# NUMBER THEORY IN THE AMERICAS

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## An overview of workshop

The workshop "Number Theory in the Americas" was held at the BIRS-CMO facility in Oaxaca, Mexico from August 11-16, 2019. The workshop brought together teams of researchers at different career stages (ranging from graduate students to senior faculty) to work together in small groups on new research in number theory. The project groups were intentionally formulated to consist of researchers representing different countries who had not previously worked together. The workshop was conducted entirely in Spanish in order to make it easier for researchers from Latin American countries to participate.

The bulk of the time was devoted to working in the project groups. However, there were several conference-wide activities that were designed to facilitate a greater sense of community among the conference participants. On the first day, each participant gave a 3-minute speed talk in order to introduce themselves and their research to all of the other participants. On the final day, each project group gave a 15-minute progress report detailing what it accomplished during the week. There was also a panel on mathematical careers in different countries, which featured panelists who recently obtained permanent academic positions in Colombia, Chile, Costa Rica, Brazil, the United States, Canada, and France.

The feedback from the conference participants was overwhelmingly positive. As of October 2019, every group expects to publish a paper based on what it accomplished during the workshop. Several of these papers have already appeared on the arXiv.

## Project Group: Analytic number theory

Clara Aldana, Emanuel Carneiro, Carlos Andres Chirre Chávez,
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At the simplest level, an upper bound sieve of what we may call "Selberg type" consists of a choice of coefficients $\rho(d)$, $d \leq D$, with $\rho(1) = 1$, such that

$$S = \sum_{n \leq N} \left( \sum_{d|n} \mu(d)\rho(d) \right)^2$$

is as small as possible. (On the name: it is clear that $S$ gives an upper bound on the number of integers $n \leq N$ without prime factors $\leq D$.) It is easy to show that

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$S = M \cdot N + O_\rho(D^2)$, where

$$M = M(D) = \sum_{d_1,d_2} \frac{\mu(d_1)\mu(d_2)}{[d_1,d_2]} \rho(d_1)\rho(d_2).$$

We must thus minimize $M$.

The choice of $\rho(d)$ for given $D$ for which $M$ is minimal was found by Selberg in 1947. However, this choice depends heavily on the divisibility properties of $d$. For quite a few applications, it is better to restrict the search to functions $\rho(d)$ that are scaled versions of a given continuous function $h : [0,\infty) \to \mathbb{R}$:

$$\rho(t) = \begin{cases} h\left(\frac{\log(D/t)}{\log D}\right) & \text{if } t \leq D, \\ 0 & \text{if } t > D. \end{cases} \tag{1}$$

The question is then: for which function $h$ is $M = M_h(D)$ minimal as $D \to \infty$? And how small is this value of $M$ compared that given by Selberg's choice?

We can also consider the analogous problem for functions $\rho$ that obey $\rho(t) = 0$ for $t > y_2$ and $\rho(t) = 1$ for $t < y_1$, where $y_1$ and $y_2$ are two parameters.

This kind of sieves was studied in depth from the late 60s to the early 80s [BV68], [Mot74], [Gra78], [Jut79b], [Jut79a], [Mot83] and then laid half-dormant until its use in the work of Goldston-Yildirim (starting in the late 90s) and much of what followed (see [May16] and in particular [Pol14], but also [Vat18]).

In summary: it was known ever since [Gra78] that the choice $h(x) = x$ in (1) gave an $M(D) = M_h(D)$ with the same main term as $D \to \infty$ as the value of $M(D)$ given by Selberg's sieve. Thus, $h(x) = x$ was used in practice, although the lower-order terms of $M(D)$ for that choice never seem to have been worked out. In the two parameter-case, Barban and Vehov had proved that the main term was of the right order of magnitude for $h(x) = x$, but no more seems to have been proved.

We have succeeded in proving that the choice $h(x) = x$ is the sole one giving the right main term under very broad conditions, whether for the one- or the two-parameter problem. Moreover, we seem to have shown that the main secondary term (smaller than the main term by a factor of size about $1/L$ for $L = \log D$, or $L = \log(y_1/y_2)$) is best for $h(x) = x$ among all functions as above, even when we consider functions of the form $h(x) = h_1(x) + h_2(x)/L$ (which do give the right main term). This secondary term is negative in the one-parameter case, but it is strictly larger than the secondary-term in $M(D)$ for Selberg's choice.

