# Conformal Geometry

Thomas Branson (University of Iowa),
Michael Eastwood (University of Adelaide),
McKenzie Wang (McMaster University).

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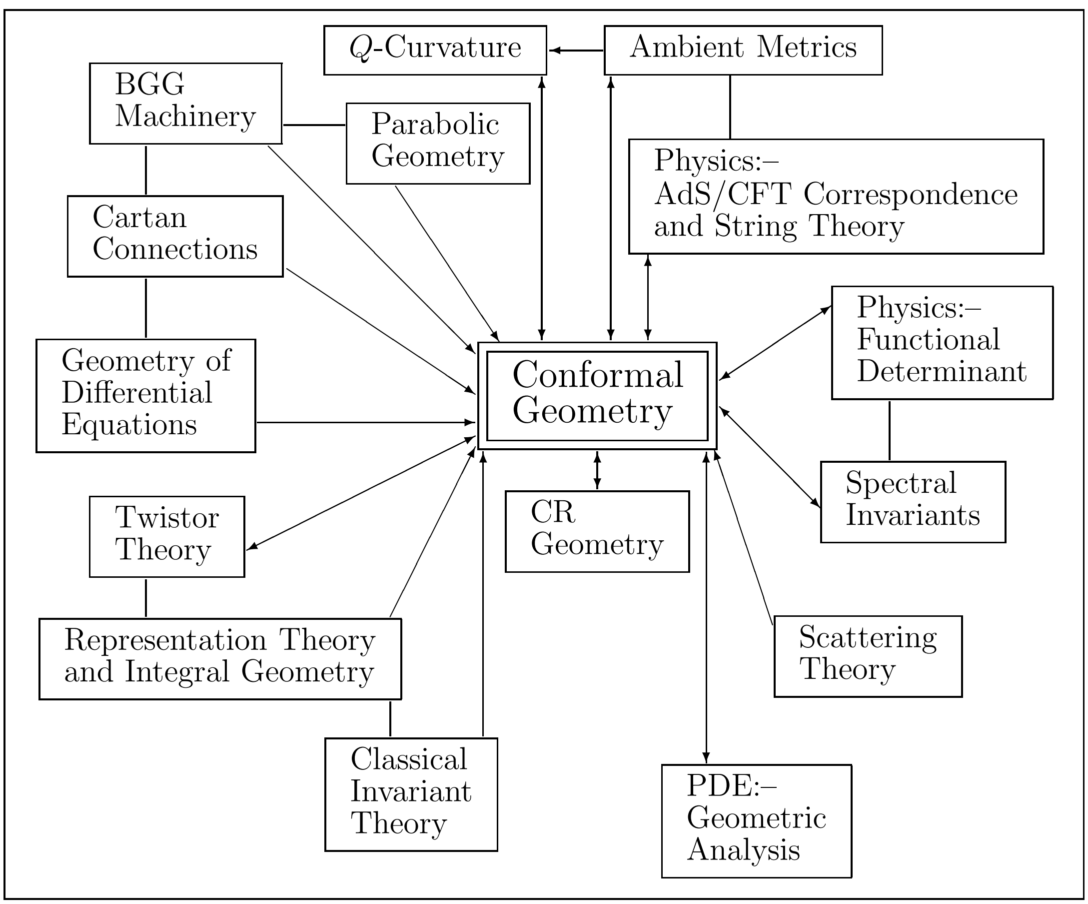
The workshop organisers see conformal geometry as central in the following circle of ideas:–




Arrows in this diagram indicate input from one topic to another. Closely related topics are joined by lines. Conformal geometry is highly analogous to CR geometry, so their boxes are close together and arrows run in both directions. The left hand side of the diagram is largely algebraic. At the top of the diagram, <u>$Q$-curvature</u> and <u>ambient metrics</u> are specific aspects of conformal geometry, which are separated for special attention. The right hand side of the diagram is more concerned with applications in geometric analysis and physics. The workshop touched on all aspects of this diagram and the discussion below will refer to topics in the diagram by underlining them.

