On the Best Constant in the Moser-Onofri-Aubin Inequality

Nassif Ghoussoub [1] and Chang-Shou Lin [2]

[1] *Department of Mathematics, University of British Columbia,*
Vancouver, BC V6T1Z2, Canada

[2] *Department of Mathematics, Taida Institute for Mathematical Sciences,*
National Taiwan University, Taipei, 106, Taiwan

Abstract

Let S^2 be the 2-dimensional unit sphere and let J_α denote the nonlinear functional on the Sobolev space $H^{1,2}(S^2)$ defined by

$$J_\alpha(u) = \frac{\alpha}{16\pi}\int_{S^2} |\nabla u|^2\, d\mu_0 + \frac{1}{4\pi}\int_{S^2} u\, d\mu_0 - \ln \int_{S^2} e^u\, \frac{d\mu_0}{4\pi},$$

where $d\mu_0 = \sin\theta\, d\theta \wedge d\phi$. Onofri had established that J_α is non-negative on $H^1(S^2)$ provided $\alpha \geq 1$. In this note, we show that if J_α is restricted to those $u \in H^1(S^2)$ that satisfies the Aubin condition:

$$\int_{S^2} e^u x_j d\mu_0 = 0 \quad \text{for all } 1 \leq j \leq 3,$$

then the same inequality continues to hold (i.e., $J_\alpha(u) \geq 0$) whenever $\alpha \geq \frac{2}{3} - \epsilon_0$ for some $\epsilon_0 > 0$. The question of Chang-Yang on whether this remains true for all $\alpha \geq \frac{1}{2}$ remains open.

1 Introduction

Let S^2 be the 2-dimensional unit sphere with the standard metric g_0 whose corresponding volume form $d\omega := \frac{d\mu_0}{4\pi}$ is normalized so that $\int_{S^2} d\omega = 1$. For $\alpha > 0$, we consider the following nonlinear functional on the Sobolev space $H^{1,2}(S^2)$:

$$J_\alpha(u) = \frac{\alpha}{16\pi}\int_{S^2} |\nabla_{g_0} u|^2\, d\omega + \int_{S^2} u\, d\omega - \ln \int_{S^2} e^u\, d\omega.$$

The classical Moser-Trudinger inequality [14] yields that J_α is bounded from below in $H^1(S^2)$ if and only if $\alpha \geq 1$. In [15], Onofri proved that the infimum is actually equal to zero for $\alpha = 1$, by using the conformal invariance of J_1 to show that

$$\inf_{u\in\mathcal{M}} J_1(u) = \inf_{u\in H^1(S^2)} J_1(u) = 0, \tag{1.1}$$

where $\mathcal{M}$ is the submanifold of $H^1(S^2)$ defined by

$$\mathcal{M} := \left\{ u \in H^1(S^2)\,;\ \int_{S^2} e^u\, \mathbf{x}\, dw = 0 \right\}, \tag{1.2}$$

with $\mathbf{x} = (x_1, x_2, x_3) \in S^2$, on which the infimum of J_1 is attained. Other proofs were also given by Osgood-Phillips-Sarnak [16] and by Hong [11].

Prior to that, Aubin [1] had shown that by restricting the functional J_α to $\mathcal{M}$, it is then again bounded below by — a necessarily non-positive — constant C_α, for any $\alpha \geq \frac{1}{2}$. In their work on Nirenberg's prescribing Gaussian curvature problem on S^2, Chang and Yang [5, 6] showed that C_α can be taken to be equal to 0 for $\alpha \geq 1 - \epsilon_0$ for some small ϵ_0. This led them to the following

Conjecture 1: If $\alpha \geq \frac{1}{2}$ then $\inf\limits_{u \in \mathcal{M}} J_\alpha(u) = 0$.

Note that this fails if $\alpha < \frac{1}{2}$, since the functional J_α is then unbounded from below (see [9]). In this article, we want to give a partial answer to this question by showing that this is indeed the case for $\alpha \geq \frac{2}{3}$ and slightly below that.

