Current Trends in Arithmetic Geometry and Number Theory

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1 Arithmetic geometry and number theory

The area of arithmetic geometry is motivated by studying questions in number theory through algebraic geometry and representation theory, a viewpoint which was hinted at in the 19th century and which has been brought to fruition very successfully this century. Due to the variety of the techniques and theory required, it is an area which maintains deep interconnections with other branches of mathematics such as algebra, analysis, and topology.

Arithmetic geometry has witnessed many significant results in past decades. Some of the better known examples include the proofs of the Mordell conjecture [5], Fermat's Last Theorem [10], and the modularity conjecture [3]. Other examples include progress towards the Birch and Swinnerton-Dyer conjecture (now a Clay Millenium Prize), the Langland's conjecture for GL_n over function fields [8] (work awarded a Fields medal in 2002), and results revolving around the Langlands' and Serre's conjectures.

2 p-adic methods

The field of p-adic numbers $\mathbb{Q}_p$ is obtained by completing the rationals with respect to a non-archimedean absolute value associated to a prime p which measures distance in terms of divisibility by p. It is analogue of the real numbers in terms of analysis (but with some important differences) and allows one to focus one's attention on the contribution of a single prime p from the point of view of arithmetic.

The use of p-adic methods in arithmetic geometry has been pervasive. Many recent developments in number theory have relied crucially on the use of p-adic methods. These arise in many forms, such as via p-adic representation theory, p-adic L-functions, and p-adic geometry. These have figured prominently in the recent progress on the Birch and Swinnerton-Dyer conjecture, the proof of Fermat's Last Theorem, and the modularity conjecture.

This workshop brought together both experts and newcomers to these areas of number theory. There were two components to the workshop:

- Three lectures per day on recent developments in the field (a total of 12 lectures), consisting of various mathematicians reporting on their research in the field.
- A series of instructional lectures on Φ-Γ modules, period rings, and their applications. These lectures were aimed at those who are not specialists in the field, and this series consisted of 2 lectures per day (a total of 9 lectures). The lectures were given by Brian Conrad, Adrian Iovita, Nathalie Wach, Pierre Colmez, Laurent Berger, and Kiran Kedlaya (in order of presentation).

The setting and support provided by BIRS greatly facilitated the aims of the scientific activities of the workshop. The travel expenses for junior participants were offset by support from MSRI and some of the conference organizers. The atmosphere was particularly conducive to extensive mathematical discussion among the participants outside of the time of lectures, both on the topics of the lectures as well as on their own current work.

3 p-adic Representation Theory

In number theory, much effort is devoted to the study of arithmetic properties of algebraic varieties over global fields (that is, finite extensions of the rational field or of a rational function field over a finite field) and over local fields (that is, finite extensions of the field $\mathbb{Q}_p$ or a formal Laurent series field over a finite field). A basic example is the problem of determining the rank of the group of rational points on an elliptic curve over a global field. The arithmetic properties of such algebraic varieties are often encoded in representation spaces (such as étale cohomology) for the absolute Galois group G_K of the global base field K. Somewhat abstracting the underlying structure of these basic examples, one is led to study the theory of *p-adic representations* of such Galois groups; these are continuous representations of the Galois group on finite-dimensional p-adic vector spaces, that is, vector spaces over $\mathbb{Q}_p$. Further development of the theory (inspired by the local Langlands conjectures) shows that it is equally important to study the representation theory of groups of the form $G(k)$ where k is a local field and G is a reductive algebraic group over k; historically such representations were on complex vector spaces, but in recent years there has developed a rich theory of such representations on (infinite-dimensional) p-adic vector spaces (where p is the residue characteristic of k).

Many examples of p-adic representations arise naturally from geometry in the sense that given a variety over a global field one can attach p-adic vector spaces with prescribed dimensions (arising from the étale cohomology groups of the variety) and on these the Galois group naturally acts in a continuous and linear way.