In contrast with the more familiar Riemannian geometry, there are clear difficulties concerning the basic local geometry, symmetry, and invariance in conformal geometry. These difficulties may be traced to differences in the 'flat model': Riemannian geometry is a curved version of Euclidean geometry whereas conformal geometry is a curved version of the sphere $S^n$ as a homogeneous space for $\mathrm{SO}(n+1,1)$. In particular, the isotropy subgroup in the Euclidean case is $\mathrm{SO}(n)$ and in Riemannian geometry there is a principal $\mathrm{SO}(n)$-connection on the frame bundle responsible for the usual Riemannian curvature. On the conformal sphere, however, the isotropy subgroup is a parabolic subgroup $P \leq \mathrm{SO}(n+1,1)$ whose algebraic structure and representation theory is much more subtle. There is a principal $P$-bundle induced on any conformal manifold but no principal $P$-connection. Instead one has only a 'Cartan connection' [18] (with values in $\mathfrak{so}(n+1,1)$).

Let us run now through the talks presented at the workshop and discuss how they fit in this general scheme of things. Some background is included. Also, some digressions are indulged in an effort to explain links between talks and to bring out some underlying themes. Speakers in appear in **bold type**.

**SUNDAY** Conformal differential geometry was a popular topic in the 1920s. It was then that Cartan introduced his conformal connection [18]. An equivalent formulation due to Thomas [51] followed shortly afterwards and is now understood as an important alternative viewpoint. But this was only the beginning of a theory of conformal differential geometry. At that stage, obvious questions remained unanswered. There was no classification, or even a means of writing down, conformal invariants or conformally invariant differential operators. A few intriguing examples were known: in addition to the Weyl tensor, there was the conformally invariant Bach tensor [3] in 4 dimensions and various differential equations from physics, including Maxwell's equations, were known to be conformally invariant. Little further progress was made until the mid 1980s although, with hindsight, hints of the ambient metric construction and AdS/CFT correspondence may be seen in articles of Thomas [52], Schouten-Haantjes [49], and Dirac [21]. Some renewed stimulus was also provided by Branson's finding [7] new conformally invariant operators with a more significant dependence on the Ricci tensor. There are some separate issues of geometric analysis concerning special Riemannian metrics and the Yamabe problem. We shall return to them in the discussion of Wednesday below. Laying them on one side for the moment, the real breakthrough in conformal geometry was presaged by developments in CR geometry. A real hypersurface in a complex manifold inherits a CR structure as a remnant of the ambient Cauchy-Riemann equations. There are clear parallels between CR geometry and conformal geometry. In particular, there is a Cartan connection in the CR case due to Chern-Moser [19] and Tanaka [50]. In 1979, Fefferman [26] developed the ambient metric construction for CR structures. It is a formal construction, which attempts to associate to every non-degenerate CR manifold a Kähler-Einstein manifold of higher dimension. As a consequence, CR invariants may be built from Lorentzian invariants of the ambient Kähler-Einstein manifold. Though the construction breaks down at a certain order, it builds CR invariants below a certain 'weight'. Fefferman also formulated a purely algebraic problem whose positive solution would show that all CR invariants below the critical weight arise in this way. This would be sufficient to predict the form of the coefficients of the asymptotic expansion up to the log term of the Bergman kernel of a strictly pseudo-convex domain. Since the isotropy subgroup $P \leq \mathrm{SU}(n+1,1)$ of the flat model (a hyper-quadric in $\mathbb{CP}_{n+1}$) is a parabolic subgroup and since the algebraic problems concerned the invariant theory of $P$, Fefferman dubbed these matters of 'parabolic invariant theory'. He was able to solve the algebraic problems subject to suitable restrictions on the degree and weight of the invariant.