As mentioned above, Aubin had proved that for all $\alpha \geq \frac{1}{2}$, the functional J_α is coercive on $\mathcal{M}$, and that it attains its infimum on some function $u \in \mathcal{M}$. Accounting for the Lagrange multipliers, and setting $\rho = \frac{1}{\alpha}$, the Euler-Lagrangian equation for u is then

$$\Delta_{g_0} u + 8\pi\rho \left(\frac{e^u}{\int_{S^2} e^u\, dw} - 1 \right) = \sum_{j=1}^{3} \alpha_j x_j e^u \quad \text{on } S^2.$$

In [6], Chang and Yang proved however that $\alpha_j, j = 1, 2, 3$ necessarily vanishes. Thus u satisfies – up to an additive constant – the equation $\Delta_{g_0} u + 8\pi\rho(e^u - 1) = 0$ on S^2, equivalently

$$\Delta u + 2\rho(e^u - 1) = 0 \quad \text{on } S^2, \tag{1.3}$$

where now the Laplacian $\Delta := 4\pi\Delta_{g_0}$ corresponds to the metric on the unit sphere whose volume form is $d\mu_0 = \sin\theta\, d\theta \wedge d\phi$.

Here is the main result of this note.

Theorem 1.1. *If $1 < \rho \leq \frac{3}{2}$ and u is a solution of* (1.3)*, then $u \equiv 0$ on S^2.*

This then clearly gives a positive answer to Conjecture 1 for $\alpha \geq \frac{2}{3}$.

2 The axially symmetric case

The proof of Theorem 1.1 relies on the fact that the conjecture has been shown to be true in the axially symmetric case. In other words, the following result holds.

Theorem A . *Let u be a solution of* (1.3) *with $1 < \rho \leq 2$. If u is axially symmetric, then $u \equiv 0$ on S^2.*

Theorem A was first established by Feldman, Froese, Ghoussoub and Gui [9] for $1 < \rho \leq \frac{25}{16}$. It was eventually proved for all $1 < \rho \leq 2$ by Gui and Wei [10], and independently by Lin [12]. Note that this means that the following one-dimensional inequality holds:

$$\frac{1}{2}\int_{-1}^{1}(1-x^2)|g'(x)|^2\,dx + 2\int_{-1}^{1} g(x)\,dx - 2\ln\frac{1}{2}\int_{-1}^{1} e^{2g(x)}dx \geq 0,$$

for every function g on $(-1,1)$ satisfying $\int_{-1}^{1}(1-x^2)|g'(x)|^2dx < \infty$ and $\int_{-1}^{1} e^{2g(x)}xdx = 0$.

We now give a sketch of the proof of Theorem A that connects the conjecture of Chang-Yang to an equally interesting Liouville type theorem on R^2. For that, we let Π denote the stereographic projection $S^2 \to \mathbb{R}^2$ with respect to the North pole $N = (0,0,1)$:

$$\Pi(x) := \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right).$$

Suppose u is a solution of (1.3), and set

$$\tilde{u}(y) := u(\Pi^{-1}(y)) \quad \text{for } y \in \mathbb{R}^2.$$

Then $\tilde{u}$ satisfies

$$\Delta\tilde{u} + 2\rho\, J(y)\left(e^{\tilde{u}} - 1\right) = 0 \quad \text{in } \mathbb{R}^2,$$

where $J(y) := \left(\frac{2}{1+|y|^2}\right)^2$ is the Jacobian of Π. By letting

$$v(y) := \tilde{u}(y) + \rho\log\left((1+|y|^2)^{-2}\right) + \log(8\rho) \quad \text{for } y \in \mathbb{R}^2, \tag{2.1}$$

we have that v satisfies

$$\Delta v + (1+|y|^2)^l e^v = 0 \quad \text{in } \mathbb{R}^2, \tag{2.2}$$

where $l = 2(\rho - 1)$.

Let v be a solution of (2.2) and suppose $\beta_l(v)$ is finite, where

$$\beta_l(v) = \frac{1}{2\pi}\int_{\mathbb{R}^2}(1+|y|^2)^l e^v dy. \tag{2.3}$$

Then $v(y)$ has the following asymptotic behavior at ∞:

$$v(y) = -\beta_l(v)\log|y| + O(1). \tag{2.4}$$

We refer to [7] for a proof of (2.4), which once combined with the Pohozaev identity yields the following result.