A conjecture of Fontaine and Mazur [7] gives local conditions which are necessary and sufficient to ensure that an irreducible p-adic representation arises from geometry. Roughly speaking, a p-adic representation is geometric if it is unramified outside a finite set S of places of K and its restriction to a decomposition group at a place of S is potentially semi-stable. By a theorem of Grothendieck, only the places $v \mid p$ impose an actual condition. The predicted dimensions of the variety are determined from the p-adic Hodge-Tate weights of the p-adic representation.

In the 1960's, Tate discovered that the p-adic étale cohomology of a smooth projective variety over a p-adic field should have a canonical structure analogous to the Hodge filtration on topological cohomology for smooth projective varieties over the complex numbers (and Tate constructed such a filtration in the case of degree-1 étale cohomology for an abelian variety with good reduction). Partly motivated by Tate's discovery, as well as Grothendieck's "mysterious functor" relating de Rham cohomology and crystalline cohomology in degree 1, Fontaine was led to develop the theory of so-called period rings (thereby leading to the creation of p-adic Hodge theory). These period rings and functors defined with them are the fundamental mechanism for distinguishing the "good" p-adic representations of local Galois groups G_{K_v} when $v|p$. The periods rings, typically denoted B, are topological K_v-algebras equipped with a continuous linear action of G_{K_v} as well as extra G_{K_v}-equivariant linear-algebra structure (such as a grading, or a filtration, etc). For each such B, one defines $D_B(V) = (B \otimes_{\mathbf{Q}_p} V)^{G_{K_v}}$; this is a finite-dimensional vector space over K_v with dimension at most the $\mathbf{Q}_p$-dimension of V and with auxiliary structure inherited from B. When equality holds then one says that V is B-*admissible*, and for various choices of B one gets various special classes of p-adic Galois representations (such as de Rham representations, crystalline representations, and semistable representations). "All" p-adic Galois representations arising from the étale cohomology of algebraic varieties over local fields are B-admissible for a suitable B, where the "best" choice of B depends on the geometry of the variety under consideration.

In many situations the functor D_B is fully faithful into a category of "semi-linear-algebra" structures, and V may be recovered from $D_B(V)$. This state of affairs inspires the problem of determining the essential image of D_B for various B (solved by Fontaine and Colmez in the case $B = B_{\mathrm{st}}$), and more generally it motivates the philosophy of translating hard problems concerning p-adic Galois representations into (hopefully more tractable) problems in a suitable "semi-linear algebra" category.

For example, in the early days of the subject, Fontaine gave a more tractable way to work with p-adic representations of a local Galois group G_{K_v} by identifying this with the category of étale (Φ, Γ)-modules. These are modules (over a certain ring) endowed with a semi-linear "Frobenius" endomorphism Φ and a semi-linear action of a certain "explicit" quotient Γ of G_{K_v} such that Φ commutes with the action of Γ and has slope 0.

Extending work by Cherbonnier and Colmez [4], Berger shows how to associate to every p-adic representation V an invariant $D^{\dagger}_{\mathrm{rig}}(V) := (B^{\dagger}_{\mathrm{rig}}(V) \otimes V)^G$ where $B^{\dagger}_{\mathrm{rig}}$ is the Robba ring . The Robba ring arises in the theory of p-adic differential equations in such a way that the invariant $D^{\dagger}_{\mathrm{rig}}(V)$ can be interpreted as a p-adic differential equation. Berger's construction uses Fontaine's theory of (Φ, Γ)-modules and allows one to recognize semi-stable and crystalline representations in the sense that $D_{\mathrm{st}}(V)$ and $D_{\mathrm{cris}}(V)$ can be constructed from $D^{\dagger}_{\mathrm{rig}}(V)$.

If V is de Rham, the associated p-adic differential equation has much better behavior than one might have expected *a priori*. This allows one to obtain a classical p-adic differential equation, and using a recent theorem of André (also proved independently by Kedlaya and Mebkhout) this allows one to show that every de Rham representation is potentially semi-stable.

The research talks which touched upon recent progress in the area of p-adic representation theory were those by Jeremy Teitelbaum, Gebhard Böckle, Matt Emerton, and Kiran Kedlaya.