The real breakthrough, alluded to above, was the ambient metric construction of Fefferman and Graham [28]. It is the (more difficult) conformal counterpart of Fefferman's ambient construction in the CR case [26]. Many of the recent developments in conformal differential geometry are based on this construction. It also lies at the heart of the spectacular AdS/CFT correspondence [44, 54] in the physics of string theory. Our first speaker on Sunday was **Robin Graham**. His talk concerned some new analytic aspects of the ambient metric construction. The most basic of questions concerning the ambient metric construction is whether the formal asymptotics detailed by Fefferman and Graham are attached to a genuine analytic construction. There has been a lot of recent work in this direction especially by Anderson [1] but this aspect was not seriously discussed at the workshop. Instead, a sufficient starting point for Graham was his older work with Lee [35], which shows that every conformal metric on the sphere sufficiently close to the round metric is the conformal infinity of a unique asymptotically hyperbolic Einstein metric on the ball. The usual Dirichlet-to-Neumann mapping is obtained by taking a function on the sphere as the Dirichlet data for a harmonic function in the ball and then restricting the normal derivative back to the sphere. Graham considers a natural non-linear version of this construction starting with a deformation of the conformal metric on the sphere at infinity and picking off the conformally invariant part of the volume renormalisation [33]. This is an interesting construction even starting on the round sphere. In this case he uses an analysis of intertwining operators from representation theory [10] to see what is happening. What is happening more generally is clearly a very interesting but currently unanswered problem.

A little surprisingly, there was not a lot of discussion at this workshop concerning Branson's $Q$-curvature [8, 9]. A closely related entity, however, is the Fefferman-Graham obstruction tensor. This is a conformally invariant tensor defined in even dimensions generalising the Bach tensor [3] in four dimensions. Last year, Graham and Hirachi [36] showed that the metric variation of the integral of $Q$-curvature is the obstruction tensor. In their talks on Monday, **Rod Gover** and **Larry Peterson** showed how to obtain the obstruction tensor via certain conformally invariant differential operators constructed via 'tractor calculus'. Gover explained how some basic properties of the obstruction tensor could be seen from this point of view (e.g. that it vanishes for metrics conformal to Einstein) and Peterson explained how to use this method to teach a computer (using Lee's Ricci program [40]) to find explicit formulae for the obstruction tensor (in low dimensions). This should be compared with earlier work by the same two authors [31] on explicit formulae for $Q$ via 'tractors' and computer assisted calculations. There now follows a brief digression on the tractor calculus.

As already mentioned, the natural connection in conformal geometry is a Cartan connection. It was first introduced in [18] as an $\mathfrak{so}(n+1,1)$-valued 1-form on the total space of a principal $P$-bundle naturally attached to any conformal manifold as a certain bundle of second order frames. Here, $P$ is the stabiliser subgroup of $\mathrm{SO}(n+1,1)$ under its action on the $n$-sphere as conformal motions. A completely equivalent formulation of the Cartan connection arises as the induced connection on the vector bundle $\mathbb{T}$ arising from the defining representation of $\mathrm{SO}(n+1,1)$ on $\mathbb{R}^{n+2}$. In [51], Thomas independently and directly defined this bundle and its connection. Bearing in mind that this was before 'vector bundles' were standard mathematical objects, it was not so clear at the time that this construction was equivalent to Cartan's. A modern explanation of Thomas's construction is given in [4]. There are certainly some aspects to conformal geometry that are better pursued from the Thomas viewpoint. Thomas himself already understood [52] that the basic connection is an inadequate substitute for the Levi-Civita connection in Riemannian geometry. Roughly speaking, the Levi-Civita connection, as a connection on tensor bundles, can be iterated and so invariantly captures all the higher jets of a Riemannian metric. This is not the case for the Cartan or Thomas connections. It is this problem that the Fefferman-Graham ambient metric construction overcomes (to all orders in odd dimension and to some finite but obstructed order in even dimensions (see [30] for how to push past the obstruction in even dimensions)). The Thomas construction, or 'tractor calculus' as it is now known, is more closely related to the ambient metric construction [13] than is the Cartan connection. It seems to be more and more involved in progress in conformal geometry.