Lemma 2.1. *Let $l > 0$ and v be a solution of (2.2) such that $\beta_l(v) < +\infty$. Then*

$$4 < \beta_l(v) < 4(l+1).$$

Proof. Multiplying (2.2) by $y \cdot \nabla v$ and integrating by parts on $B_R = \{y \,|\, |y| < R\}$, we have

$$\int_{\partial B_R} (y \cdot \nabla v) \frac{\partial v}{\partial \nu} ds - \frac{1}{2} \int_{\partial B_R} (y \cdot \nu) |\nabla v|^2 ds = - \int_{B_R} (1 + |y|^2)^l \, y \cdot \nabla e^v dy$$
$$= (l+2) \int_{B_R} (1 + |y|^2)^l e^v dy - l \int_{B_R} (1 + |y|^2)^{l-1} e^v dy - \int_{\partial B_R} (1 + |y|^2)^l (y \cdot \nu) e^v ds.$$

By letting $R \to +\infty$ in the above formula and by using (2.4), we obtain that

$$(l+2) \int_{\mathbb{R}^2} (1 + |y|^2)^l e^v dy - l \int_{\mathbb{R}^2} (1 + |y|^2)^{l-1} e^v dy = \pi \beta_l^2(v).$$

Note now that

$$\begin{aligned} 4\pi \beta_l(v) &= 2 \int_{\mathbb{R}^2} (1 + |y|^2)^l e^v dy \\ &< (l+2) \int_{\mathbb{R}^2} (1 + |y|^2)^l e^v dy - l \int_{\mathbb{R}^2} (1 + |y|^2)^{l-1} e^v dy \\ &< (l+2) \int_{\mathbb{R}^2} (1 + |y|^2)^l e^v dy = 2\pi(2l+2) \beta_l(v), \end{aligned}$$

which means that

$$4\pi \beta_l(v) < \pi \beta_l^2(v) \leq 2\pi(2l+2)\beta_l(v), \quad \text{i.e.,} \quad 4 < \beta_l(v) < 4(l+1).$$

□

Note that by using (2.1) with $u \equiv 0$, equation (2.2) always has a special axially symmetric solution, namely

$$v^*(y) = -2\rho \log(1 + |y|^2) + \log(8\rho) \quad \text{for } y \in \mathbb{R}^2, \tag{2.5}$$

where

$$\beta_l(v^*) = 4\rho = 2(l+2). \tag{2.6}$$

An open question that would clearly imply the conjecture of Chang and Yang is the following:

Conjecture 2: Is v^* the only solution of (2.2) with $\beta_l = 2(l+2)$?

Note that it is indeed the case if $\ell < 0$ (i.e., $\rho < 1$ and $\alpha > 1$), since then we can employ the method of moving planes to show that $v(y)$ is radially symmetric with respect to the origin, and then conclude that $u(x)$ is axially symmetric with any line passing through the origin. Thus $u(x)$ must be a constant function on S^2. Equation (1.3) then yields $u = 0$, which implies $J_\alpha \geq 0$ on $\mathcal{M}$. By passing to the limit as $\alpha \to 1$, we recover the Onofri inequality.

When $l > 0$ (i.e., $\rho > 1$ and $\alpha \leq 1$), the method of moving planes fails and it is still an open problem whether any solution of (2.2) with $\beta_l = 2(l+2)$ is equal to v^* or not. The following uniqueness theorem reduces however the problem to whether any solution of (2.2) is radially symmetric.

Theorem B . *Suppose $l > 0$ and $v_i(y) = v_i(|y|), i = 1, 2,$ are two solutions of (2.2) satisfying*

$$\beta_l(v_1) = \beta_l(v_2). \tag{2.7}$$

Then $v_1 = v_2$ under one of the following conditions:

(i) $l \le 1$ or

(ii) $l > 1$ and $2l < \beta_l(v_i) < 2(2 + l)$ for $i = 1, 2$.

See [12] for a proof of Theorem B. In order to show how Theorem B implies Theorem A, we suppose u is a solution of (1.3) that is axially symmetric with respect to some direction. By rotating, the direction can be assumed to be $(0, 0, 1)$. By using the stereographic projection as above, and setting v as in (2.1), we have

$$\begin{cases} v(y) = -4\rho \log |y| + O(1), \\ \frac{1}{2\pi} \int_{\mathbb{R}^2} (1 + |y|^2)^l e^v dy = 4\rho = 4 + 2l. \end{cases} \tag{2.8}$$

If $l \le 1$, i.e., $\rho \le \frac{3}{2}$, then $v = v^*$ by (i) of Theorem B, and then $u \equiv 0$. If $l > 1$, then by noting that

$$2l < 4\rho = 4 + 2l = \beta_l(v) < 4 + 4l,$$

we deduce that $v = v^*$ by (ii) of Theorem B, which again means that $u \equiv 0$.

3 Proof of the main theorem

We shall prove Theorem 1.1 by showing that if $\rho \le \frac{3}{2}$, then any solution of (1.3) is necessarily axially symmetric. We can then conclude by using Theorem A.