4 p-adic L-functions

L-functions are meromorphic functions of a complex variable, and they are obtained from algebraic objects by means of infinite products or sums. It is a mysterious phenomenon that L-functions capture arithmetic information about the algebraic object. The mechanism behind this is only well-understood in a few cases, but in general settings there are many conjectures which predict the precise relationship between the analytic properties of associated L-function and the arithmetic properties of the algebraic object.

The most celebrated example of an L-function is the Riemann zeta-function. This is known to capture information about the integers. Another well-known example is the L-function attached to an elliptic curve over $\mathbb{Q}$. The conjectural properties of this L-function encode the arithmetic of the elliptic curve, as is made precise by the Birch and Swinnerton-Dyer conjecture.

A natural point of view which has gained much attention is to try to construct and study L-functions which are meromorphic functions of a p-adic variable. One of the first attempts was by Kubota and Leopoldt. They constructed a p-adic zeta-function motivated by classical congruences. The construction of p-adic L-functions has been considerably generalized to broader contexts. Iwasawa formulated a Main Conjecture for his p-adic L-functions which was later proved by Mazur and Wiles. This conjecture became the paradigm for Main Conjectures in many other settings.

Several of the research talks reported recent progress on subjects related to p-adic L-functions and various Main Conjectures. These included the talks of Haruzo Hida, Eric Urban, Adebisi Agboola, and Robert Pollack.

5 p-adic Geometry

In the case of a p-adic representation arising from geometry, the geometrical properties of the variety in question are reflected in the properties of the representation and the associated automorphic form. Of particular interest are the geometrical properties of the variety as a p-adic analytic object. This is a viewpoint which gained prominence through work of Robert Coleman and Nicholas Katz on p-adic modular forms in the classical case.

The research talks in this and related areas were those given by Ehud de Shalit, Graham Herrick, Robert Coleman, Mak Trifkovic, and Thomas Zink.

6 Instructional lectures

The instructional component provided an introduction to the recent proof by Berger [2] of the p-adic Monodromy Conjecture of Fontaine described above. This conjecture states that every p-adic de Rham representation is potentially semi-stable. A good account of Berger's work can be found the featured review [9] by Adolfo Quirós.

1. **Brian Conrad** p-adic Hodge theory I
2. **Brian Conrad** p-adic Hodge Theory II
3. **Adrian Iovita** Introduction to Φ-Γ-modules I
4. **Adrian Iovita** Introduction to Φ-Γ-modules II
5. **Nathalie Wach** $\Phi - \Gamma$-modules of finite height
6. **Pierre Colmez** Overconvergent Φ-Γ-modules
7. **Laurent Berger** p-adic Galois representations and p-adic differential equations I
8. **Laurent Berger** p-adic Galois representations and p-adic differential equations II
9. **Kiran Kedlaya** Frobenius slope filtrations and Crew's conjecture

7 The participants

1. Agboola, Adebisi (University of California Santa Barbara)
2. Berger, Laurent (Harvard University)
3. Boeckle, Gebhard (Institute for Experimental Mathematics)
4. Chang, Seunghwan (Brandeis University)
5. Chen, Imin (SFU)
6. Chenevier, Ga'tan (Ecole Normale Superieure)
7. Coleman, Robert (Univ of California Berkeley)
8. Colmez, Pierre Conrad, Brian (Univ. Michigan - Ann Arbor)
9. Cornut, Christophe (Jussieu)
10. Diamond, Fred (Brandeis University)
11. Edixhoven, Bas (Mathematisch Instituut, Leiden)
12. Emerton, Matthew (Northwestern University)