¿From the Cartan point of view, however, conformal differential geometry arises as a sort of 'curved version' of the flat model, which is the $n$-sphere under the action of $\mathrm{SO}(n+1,1)$ (the $n$-sphere is regarded as the space of generators of the null cone in $\mathbb{R}^{n+2}$ with its standard Lorentzian metric). It was pondered for some time whether there are 'curved versions' of all homogeneous spaces $G/P$ where $G$ is an arbitrary Lie groups and $P$ is a parabolic subgroup. For some years, there were several known geometries (projective, conformal, CR, Cartan's five variables [17], …) but no unified theory. The term parabolic geometry was already being employed before the general formulation, which now exists thanks to Morimoto [46] and Čap-Schichl [14]. The theory is now highly developed. **Andreas Čap** and **Jan Slovák** are writing a comprehensive text on the subject [15]. Čap gave a survey of the main points, especially Lie algebra cohomology and how it is used to normalise the Cartan connections and explore where lies the torsion and curvature of a normalised connection.

The final speaker on Monday was **Gestur Ólafsson** who talked about a certain integral geometric transform in representation theory. Though this was a little to one side of the main theme of the workshop, there are some links as follows. Twistor theory [38] is an alternative approach to basic physics introduced and developed by Roger Penrose over the past 30 years or so. A basic idea in twistor theory is that space-time should be view as a space of preferred 'cycles' in another, more fundamental, space: its 'twistor space'. All basic physics should then be seen as constructions on twistor space and the physical ramifications obtained by integral geometry. Usually, the twistor space has some complex structure (at least a CR structure) but the complex integral geometry is closely parallel to classical real integral geometry starting with the Radon and Funk transforms (now familiar in medical imaging). Ólafsson's talk was concerned with the real integral geometry of complex cycles in a complexified non-Riemannian symmetric space. There is a mathematical toolbox here that impinges of conformal geometry through the ideas of twistor theory and also because representation theory is finding its way more and more into conformal geometry.

**MONDAY** Conformal differential geometry is very much tied to physics. This is true on a classical level, where the most basic equations of physics (for example, the Dirac equation) are conformally invariant. It is also true on a quantum level and especially through the AdS/CFT correspondence [44, 54] linking string theory and supergravity on a 'bulk' Einstein manifold of negative scalar curvature and conformal field theory on its 'conformal infinity'. Monday's talks were largely devoted to physics.

For the benefit of mathematicians, **Louise Dolan** gave a very useful survey of her joint work [22] with Nappi and Witten from a couple of years ago. The idea is that there are equations that one can write down (due to Deser), in some sense between the usual massless and massive field equations on space-time. It turns out that these equations in the bulk have surprising consequences at infinity under the AdS/CFT correspondence, producing a known basic series of conformally invariant operators. These 'partially massless fields' are certainly intriguing but have not yet found their way into mathematics. Maybe this could be done now. There are further clues to a mathematical theory in that the inner product of the space of fields turns out to be sometimes negative (not physical).

**Don Page** talked about construction of Einstein metrics on certain compact manifolds (such as $S^2 \times S^n$) especially in odd dimensions. Like the classical Kerr metric, these metric admit many (conformal) symmetries. Finding explicit Einstein metrics is a difficult business. Nicely symmetrical examples (cf. [53]) are good for testing the behaviour of fields subject to invariant equations.

**Lionel Mason**'s talk was very much based on the ideas of twistor theory. From the mathematical point of view he was talking about connections on $\mathbb{RP}_2$ with zero holonomy. He has worked out the Penrose-Ward transform for such connections and there are two parts to the twistor data. One is a field on $\mathbb{RP}_2^*$ much like the Funk-Radon transform would give. But the other part (to which the first part is coupled) takes the form on a holomorphic vector bundle on $\mathbb{CP}^*$. This is similar in spirit to Ólafsson's talk yesterday in that there is a unexpectedly useful complexification.

