \in (\lambda_{2k}, \lambda_{2k+2})$; and there exist exactly $2k$ solutions for $(S)_\lambda$ with $\lambda \in (\lambda_{2k+1}, \lambda_{2k-1})$; further, there exist infinitely multiple solutions for $(S)_{\lambda_*}$. Therefore, it is reasonable to believe that the multiplicity of solutions for the general $(S)_\lambda$ greatly depends on the permittivity profile $f(x)$, the dimension N and the value of λ.

Remark 5.2. Our results show that for $f(x) = |x|^\alpha$ with $N \geq 8$ and $0 \leq \alpha \leq \alpha^{**}$, then there does not exist any classic solution for $(S)_{\lambda^*}$, where $\lambda^* = (2+\alpha)(3N+\alpha-4)/9$; but for other cases of N and α, there exists a unique solution for $(S)_{\lambda^*}$. Therefore, for $N \geq 8$ it seems from these results that whether there exist solutions for $(S)_{\lambda^*}$ depends on the varying permittivity profile $f(x)$. However, we conjecture that for $N \geq 8$ there is no solution for $(S)_{\lambda^*}$ if the permittivity profile $f(x) > 0$ on Ω.

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