13. Goren, Eyal (McGill University)
14. Greenberg, Ralph (University of Washington)
15. Herrick, Graham (Northwestern University)
16. Hida, Haruzo (UCLA)
17. Iovita, Adrian (University of Washington)
18. Kassaei, Payman (McGill University)
19. Kedlaya, Kiran (MIT)
20. Khare, Chandrashekhar (Tata Institute of Fundamental Research)
21. Kisin, Mark (Westfalische Wilhelms Universitat)
22. Mailhot, Jim (University of Washington)
23. Marshall, Susan (University of Texas)
24. Nichifor, Alexandra (University of Washington)
25. Niziol, Wieslawa (University of Utah)
26. Pollack, Robert (University of Chicago)
27. Ramakrishna, Ravi (McGill University)
28. Savitt, David (IHES, Le Bois-Marie)
29. Schneider, Peter (Mathematisches Institut Muenster)
30. de Shalit, Ehud (Hebrew University, Giv'at-Ram)
31. Skinner, Chris (Univ. Michigan, Ann Arbor)
32. Teitelbaum, Jeremy (University of Illinois at Chicago)
33. Trifkovic, Mak (McGill University)
34. Urban, Eric (LAGA-CNRS)
35. Vatsal, Nike (UBC)
36. Virdol, Cristian (University of Southern California)
37. Wach, Nathalie (IRMA, Univ. Strasbourg)
38. Wortmann, Sigrid (Mathematisches Institut Heidelberg)
39. Zink, Thomas (Universitat Bielefeld)

8 Titles and Abstracts of talks

Adebisi Agboola (UC Santa Barbara)

Title: Anticylotomic Main Conjectures for CM elliptic curves

Abstract: (joint with Ben Howard) This is a report on joint work with B. Howard. We shall discuss the Iwasawa theory of a CM elliptic curve E in the anticyclotomic Z_p extension of the CM field, where p is a prime of good, ordinary reduction for E. When the complex L-function of E vanishes to odd order, work of Greenberg shows that the Pontryagin dual of the p-power Selmer group over the anticyclotomic Z_p-extension is not a torsion Iwasawa module. We shall show that the dual of the Selmer group is a rank one Iwasawa module, and we shall prove one divisibility of an Iwasawa Main Conjecture for its torsion submodule.

Gebhard Böckle (ETH-Zurich)

Title: A conjecture of de Jong

Abstract: (joint with Chandrashekhar Khare) Let X be a smooth curve over a finite field k of characteristic p and $\overline{X}$ its base change to the algebraic closure $\overline{k}$ of k. In this context, one has the following conjecture of de Jong: Let $\rho : \pi_1(X) \to \mathrm{GL}_n(F_l[[T]])$ be continuous and $l \neq p$. Then $\rho(\pi_1(\overline{X}))$ is finite. In the first part of the talk I plan to present various reformulations and consequences of the conjecture of de Jong as well as known results. for instances, assuming the conjecture, one easily obtains an analogue of Serre's conjecture for function fields. In the second part, I will sketch an approach to proving the conjecture in many cases. It is based on the methods used in the modularity proofs by Wiles, Taylor, et al. as well as on the recent results of Lafforgue on the Langlands conjecture.

Robert Coleman (UC Berkeley)

Title: $X_0(125)$

Abstract: Suppose p is a prime and $(p, N) = 1$. The minimal model of $X_0(pN)$ over Z_p was determined by Deligne-Rapoport. It is also (semi-)stable. Edixhoven, Katz and Mazur determined the minimal model of $X_0(p^n N)$ and interpreted it moduli theoretically. It is not semi-stable in general. Edixhoven determined the stable model of $X_0(p^2 N)$ but gave no moduli-theoretic interpretation of his results. Results of McMurdy and myself interpret Edixhoven's model, compute the stable model of $X_0(125)$ and suggest that something very interesting is going on.

Matt Emerton (Northwestern University)

Title: On the ramification of Hecke algebras at Eisenstein primes

Abstract: (Joint with Frank Calagari) In his celebrated "Eisenstein ideal" paper, Mazur considers the completion of the Hecke algebra acting on weight two modular forms of prime level N at the so-called Eisenstein maximal ideals (the ideals of fusion between cuspforms and the weight two Eisenstein series). He shows that any such completion is finite, flat, monogenic Z_p-algebras (here p is the residue characteristic of the Eisenstein maximal ideal under consideration) and raises the question of computing its rank over Z_p. In this talk I will explain how to relate this rank to class field theoretic properties of the algebraic number field $Q((-N)^{1/p})$. When $p = 2$, we are able to determine the rank completely: it equals $2^{(m-1)} - 1$, where 2^m denotes the two-power part of the class number of $Q((-N)^{1/2})$. If p is odd, then our results are less definitive. However, when combined with earlier work of Merel, they do yield the following theorem: If $p \geq 5$, and $N \equiv 1 \pmod{p}$, then the p-part of the class group of $Q(N^{1/p})$ is cyclic only if $\prod_{\ell=1}^{(N-1)/2} \ell^\ell$ is not a pth power $\pmod{N}$. The method of proof depends on identifying the Eisenstein completions of the Hecke ring with certain universal deformation rings of Galois representations. This identification (which we prove following the method of Wiles) is interesting in itself, since it allows us to recover all of Mazur's structural results concerning these Eisenstein completions while avoiding any analysis of the arithmetic of the Jacobian $J_0(N)$.