**Spyros Alexakis** gave an excellent review of how $Q$-curvature has recently been used (especially in 4 dimensions by Chang, Yang, and co-workers) in geometric analysis. Unfortunately, none of the researchers directly involved in this area were able to attend the workshop. Here is a digression on $Q$-curvature. It is a scalar Riemannian invariant canonically defined in even dimensions. In four dimensions $Q = \frac{1}{6}R^2 - \frac{1}{2}R^{ab}R_{ab} - \frac{1}{6}\Delta R$. Though there is a now a straightforward definition [29] in terms of the ambient metric, explicit formulae in higher dimensions are complicated [31] or unavailable. A characterisation of $Q$ is also unavailable but one of its key properties is that, under conformal rescaling of the metric $g_{ab} \mapsto \widehat{g}_{ab} = \Omega^2 g_{ab}$ we have $\widehat{Q} = Q + P \log \Omega$, where $P$ is a *linear* operator. Conformal invariance of $P$ is forced. It is the Graham-Jenne-Mason-Sparling operator [34]. It also follows that $\int_M Q$, for $M$ a compact manifold, is conformally invariant. It has been asserted in the physics literature that these properties force $Q$ to have to form $Q = E + I + D$ where $E$ is the Euler density or Pfaffian, $I$ is a conformal invariant, and $D$ is a divergence (and so does not contribute to $\int_M Q$). Alexakis's PhD thesis, currently in preparation, proves this assertion. It is not at all an easy argument and he was only able to sketch one ingredient (that if an identity on a Riemannian manifold holds formally (in the sense of Weyl's classical invariant theory), then it holds in all dimensions and so one can extract consequences by looking at the leading order terms as the dimension goes to infinity!).

Penrose's 'Weyl curvature hypothesis' states that the initial singularity in space-time (i.e. the 'big bang') is qualitatively different from final singularities (i.e. 'black holes' and the 'big crunch'). Specifically, at the initial singularity the conformal curvature should be smooth up to the boundary such as one might obtain by taking a smooth metric and rescaling it by a conformal factor that blows up at the boundary. But how can one recognise this phenomenon? There are no preferred coordinates and nor even a given boundary. One might choose bad coordinates and/or a bad conformal gauge. **Paul Tod** suggested various conformally invariant tests for the Weyl curvature hypothesis but complete recognition is still a problem.

**Maciej Dunajski** also spoke on material derived from twistor theory. Beginning with the Ward correspondence for instantons, there are twistor description of various integrable systems. All of them involve families of rational curves in an associated 'twistor space'. In higher dimensions, 'paraconformal' or 'almost Grassmannian' structures seem to provide a suitable generalisation of conformal geometry in dimension four. Dunajski discussed a refinement of such structures, namely manifolds for which the tangent bundle has the form $S' \otimes \bigodot^k S$ where $S$ and $S'$ are rank 2 vector bundles.

**TUESDAY** A fundamental observation in even-dimensional conformal geometry is that the Hodge $\star$-operator is conformally invariant on forms of middle degree. For example, in 4 dimensions the 2-forms canonically split into self-dual and anti-self-dual types. Many of the special features of 4-dimensional geometry can be traced to this fact. Moreover, the general structure of conformally invariant differential operators follows similar patterns to the de Rham sequence. These are the Bernstein-Gelfand-Gelfand sequences (introduced in [24] in dimension 4). There is now an extensive theory, referred to as BGG machinery and largely due to Čap-Slovák-Souček [16] and Calderbank-Diemer [12]. Though the BGG sequence cannot usually be locally exact in the curved setting, it has been noticed that sometimes there are subcomplexes that have this property on special manifolds (for example, CR manifolds or self-dual manifolds in dimension 4). **Vladimír Souček** presented many examples of this phenomenon (found in joint work with Čap). Representation theory is involved, this time to see that curvature cannot possibly interfere with local exactness if the relevant representation is not present. These are exactly the sort of sequences that appear in trying to deform a given parabolic geometry (and in good circumstances this 'deformation complex' is elliptic).