We shall need the following lemma.

Lemma 3.1. *Let Ω be a simply connected domain in $\mathbb{R}^2$, and suppose $g \in C^2(\Omega)$ satisfies*

$$\begin{cases} \Delta g + e^g > 0 \quad \text{in } \Omega \quad \text{and} \\ \int_\Omega e^g dy \le 8\pi. \end{cases}$$

Consider an open set $\omega \subset \Omega$ such that $\lambda_{1,g}(\omega) \le 0$, where $\lambda_{1,g}(\omega)$ is the first eigenvalue of the operator $\Delta + e^g$ on $H_0^1(\omega)$. Then, we necessarily have that

$$\int_\omega e^g dy > 4\pi. \tag{3.1}$$

Lemma 3.1 was first proved in [2] by using the classical Bol inequality. The strict inequality of (3.1) is due to the fact that $\Delta g + e^g > 0$ in Ω. See [3] and references therein.

Remark 3.2. We note that Lemma 3.1 can be applied even when ω is unbounded. Indeed, for simplicity, we shall assume –as will be the case in the application below to the proof of Theorem 1.1– that for some $\beta \geq 2$, we have

$$g(y) = -\beta \log |y| + O(1) \quad \text{at } \infty.$$

We shall also assume that the corresponding null-eigenfunction φ in ω, i.e.,

$$\begin{cases} \Delta\varphi + e^g \varphi = 0 & \text{in } \omega, \\ \varphi|_{\partial\omega} = 0, \end{cases}$$

is bounded in $\overline{\omega}$. Without loss of generality, we may also assume that $0 \notin \overline{\omega}$. Now set

$$\hat{g}(x) = g(\frac{x}{|x|^2}) - 2\log|x| \quad \text{and} \quad \hat{\varphi}(x) = \varphi(\frac{x}{|x|^2}) \quad \text{for } x \in \omega^* = \{y = \frac{x}{|x|^2};\ x \in \omega\}.$$

Since $\beta \geq 2$, $e^{\hat{g}}$ is a Hölder function at $0 \in \overline{\omega^*}$, and $\hat{g}$ and $\hat{\varphi}$ satisfy

$$\Delta\hat{g} + e^{\hat{g}} > 0 \quad \text{in } \omega^* \setminus \{0\} \quad \text{and} \quad \Delta\hat{\varphi} + e^{\hat{g}}\hat{\varphi} = 0 \quad \text{in } \omega^*.$$

By the boundedness of $\hat{\varphi}$, $\hat{\varphi}$ is continuous on $\overline{\omega^*}$. If $0 \in \omega^*$, then by noting that $\hat{g}$ satisfies $\Delta\hat{g} + e^{\hat{g}} \geq (\beta - 2)\delta_0$, where δ_0 is the Dirac measure at 0 and $\beta - 2 \geq 0$, we can then apply a version of Lemma 3.1 where $\hat{g}$ can have a singularity (see [3]), to deduce that

$$\int_{\omega^*} e^{\hat{g}(x)} dx = \int_{\omega} e^{g(x)} dx \geq 4\pi.$$

We note that in the application of the lemma to the proof of Theorem 1.1, we have that φ is bounded on all of $\mathbb{R}^2$.

Now we are in the position to prove the main theorem.

Proof of Theorem 1.1. Suppose $u(x)$ is a solution of (1.3). Let ξ_0 be a critical point of u. Without loss of generality, we may assume $\xi_0 = (0, 0, -1)$. By using the stereographic projection Π as before and letting

$$v(y) := u(\Pi^{-1}(x)) - 2\rho \log(1 + |y|^2) + \log(8\rho),$$

v satisfies (2.2) and

$$\nabla v(0) = 0. \tag{3.2}$$

Set

$$\varphi(y) := y_2 \frac{\partial v}{\partial y_1} - y_1 \frac{\partial v}{\partial y_2}.$$

Then φ satisfies

$$\Delta\varphi + (1+|y|^2)^l e^v \varphi = 0 \quad \text{in } \mathbb{R}^2. \tag{3.3}$$

By (2.1), it is easy to see φ is bounded in $\mathbb{R}^2$. If $\varphi \not\equiv 0$, then by (3.2),

$$\varphi(y) = Q(y) + \text{higher order terms} \quad \text{for } |y| \ll 1,$$

where $Q(y)$ is a quadratic polynomial of degree m with $m \geq 2$, that is also a harmonic function, i.e., $\Delta Q = 0$. Thus, the nodal line $\{y \,|\, \varphi(y) = 0\}$ divides a small neighborhood of the origin into at least four regions. Let γ_i, $i = 1, 2, 3, 4$, be four branches of nodal line of φ issuing from the origin. If γ_i does not intersect with γ_j, $i \neq j$, then $\mathbb{R}^2 \setminus \bigcup_{i=1}^{4} \gamma_i$ contains at least four simply-connected components. See Figure 1 below. If γ_i intersects with some γ_j, then $\mathbb{R}^2 \setminus \bigcup_{i=1}^{4} \gamma_i$ contains at least three simply-connected components. See Figure 2.