Our approach is mainly complex-analytic, with the optimization problem reducing to a simple application of Cauchy-Schwarz. Higher-dimensional variants (with "higher-dimensional" understood in the sense of sieve theory) ought to yield to the same approach.

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## Project Group: Number theory and geometry

Mikhail Belolipetsky, Matilde Lalín, Plinio Murillo, Lola Thompson

### 1. General overview

A *Salem number* is a real algebraic integer $\lambda > 1$ such that all its Galois conjugates except $\lambda^{-1}$ have absolute value equal to 1. Salem numbers appear in many areas of mathematics including algebra, geometry, dynamical systems and number theory. They are closely related to the celebrated Lehmer's problem about the smallest Mahler measure of a non-cyclotomic polynomial. We refer to [5] for a survey of research on Salem numbers.

It has been known for some time that exponential lengths of the closed geodesics of an arithmetic hyperbolic $n$-dimensional manifold or orbifold are given by Salem numbers. For $n = 2$ and 3 this relation is described in the book by C. Maclachlan and A. Reid [4, Chapter 12]. More recently, it was elaborated and generalized to higher dimensions by V. Emery, J. Ratcliffe and S. Tschantz in [2]. In particular, their Theorem 1.1 implies that for a non-compact arithmetic hyperbolic $n$-orbifold $\mathcal{O}$, a closed geodesic of length $\ell$ corresponds to a Salem number $\lambda = e^{\ell}$ if the dimension $n$ is even, and to a so called square-rootable Salem number $\lambda = e^{2\ell}$ if $n$ is odd. The degrees of these Salem numbers satisfy $\deg(\lambda) \leq n + 1$. A natural question arises: *What proportion of Salem numbers of a given degree are associated to a fixed orbifold $\mathcal{O}$?*

To this end, let us recall some results about the distribution of algebraic integers. This field has a long history, so we will mention only the more recent results which are relevant to our work. In a beautiful paper [6], W. Thurston, motivated by study of entropy of one dimensional dynamical systems, encountered limiting distributions of conjugates of Perron numbers, a class which includes Salem numbers as a subset. His experiments led to a set of interesting problems and conjectures, some of which were successfully resolved by F. Calegary and Z. Huang in [1]. Later on some ideas from their approach helped F. Götze and A. Gusakova to compute the asymptotic growth of Salem numbers. More precisely, let

$$\mathrm{Sal}_m(Q) := \{\alpha \in \mathrm{Sal}_m : \alpha \leq Q\},$$

where $\mathrm{Sal}_m$ denotes the set of all Salem numbers of degree $2(m+1)$ (note that degree of a Salem number is always even). It was shown in [3] that $\#\mathrm{Sal}_m(Q) = \omega_m Q^{m+1} + O(Q^m)$, with an explicit positive constant $\omega_m$. It is remarkable that this result was established only very recently, as it allows us to play this formula against the distribution of closed geodesics of an arithmetic $n$-orbifold with $n = 2m+1$. We also come up with a related question about distribution of square-rootable Salem numbers. We were able to answer the questions in the first non-trivial case when the degree of the Salem numbers is 4 and the corresponding dimension of arithmetic orbifolds is 3.

Our main result is the following theorem:

**Theorem 1.1.**

A. *Let $\mathcal{O}_D$ be a non-compact arithmetic hyperbolic 3-orbifold associated to a Bianchi group $\Gamma_D = \mathrm{PSL}(2, \mathfrak{o}_K)$, where $\mathfrak{o}_K$ is the ring of integers of an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-D})$, $D$ is a square-free positive integer. Then $\mathcal{O}_D$ generates*

$$cQ^{1/2} + O(Q^{1/4})$$

*square-rootable Salem numbers of degree 4, where $c = \frac{\pi}{4\sqrt{D}}$ if $D \equiv 1, 2 \bmod 4$ and $c = \frac{\pi}{2\sqrt{D}}$ if $D \equiv 3 \bmod 4$.*

B. *The number of Salem numbers of degree 4 that are square-rootable over $\mathbb{Q}$ and less than or equal to $Q$ is*

$$\frac{4}{3}Q^{3/2} + O(Q).$$

The theorem implies that, in the logarithmic scale, a given 3-orbifold $\mathcal{O}_D$ generates asymptotically 1/4 of all Salem numbers of degree 4 and asymptotically 1/3 of the square-rootable Salem numbers of degree 4.

1.1. **Original plan.** The original proposal for the group led by Misha Belolipetsky was to study Thurston's results and observations on distribution of Perron numbers, consider related problems for Salem numbers, and investigate possible relation to Lehmer's problem.