As already mentioned, CR geometry of hypersurface type is closely analogous to conformal geometry. Remarkably, there are other examples of CR geometry that fit into this scheme, most notably the CR geometry of real codimension 2 submanifolds in $\mathbb{C}^4$ due to Schmalz and Slovák [48], which was only found via the general parabolic theory (and almost surely would have been impossible to find without the general theory). But is is far from clear whether parabolic geometry is the end of the road. More specifically, the basic construction in parabolic geometry is of a preferred Cartan connection and, until recently, the means for picking out a preferred connection was to insist that it be 'normal', a restriction best formulated with the BGG machinery. Earlier this year, Fox [27] found an example of a parabolic geometry (contact projective geometry), which possessed a canonical Cartan connection that was not necessarily normal. Instead, normality was an extra condition that showed up in the torsion of the Fox connection. In his talk on Tuesday, **Gerd Schmalz** considered the CR geometry of real codimension 2 submanifolds in $\mathbb{C}^3$. This geometry is far from parabolic: the 5-dimensional structure algebra is not even close to semisimple. Nevertheless, Schmalz reported on joint work with Beloshapka and Ezhov that constructs a canonical Cartan connection (and a normal form for the embedding analogous to Moser normal form [19] for hypersurfaces). It is interesting to note that the underlying real structure, of an 'Engel manifold' (a 4-dimensional real manifold $M$ with 2-dimensional distribution $D$ such that $[D, [D, D]] = TM$), carries no local information. In this sense it is quite close to the hypersurface case with underlying contact structure. However, the local structure algebra is quite different.

**Kengo Hirachi** continued the CR theme but back to hypersurface type. Of course, the prime example such geometry is found on the boundary of a strictly pseudo-convex domain in $\mathbb{C}^n$. Hirachi explained how to formulate the volume expansion with respect to the Einstein-Kähler or Bergmann metric. This is a parallel development to the conformal case [33] but there are new features: if the domain is taken to be a strictly pseudo-convex neighbourhood of the zero section of a sufficiently ample vector bundle, then the Chern classes of the bundle show up as CR invariants on the boundary.

There were two talks in the evening given by graduate students. Continuing Hirachi's volume renormalisation in the CR setting, his student **Neil Seshadri** talked about another aspect of the volume expansion: there are two interesting terms in the expansion and Seshadri talked about the coefficient of the 'log' term. As a digression, it is worth noting that there is a direct link between CR geometry and conformal geometry. Fefferman [25] (in the embedded case and Burns-Diederich-Shnider [11] intrinsically) found that a non-degenerate CR structure on a manifold $M$ determines a canonical circle bundle over $M$, the total space of which is now known as the Fefferman space. This is naturally equipped with a conformal structure, which encodes the underlying CR structure. Conformally invariant constructions on the Fefferman space gives rise to CR invariant constructions on $M$. For example, the Chern-Moser chains on $M$ arise as the images of null geodesics and CR $Q$-curvature pulls back to the conformal $Q$-curvature. However, the conformal structures that arise are special [32] and so it is wise to proceed independently. It is unknown in the CR case whether $\int_M Q$ can be non-zero.

The other evening talk was by **Doojin Hong**, a student of Branson. He talked about generating the spectrum for spinors over $S^1 \times S^n$ with Lorentzian metric. This should be compared with spectrum generating [10] for densities on $S^n$ as an example of principal series representations and their intertwinors (and as used by Graham in the opening talk of this conference (and as may be used to understand the GJMS-operators operators on the sphere)).

**WEDNESDAY** The first talk was given by **Claude LeBrun** who discussed 'optimal metrics' on 4-manifolds, defined on an $n$-manifold $M$ as a metric such that $\int_M |R|^{n/2}$ is absolutely minimised, where $|R|$ denotes the pointwise norm of the Riemann curvature tensor. The exponent $n/2$ is forced by requiring invariance under constant rescaling of the metric and already one can see that dimension 4 is special since then we are talking about the $L^2$ norm of the Riemann curvature. From the integral formulae for the Euler characteristic and signature, it follows immediately that there are two special ways of guaranteeing an optimal metric in 4 dimensions: Einstein metrics are always optimal and so are scalar-flat anti-self-dual metrics. LeBrun uses twistor theory to construct optimal metrics of this second type on the connected sum of 6 of more copies of the complex projective plane with its opposite orientation. Earlier results say that these manifolds are good test cases for the existence of optimal metrics. Twistor methods signal conformal geometry. The point is that for a metric to be anti-self-dual is a conformally invariant condition and these are exactly the manifolds with a twistor space. LeBrun uses the behaviour of the Green's function of the Yamabe operator to determine the sign of the Yamabe constant of a four manifold, this being reflected in the twistor space by dint of Atiyah's construction [2]. He uses this information to detect a change of sign in a family of conformal metrics and hence to find a find a metric with vanishing Yamabe constant. The details are in [39].