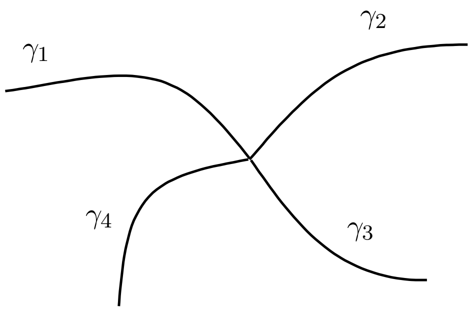

Fig.1

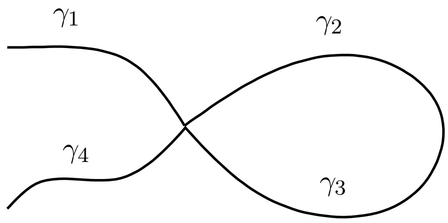

Fig.2

If there are more branches of nodal line of φ issuing from the origin, then $\mathbb{R}^2 \setminus \{\varphi = 0\}$ is divided into more components of simply-connected domains. Therefore, we conclude that $\mathbb{R}^2$ is divided by the nodal line $\{y \,|\, \varphi(y) = 0\}$ into at least 3 regions, i.e.,

$$\mathbb{R}^2 \setminus \{y \,|\, \varphi(y) = 0\} = \bigcup_{j=1}^{3} \Omega_j.$$

In each component Ω_j, the first eigenvalue of $\Delta + (1+|y|^2)^l e^v$ being equal to 0. Let now

$$g := \log\left((1+|y|^2)^l e^v\right).$$

By noting that

$$\Delta g + e^g > 0 \quad \text{in } \mathbb{R}^2,$$

Lemma 3.1 then implies that for each $j = 1, 2, 3$,

$$\int_{\Omega_j} e^g dy = \int_{\Omega_j} (1+|y|^2)^l e^v dy > 4\pi.$$

It follows that

$$8\pi\rho = \sum_{j=1}^{3} \int_{\Omega_j} (1+|y|^2)^l e^v dy > 12\pi,$$

which is a contradiction if we had assumed that $\rho \le \frac{3}{2}$. Thus we have $\varphi(y) = 0$, i.e., $v(y)$ is axially symmetric. By Theorem A, we can conclude $u \equiv 0$. □

Remark 3.3. If we further assume that the antipodal of ξ_0 is also a critical point of u, then $\mathbb{R}^2 \setminus \{y \,|\, \varphi(y) = 0\} = \bigcup_{j=1}^{m} \Omega_j$, where $m \ge 4$. Lemma 3.1 then yields

$$8\pi\rho = \int_{\mathbb{R}^2} (1+|y|^2)^l e^v dy \ge \sum_{j=1}^{m} \int_{\Omega_j} (1+|y|^2)^l e^v dy > 4m\pi \ge 16\pi,$$

which is a contradiction whenever $\rho \le 2$. By Theorem A, we have again that $u \equiv 0$. For example, if u is even on S^2 (i.e., $u(z) = u(-z)$ for all $z \in S^2$), then the main theorem holds for $\rho \le 2$.

Remark 3.4. If v is a solution of (2.2) with $\beta_l(v) \le 6$, and 0 is a critical point of v, then by the same proof of Theorem 1.1, we can conclude v is radially symmetric in $\mathbb{R}^2$. Furthermore, if $v(x_1, x_2)$ is even in both x_1 and x_2, then v is radially symmetric if $\beta_l(v) \le 8$.