1.2. **Work progress.** On the first day, after a consolidated bibliographical search, the group encountered two relevant articles — a paper by F. Calegary and Z. Huang [1] and a preprint by F. Götze and A. Gusakova [3]. The latter is very recent and comes particularly close to the scope of the project.

For the next few days the group focused on studying the moments of the distribution of Salem numbers, with a potential application to Lehmer's problem. The idea was to try to exploit some methods from "On a conjecture for $\ell$-torsion in class groups of number fields: from the perspective of moments" by Lillian B. Pierce, Caroline L. Turnage–Butterbaugh and Melanie Matchett Wood. It was a great opportunity to closely study this nice work and related papers, but unfortunately it did not led to any interesting conclusions. Apparently, the statistical information available is not sufficient for making desired connections.

At this point, the group activity switched to another direction related to arithmetic hyperbolic spaces. It has been known for some time that exponential lengths of the closed geodesics of an arithmetic hyperbolic $n$-dimensional manifold or orbifold are given by Salem numbers. For $n = 2$ and 3 this relation is described in the book by C. Maclachlan and A. Reid [4, Chapter 12]. More recently, it was elaborated and generalized to higher dimensions by V. Emery, J. Ratcliffe and S. Tschantz in [2]. In particular, their Theorem 1.1 implies that for a non-compact arithmetic hyperbolic $n$-orbifold $\mathcal{O}$, a closed geodesic of length $\ell$ corresponds to a Salem number $\lambda = e^{\ell}$ if the dimension $n$ is even, and to a so called square-rootable Salem number $\lambda = e^{2\ell}$ if $n$ is odd. The degrees of these Salem numbers satisfy $\deg(\lambda) \leq n + 1$. A natural question arises: *What proportion of Salem numbers of a given degree are associated to a fixed orbifold* $\mathcal{O}$*?*

The group was able to answer this question for 3-dimensional non-compact arithmetic orbifolds — see Theorem 1 in the previous section. The higher dimensions and compact arithmetic 3-orbifolds are associated to Salem numbers of higher degree, and this remains open for the future study.

1.3. **Current state.** The results on non-compact arithmetic hyperbolic 3-orbifolds, Bianchi groups, and Salem numbers of degree 4 are currently under preparation for publication.

1.4. **Future plans.** The project highlighted two challenging problems for future research — multiplicities in geodesic spectrum of arithmetic hyperbolic $n$-orbifolds for $n > 3$, and distribution of square-rootable Salme numbers of degree $d > 4$. The group plans to continue investigation in both directions.

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**Project Group: Additive combinatorics**

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During the last 15 years the methods and tools from Additive Combinatorics have found many important applications in Analytic Number Theory. One example is the application of the *sum-product phenomenon in finite fields* in the study of exponential sums. Specifically, let $F_p$ be the field of prime order $p$ and $F_p^*$ be its multiplicative subgroup. Let $H$ be a subgroup of $F_p^*$ with $\#H > p^{\varepsilon}$ and let $a$ be an integer with $\gcd(a,p) = 1$. The problem of obtaining nontrivial upper bounds for the exponential sum

$$S = \sum_{x \in H} e_p(ax)$$

is a classical problem with a variety of results and applications in number theory. The result of Gauss implies that if $\#H = (p-1)/2$, then $|S| = p^{1/2}$. From the work of Hardy and Littlewood on the Waring problem it is known that $|S| < p^{1/2}$, which is non-trivial when $\#H > p^{1/2}$. The problem of obtaining nontrivial bounds for $\#H < p^{1/2}$ has been a subject of much investigations, culminating in the work of Bourgain, Konyagin and Glibichuk [2]. Using the sum-product estimate and other tools from additive combinatorics, they proved that if $H$ is a subgroup of $F_p^*$ with $\#H > p^{\varepsilon}$, then

$$\Big|\sum_{x \in H} e_p(ax)\Big| \le \#H \cdot p^{-\delta},$$

where $\delta = \delta(\varepsilon) > 0$. Prior to their work, this estimate had been only known under the assumption $\#H > p^{1/4+\varepsilon}$ due to Konyagin. In the limiting case $\#H \sim p^{1/4}$ Bourgain and Garaev [1] obtained an explicit bound, based on explicit sum-product estimates. Another topic of interest, is the double sum involving intervals and subgroups, that is the sum of the form

$$\sum_{n=L+1}^{L+N} \sum_{x \in H} e_p(anx).$$

This sum has also been investigated in a series of works.