Graham Herrick (Northwestern University)

Title: Cusp forms mod p and conjectural slope formulae

Abstract: This talk gave an explicit formula for the slopes of classical modular forms on $\Gamma_0(N)$ whose associated Galois representation is reducible when restricted to a decomposition group.

Haruzo Hida (UCLA)

Title: Anticyclotomic Main Conjecture

Abstract: Non-vanishing modulo p of almost all twisted Hecke L-values combined with the technique I invented with Tilouine in 1993 gives the divisibility of the anti-cyclotomic Iwasawa power series by the anticyclotomic p-adic Hecke L-function. Under the assumption of Taylor-Wiles-Fujiwara (which gives the identification of the Hilbert modular p-ordinary Hecke algebra with a Galois deformation ring), we can prove the reverse divisibility via the integrality theory of definite and indefinite theta series (an integral Jacquet-Langlands-Shimizu correspondence). I will give a sketch of the proof of the reverse divisibility.

Robert Pollack (University of Chicago)

Title: Relations between congruences of modular forms and the Main Conjecture

Abstract: (joint with Matthew Emerton, Tom Weston) This talk discussed a generalization to arbitrary weight of a result of Greenberg-Vatsal concerning μ and λ invariants in Hida families.

Ehud de Shalit (Hebrew University, Givát-Ram)

Title: The p-adic monodromy-weight conjecture for p-adically uniformized varieties

Abstract: (joint with Peter Schneider) The p-adic monodromy-weight conjecture asserts that the monodromy filtration and the weight filtration on the Hyodo-Kato cohomology of a smooth proper variety over Q_p with semistable reduction, coincide. It is a p-adic analogue of an l-adic conjecture due to Deligne. Both conjectures are in general open, but for p-adically uniformized varieties, they were recently proved by T. Ito (using l-adic cohomology) and, independently, by Gil Alon and the speaker (using p-adic cohomology). We shall explain the conjecture and the main ideas behind the p-adic proof. [In fact, as Ito observed, for p-adically uniformized varieties the p-adic and l-adic conjectures imply each other].

Jeremy Teiltelbaum (University of Illinois at Chicago)

Title: An update on p-adic analytic representation theory

Abstract: We will describe some results on the structure of p-adic analytic representations and relations with arithmetic, especially p-adic Galois representations.

Mak Trifkovic (McGill University)

Title: Elliptic curves over imaginary quadratic fields and p-adic constructions of rational points

Abstract: We will discuss a conjectural p-adic analytic construction of rational points on elliptic curves defined over an imaginary quadratic field K, using mixed period integrals a là Darmon. The main difference relative to curves over Q is the nature of the (merely conjectural) modularity: the form corresponding to E/K is a harmonic form on a suitable quotient of H^3, which is not an algebraic variety. We will present the computational evidence for some of the conjectures.

Eric Urban

Title: Deformations of Eisenstein series and applications

Abstract: (joint with Chris Skinner) This talk described work on divisibility of special values of various L-functions attached to automorphic forms on $GU(2,2)$ over K an imaginary quadratic field.

Thomas Zink (Universitat Bielefeld)

Title: Higher displays in the crystalline cohomology over an artinian ring

Abstract: We introduce the tensor category of higher displays over an artinian ring. The subcategory of nilpotent displays is equivalent to the category of formal p-divisible groups. The de Rham-Witt complex defines the structure of a display on the first crystalline cohomology group of a proper and smooth scheme. We show how a higher display structures should be defined on higher crystalline cohomology groups and obtain these structures for liftable schemes.

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