Remark 3.5. One can actually show that Conjecture 1 holds for $\rho \le \frac{3}{2} + \epsilon_0$ for some $\epsilon_0 > 0$. Indeed, it suffices to show that for α smaller but close to $\frac{2}{3}$, the functional J_α is non-negative. Assuming not, then there exists a sequence of $\{\alpha_k\}_k$ such that $\frac{1}{2} < \alpha_k < \frac{2}{3}$, $\lim_k \alpha_k = \frac{2}{3}$ and $\inf_{\mathcal{M}} J_{\alpha_k}(u) < 0$. Since J_α is coercive for each $\alpha > \frac{1}{2}$, a standard compactness argument yields the existence of a minimizer $u_k \in \mathcal{M}$ for J_{α_k}. Moreover, $\|u_k\|_{H^1} < C$ for some positive constant independent of k. Modulo extracting a subsequence, u_k then converges weakly to some u_0 in $\mathcal{M}$ as $k \to \infty$, and u_0 is necessarily a minimizer for $I_{\frac{2}{3}}$ in $\mathcal{M}$. By our main result, $u_0 \equiv 0$. Now, we claim that u_k actually converges strongly in H^1 to $u_0 \equiv 0$. This is because – as argued by Chang and Yang – the Euler-Lagrange equations are then

$$\frac{\alpha_k}{2} \Delta u_k - 1 + \frac{1}{\lambda_k} e^{u_k} = 0 \tag{3.20}$$

where $\lambda_k = \int_{S^2} e^{u_k} dw < C$ for some positive constant C. Multiplying (3.20) by u_k and integrating over S^2, we obtain

$$\frac{\alpha_k}{2} \int_{S^2} |\nabla u_k|^2 \, dw + \int_{S^2} u_k(x) \, dw = \frac{1}{\lambda_k} \int_{S^2} e^{u_k(x)} u_k(x) \, dw. \tag{3.21}$$

Applying Onofri's inequality for u_k and using that $\|u_k\|_{H^1} < C$, we get that $\int_{S^2} e^{2u_k} \, dw$ is also uniformly bounded. This combined with Hölder's inequality and the fact that u_k converges strongly to 0 in L^2 yields that $\int_{S^2} e^{u_k} u_k \, dw \to 0$. Use now (3.21) to conclude that $\|u_k\|_{H^1} \to 0$ as $k \to \infty$.

Now, write $u = v + o(||u||)$ for $||u||$ small, where v belongs to the tangent space of the submanifold $\mathcal{M}$ at $u_0 \equiv 0$ in $H^1(S^2)$. It is easy to see that $\int_{S^2} v\mathbf{x}\, dw = 0$. We can calculate the second variation of J_α in $\mathcal{M}$ at $u_0 \equiv 0$ and get the following estimate around 0

$$J_\alpha(u) = \alpha \int_{S^2} |\nabla v|^2\, dw - 2 \int_{S^2} |v|^2\, dw + o(||u||^2).$$

Note that the eigenvalues of the Laplacian on S^2 corresponding to the eigenspace generated by x_1, x_2, x_3 are $\lambda_2 = \lambda_3 = \lambda_4 = 2$, while $\lambda_5 = 6$. Since v is orthogonal to $\mathbf{x}$, we have

$$\int_{S^2} |\nabla v|^2\, dw \geq 6 \int_{S^2} |v|^2\, dw$$

and therefore

$$J_\alpha(u) \geq (\alpha - \frac{1}{3})||u||^2 + o(||u||^2).$$

Taking $\alpha = \alpha_k$ and $u = u_k$ for k large enough, we get that $J_{\alpha_k}(u_k) \geq 0$, which clearly contradicts our initial assumption on u_k.

Concluding remarks. (i) The question whether $J_\alpha(u) \geq 0$ for $\frac{1}{2} \leq \alpha < \frac{2}{3}$ under the condition (1.2) is still open. However, in [13], it was proved that there is a constant $C \geq 0$ such that for any solution u of (1.3) with $1 < \rho \leq 2$ (i.e. $\frac{1}{2} \leq \alpha < 1$), we have

$$|u(x)| \leq C \quad \text{for all } x \in S^2.$$

(ii) Recently, Liouville type equations with singular data have attracted a lot of attention among PDErs since they are closely related to vortex condensates which appear in many physics models. One of the challenges in this line of research is to understand bubbling phenomena arising from solutions of these equations, and the past twenty years have seen many works in this direction. The most delicate case in bubbling phenomena is when more than one vortex collapse into a single point. (2.2) is one of the model equations that allows an accurate description of the bubbling behavior during such a collapse. See [4] and [8] for related details. Thus, understanding the structure of solutions to the equation (2.2) is fundamentally important. As mentioned above, it is conjectured that for $l \leq 2$, all solutions of (2.2) must be radially symmetric. This remains an open question, although a partial answer has been given recently in [4].

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