During 11 –16 August 2019 in Casa Matemática Oaxaca we have studied some fundamental results and tools in the study of exponential sums that stems from Additive Combinatorics. Furthermore, combining recent results of Murphy, Rudnev, Shkredov and Shteinikov [3] and of Petridis and Shparlinski [4] with the arguments of Bourgain and Garaev [1], we obtained several new results on single exponential sums over subgroups and double sums over intervals and subgroups. In particular, in the limiting case $\#H \sim p^{1/4}$ we significantly improved on the explicit bound of Bourgain and Garaev.

Currently, the group continues the work on the project and collaborates with Igor Shparlinsky. We are planning to finish the project later this year and prepare a research paper on the final results.

## References


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## Project Group: El número de fracciónes egipcias ternarias con denominador fijo

Carlos Alexis Gomez, Florian Luca, Enrique Treviño

### 2. El problema

Una fracción egipcia es una representación de $a/n$ de forma

$$\frac{a}{n} = \sum_{i=1}^{k} \frac{1}{x_i}$$

donde $x_1, \ldots, x_k$ son enteros positivos. El número $k$ se llama el tamaño o la longitud de la fracción egipcia. Lo que nos interesa es la función

$$\begin{aligned} f_k(n) &= \#\{a : a/n = 1/x_1 + \cdots + 1/x_k : x_1, \ldots, x_k \in \mathbb{N}\}; \\ f_k^*(n) &= \#\{a : (a,n) = 1 \ \ y \ \ a/n = 1/x_1 + \cdots + 1/x_k : x_1, \ldots, x_k \in \mathbb{N}\}. \end{aligned}$$

También nos interesa estudiar el orden de magnitud de

$$F_k(x) = \sum_{n \le x} f_k(n) \qquad \text{y de} \qquad F_k^*(x) = \sum_{n \le x} f_k^*(n).$$

### 3. Informe

Como antecedente, habiamos mencjonado el artículo [2] en el cual habiamos probado que

$$x(\log x)^3 \ll \sum_{p \le x} f_3(p) \ll x(\log x)^5.$$

Este trabajo ya salió publicado en *Research in Number Theory* December, **2019**.

Acerca del problema, hemos obtenido los siguientes resultados.

**Theorem 3.1.** *Sea* $h(n) := C/\log\log n$ *(con una constante explicita* $C \approx 1.066$*), para* $n \ge 57000$ *se tiene que:*

$$f_3(n) \le 10 n^{\frac{1}{2} + \frac{13}{4} h(n)} \log n.$$

**Corollary 3.2.** *Para* $n \ge 10^{10^{23}}$*, se tiene la siguiente desigualdad:*

$$A_3(n) < \frac{1}{100} n^{\frac{1}{2} + \frac{1}{15}}.$$

Tambin hemos estudiado las siguientes funciones:

$$f_a(n) = \#\left\{(m_1, m_2, m_3) \,:\, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\},$$

y

$$F(n) = \#\left\{(a, m_1, m_2, m_3) \,:\, \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}.$$

**Theorem 3.3.**

$$f_a(n) \leq n^{\varepsilon}\left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho}\right). \tag{2}$$

**Theorem 3.4.** *Se tiene* $F(n) \ll n^{5/6+\varepsilon}$.

Los primeros valores de $F(n) < n$ son: $F(8821) = 8590, F(11161) = 10270, F(11941) = 10120$. Es un problem abierto encontrar el máximo $n$ tal que $F(n) > n$. Sin embargo hemos probado que tal $n$ cumple $n < 10^{10^{23}}$.

**Theorem 3.5.** *Para* $n \geq 10^{10^{23}}$, *tenemos* $F(n) < \frac{1}{10}n$.

Estos resultados han sido incluido en un articulo [1] que ha sido enviado para su publicacióm.

## References


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## Project Group: Modular forms attached to totally real number fields of small degree

Adrian Barquero, Guillermo Mantilla-Soler, Roberto Miatello, Nathan Ryan

The group led by Guillermo Mantilla-Soler had initially intended to generalize a construction, made by Mantilla-Soler, on modular forms attached to real cubic fields to higher dimensional fields. At the beginning of the meeting it was not so clear what was the right space of modular forms in which the construction could work but after a couple of discussions among the members of the group there was a consensus on the space. Afterwards, writing some code in MAGMA and SAGE the group sailed out to prove a couple of theorems and made some conjectures based on computational evidence, and heuristics developed in Oaxaca. At the end of the workshop Mantilla-Soler's group had a clear road map of what to prove and how to do so; they even started typing out their theorems during the meeting. The group kept in close contact after the meeting, and just over a month ago the group posted in the arXiv https://arxiv.org/abs/1910.00202 their results. In addition to this, an article derived from this group project has been submitted to Experimental Mathematics.