**Helga Baum** was concerned with Lorentzian metrics with special conformal properties. It is still unknown whether a compact Lorentzian manifold with a group of essential conformal transformations has to be constant curvature (where essential means that there is no metric in the conformal class for which the transformations are merely isometries). In the Riemannian case, there are no such exotic examples. The Lorentzian case is much more difficult. For example, there are compact Lorentzian manifolds with non-compact isometry group. These questions are very much connected with the existence of parallel spinors and holonomy in the Lorentzian case: see [41, 42]. In the conformal case, the existence of conformal Killing spinors and solutions to other overdetermined systems of partial differential equations (sometimes collectively known as twistor equations) is very much linked to tractor calculus. Baum explained how this link could be used. Solutions to these equations constrain the underlying geometry [5] and are the source of symmetries of basic equations such as the Dirac operator [6] and Laplacian [23].

**Felipe Leitner** continued with this theme, discussing the holonomy of conformal manifolds, meaning the holonomy of their Cartan connections. This is very much related to the existence of Einstein metrics in a given conformal class: a parallel standard tractor generically defines an Einstein scale. Unlike the Riemannian case, not much is known about the holonomy of conformal manifolds. With very symmetrical manifolds it may be calculated and this gives rise to the first real examples. Leitner finds [43], for example, the conformal holonomy of SO(4) (regarded as a Riemannian manifold via its bi-invariant metric) is SO(7) as a subgroup of SO(7, 1). He also explained how special conformal holonomy gives rise to Einstein metrics in a given conformal class.

**William Ugalde** used the Wodzicki residue applied to a simple conformally invariant pseudo-differential operator to construct a natural conformally invariant differential pairing and an invariant differential operator, which looks very much like the critical GJMS-operator [34]. In dimension 4, Connes [20] obtained exactly this operator (sometimes called the Paneitz operator). In dimension 6, Ugalde has checked by direct calculation that one obtains the GJMS-operator. This is already a formidable task. It is extremely plausible that the GJMS-operators are indeed arising by Ugalde's construction. Perhaps the Wodzicki residue could be related to the scattering theory approach to the GJMS-operators [37]. Otherwise, we await a characterisation of these operators. Ugalde has checked many of the salient features of his operators but these features are insufficient precisely to pin them down.

Another popular geometry of the 1920s was projective differential geometry. Both Cartan and Thomas [4] developed the basic theory in parallel to conformal geometry. In particular, there is a canonically defined Cartan connection. **Vladimir Matveev** described his recent proof of the Lichnerowicz-Obata conjecture in projective geometry, namely that is a connected Lie group acts on a closed connected Riemannian manifold by transformations that preserve the geodesics (as unparameterised curves), then its acts by isometries or the manifold is covered by the round sphere. This is clearly parallel to the conformal Lichnerowicz conjecture (proved by Obata, Alekseevsky and Ferrand). Matveev converts the projective equivalence of two metrics into the existence of a tensor field, a self-adjoint endomorphism of the tangent bundle, satisfying a certain partial differential equation. He calls these tensors 'BM-structures'. He then embarks on a careful investigation of such structures bearing in mind how the eigenvalues of the endomorphism might coalesce. For details see [45].

After touching on Cartan connections in the geometry of differential equations [47] and reviewing the basic geometry of twistor theory, **George Sparling** described the the quantum Hall effect and how it has recently been viewed as a mathematical construction on the Riemann sphere under the action of $\mathrm{SU}(1,1)$. He speculated that this construction might be extended to a sort of 'cohomological twistor fluid' on the open orbits of $\mathrm{SU}(2,2)$ acting on $\mathbb{CP}_3$.

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