**Project Group: Arithmetic of algebraic varieties**
Maria Chara, Elisa Lorenzo Garcia, Álvaro Lozano-Robledo,
Cecilia Salgado, Tony Varilly-Alvarado

This group did not submit a project report.

**Project Group: Langlands functoriality**
Adrian Zenteno Gutierrez, Ariel Pacetti,
Daniel Barrera Salazar, Gonzalo Tornaria

The original plan of the group led by Ariel Pacetti was to improve a bound on the 2-Selmer group of elliptic curves over number fields in terms of class groups. The main references were an article due to Brumer-Kramer (Duke 1977) where re the authors provide an upper bound under some hypothesis, and one by Chao Li (Trans. AMS 2019) where, under more restrictive hypotheses, the author gives a lower bound as well.

During the workshop, we focused on improving the lower bound to include more general elliptic curves, with the main objective of understanding the behavior of the 2-Selmer group in families of quadratic twists. We managed to obtain such a bound and generalize both the lower and the upper bound for elliptic curves over general number fields. Furthermore, we computed many examples showing that the bounds provided are optimal (for some quadratic and cubic fields). The members of the group wrote a draft of the results, and we expect to post it on arxiv (and submit it for publication) in the next days.

Numerical experiments show that the bounds obtained do not hold if we remove our hypothesis, but it is an interesting problem to study whether other bounds are still valid. This might be a future collaboration between the members.

**Project Group: Diophantine approximations and fractal geometry**
Harold Erazo, Sergio Augosto Romana Ibarra, David Krumm, Diego Marques,
Carlos Gustavo Moreira, Rodolfo Joaquin Gutierrez Romo

Our original plan was to study properties of the classical Markov and Lagrange spectra from Diophantine approximations - particularly from the viewpoint of Fractal Geometry, and of some natural dynamical generalizations of these spectra.

Our group worked in two subgroups: one of them, formed by Moreira, Erazo, Romaña and Gutiérrez-Romo worked on fractal properties of the classical Markov and Lagrange spectrum, and proved a result on ther behaviour close to 3 (which is the smallest accumulation point of both spectra):

**Theorem 3.6.** *There are constants* $0 < C_1 < C_2$ *such that, for every* $n \in \mathbb{N}$*,*

$$\frac{C_1 \log(n)}{n} \leq \mathrm{HD}(L \cap [0, 3 + 2^{-n}]) = \mathrm{HD}(M \cap [0, 3 + 2^{-n}]) \leq \frac{C_2 \log(n)}{n}$$

The corresponding paper is still in preparation, and should be submitted to an international journal when it becomes ready.

The other group, with Moreira, Marques and Krumm, worked on another subject related to Dynamical Systems and Number Theory: the dynamics of transcendental entire functions which leave the set of algebraic numbers invariant. They proved the following result:

Let $T$ be the set of entire functions. We define a topology on $T$: given

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in T,$$

an open neighbourhood of $f$ consists of all functions

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \in T$$

for which $|a_n - b_n| < \epsilon_n$ for every $n$, where $(\epsilon_n)$ is a given sequence of poositive real numbers.

**Theorem 3.7.** *Let $(k_n)$ be a sequence in $\mathbb{N} \cup \{\infty\}$ and let $X$ be the set of functions $f \in T$ such that*

- $f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$, $f^{-1}(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$, *y*
- *$f$ has exactly $k_n$ orbits $n$-periodic in $\overline{\mathbb{Q}}$ for every $n$.*

*Then $X$ is dense on $T$.*

The corresponding paper is in preparation, and should also be submitted to an international journal when it becomes ready.

### Project Group: Automorphic forms

Lea Beneish, Michael Harris, Luis Lomeli, Alberto Minguez, Robin Zhang

Let $G$ be a connected split classical group over a global field $F$, with Langlands dual group $\hat{G}$. This is also a classical group. If $G$ is the special orthogonal group $SO(V)$ with $\dim V = 2n+1$ odd, then $\hat{G} = Sp(2n)$ is symplectic, and vice versa; if $G$ is even special orthogonal then so is $\hat{G}$. Thus in each case there is a standard representation $\rho : \hat{G} \to GL(N)$ for appropriate $N$. Langlands functoriality then predicts a relation between cuspidal automorphic representations of $G$ and (not necessarily cuspidal) automorphic representations of $GL(N)$. In one direction, if $\pi$ is a cuspidal automorphic representation of $G(\mathbf{A}_F)$, then the Langlands transfer $\rho_*(\pi)$, characterized almost everywhere by well-known relations between Satake parameters, is an automorphic representation of $GL(N)$. When $\pi$ is generic this was proved by Cogdell-Kim-Piatetski-Shapiro-Shahidi, using properties of $L$-functions; in general this was proved by Arthur using the stable twisted trace formula, at least when $\pi$ is tempered.

Moreover, Langlands functoriality identifies the image of the transfer $\rho_*$, as a subset of the *self-dual* representations $\Pi$ of $GL(N)$ – $\Pi \xrightarrow{\sim} \Pi^\vee$. Such $\Pi$, when they are cuspidal, are characterized by the property that the Rankin-Selberg $L$-function $L(s, \Pi \times \Pi)$ has a simple pole at $s = 1$. Since $L(s, \Pi \times \Pi)$ factors naturally as the product

$$L(s, \Pi \times \Pi) = L(s, \Pi, Sym^2) \cdot L(s, \Pi, \wedge^2),$$

this means that exactly one of the two factors has a pole. If $L(s, \Pi, \wedge^2)$ has a pole then $\hat{G}$ must be symplectic and so $G$ must be odd orthogonal (and $N$ is then even). On the other hand, if $L(s, \Pi, Sym^2)$ has a pole then $\hat{G}$ must be orthogonal, and then either $G$ is symplectic if $N$ is odd or $G = SO(N)$ if $N$ is even.

When $F$ is a number field, it was proved by Ginzburg-Rallis-Soudry that any self-dual cuspidal automorphic representation $\Pi$ of $GL(N)$ (and many $\Pi$ that are not cuspidal) does indeed come by functorial transfer from a cuspidal automorphic representation of the appropriate $G$, determined by the pole of the appropriate factor of $L(s, \Pi \times \Pi)$. This is called *automorphic descent*. The method is roughly as follows. Starting with $\Pi$ one constructs an Eisenstein representation $E(s, \Pi)$ of some large classical group $H$. The existence of a pole at $s = 1$ of either $L(s, \Pi, Sym^2)$ or $L(s, \Pi, \wedge^2)$ implies the existence of a non-trivial residual representation $R(\Pi)$ of $H$. One then studies the Fourier coefficients $R_\psi(\Pi)$ of $R(\Pi)$ as automorphic representations of a certain subgroup $G_\psi$ of $H$, the stabilizer of the linear form denoted $\psi$, that can be identified with the desired $G$. In a series of steps one proves that $R_\psi(\Pi)$ is non-trivial, irreducible, cuspidal, generic, and of multiplicity one, and that

$$\rho_*(R_\psi(\Pi)) \xrightarrow{\sim} \Pi.$$

The project of the group was to extend the results of Ginzburg-Rallis-Soudry [GRS] to global fields of positive characteristic $p$. The group spent the first two days studying [GRS] in the simplest cases, when $G$ is odd orthogonal or symplectic and the specific properties of orthogonal and symplectic groups in characteristic 2. Since automorphic descent for symplectic groups involves the Weil (oscillator) representation, we also studied how the Weil representation behaves in characteristic 2. We convinced ourselves that the methods of [GRS] extend without difficulty when $p$ is odd, but that many of the constructions need to be modified or replaced when $p = 2$. In particular, very little has been written about the global Weil representation in characteristic 2, and practically nothing about its applications to automorphic forms. So we decided to spend the rest of the week working on odd orthogonal groups, following a survey article of Soudry.

The initial steps in [GRS] are the determination of stabilizers in $H$ of linear forms corresponding to Fourier coefficients, the an analysis of double cosets in $H$, and the study of the poles of $L(s, \Pi, Sym^2)$ and $L(s, \Pi, \wedge^2)$. These were begun in Oaxaca and continue. The rest of the time in Oaxaca was spent outlining the steps of the proof that remain to be completed once the initial setup is complete, following the strategy sketched by Soudry. The computations are long and difficult but in principle they are the same over function fields as over number fields.