New Developments on Variational Methods and Their Applications

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1 Introduction

The calculus of variations has repeatedly proved itself to be a powerful and far-reaching tool for advancing our understanding of mathematics and its applications. This is no doubt due to the fact that variational methods are not merely techniques for solving individual, albeit very important, problems but are often "variational principles", i.e. they are manifestations of very general laws of nature which are valid in diverse branches of science and engineering.

Modern variational approaches to non-linear problems were initiated by mathematicians like H. Poincare, G. Birkhoff and his student M. Morse. E.g. Morse revealed the deep relationship between the number and types of critical points of functions and the topology of their level sets. Around the same time, variational approaches were also being developed and used by Ljusternik and Schnirelmann to establish the existence of 3 distinct closed geodesics on any compact surface of genus zero. These methods and results –which also mark the beginning of global analysis– were finite dimensional in nature. The development of tools to deal with infinite dimensional problems of nonlinear partial differential equations and geometry accelerated in the 1960's. The type of compactness required, often embodied in the Palais-Smale condition, was much studied and considerable progress was made. Subsequently in the mid 1980's and afterwards an understanding of how the the Palais-Smale condition can breakdown in less compact situations (like unbounded domains, limit exponent problems, etc.) emerged and now one can use this understanding to get existence results in such settings. Novel minimization arguments have also been developed and there is considerable current activity in refining and extending them so as to overcome the limitations to their applicability to present-day variational problems, ranging from geometry to pattern recognition and from superconductivity to phase transitions, etc. The subject has come of age in the last forty years, and a number of surveys and monographs have described much of the progress.

The goal of our workshop was to discuss some of the recent developments while emphasizing new applications to nonlinear problems. More often than not, progress is driven by specific applications. Novel variational techniques developed by groups or individuals concerned with these applications often do not make their way to others who may be using similar variational methods but on different types of problems. This workshop brought together senior experts such as Ivar Ekeland, Maria Esteban, Louise Nirenberg, Paul Rabinowitz, etc. and leading young researchers working in different areas of variational methods such as Yanyan Li, Yiming Long, Eric Séré, Peter Sternberg, etc. These areas include abstract variational methods such as Novikov Morse theory and nonsmooth critical point theory, geometric PDEs, and nonlinear problems from applied fields such as superconductivity and phase transitions. Participants with different areas and background had an opportunity to exchange ideas on topics ranging from abstract theories to applications so that novel variational methods can find more applications and new theories can be developed. Below are the main themes of the workshop:

2 Phase transitions and superconductivity

There are many problems in these applied areas that can be studied by variational methods. Indeed recently several new ideas such as the renormalized energy method, new reduction methods, new perturbation methods have been developed through the study of individual problems in these areas. Variational methods give a very good understanding of physical phenomena such as concentration, vortex formation and their dynamics. In this workshop, many participants spoke about their new results and ideas in this area.

One such topic is gamma-convergence in Ginzburg-Landau models of superconductivity and Allen-Cahn models of phase transition. Sylvia Serfaty presented a method to prove convergence of gradient-flows of families of energies which gamma-converge to a limiting energy. It provides lower bound criteria to obtain the convergence, which correspond to a sort of C^1-order gamma-convergence of functionals. They then apply this method to establish the limiting dynamical law of a finite number of vortices for the heat-flow of the Ginzburg-Landau energy in dimension 2. In this case, the limiting objects whose dynamics they study are the limiting vortices of the maps u_ϵ, and the limiting energy is a "renormalized energy", defined on the finite-dimensional space of possible vortex-locations. They prove that the conditions above are satisfied and thus re-obtain with a different method the result of Lin and Jerrard-Soner, that the limiting vortices follow the gradient-flow of the renormalized energy. They also obtain the analogous new result for the full Ginzburg-Landau model with magnetic effects. One extension of this method is to push it to "second order" to compare the C^2 structures of the energy-landscapes and F near critical points. This gives necessary conditions for stable/unstable critical points of the Ginzburg-Landau energy functional to converge to stable/unstable critical points of F. This is again applied in the case of Ginzburg-Landau to obtain stability results on the limiting vortex-configurations, and a nonexistence result of nontrivial stable

critical points when there are Neumann boundary condition and no magnetic field . Another extension is to apply it to Ginzburg-Landau vortex-dynamics with suitable space-time rescalings, which allow one to continue studying dynamics at times of collisions of vortices. This, coupled with a new estimate (in the case of Ginzburg-Landau) to the vortex-distances, allows one to give energy-dissipation rates at collision time and optimal estimates on those collision-times, and under certain assumptions, to extend the limiting dynamics after collision.

Montero discussed the weak Jacobians of Jerrard and Soner and showed they can be viewed as linear functionals that act on Hölder continuous, compactly supported vector fields in Ω. He and his collaborators use the limiting behavior of E_ε to identify a geometric condition on Ω that guarantees the existence of local minimizers of E_ε. This condition essentially amounts to the existence of a line segment in Ω, with endpoints in $\partial\Omega$, that locally minimizes length. He also showed an existence result for G_ε in 3-d simply connected domains when the applied field h_{ap} is not too big. In particular, for the case $h_{ap} = 0$, this provides what is perhaps the first existence result via Ginzburg-Landau theory of permanent currents in the presence of vortices.

X. Ren studied gamma convergence in a dipolymer model. A molecule in a diblock copolymer is a linear sub-chain of A monomers grafted covalently to another sub-chain of B monomers. The different type sub-chains tend to segregate locally, resulting in micro-domains rich in A and B monomers. These micro-domains form morphology patterns/phases in a larger scale. The Ohta-Kawasaki free energy of a diblock copolymer melt is a functional of the A monomer density field $u(x)$. When there is high A monomer concentration at x, $u(x)$ is close to 1; when there is high concentration of B monomers at x, $u(x)$ is close to 0. A value of $u(x)$ between 0 and 1 means that a mixture of A and B monomers occupies x. The re-scaled, dimensionless free energy of the system is

$$I(u) = \int_D \{\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{\epsilon\gamma}{2}|(-\Delta)^{-1/2}(u-a)|^2 + W(u)\}\, dx,$$

which is defined in the admissible set

$$X_a = \{u \in W^{1,2}(D) : \overline{u} = a\}$$

where $\overline{u} = \frac{1}{|D|}\int_D u\,dx$ is the average of u in D. a is a fixed constant in $(0,1)$. It is the ratio of the number of the A monomers to the number of all the monomers in a chain molecule. One can take

$$W(u) = \frac{1}{4}(u^2 - u)^2.$$

The two parameters ϵ and γ characterize the system. Ren considered the parameter range

$$\epsilon \to 0,\ \gamma \sim 1.$$

He studied two solutions: the spot solution and the ring solution of K interfaces, both in a unit disc. The spot solution models a cell in a cylindrical phase of the diblock copolymer and the ring solution models a defective lamellar phase. Using the Γ-convergence theory he showed that the spot solution exists for all $\gamma > 0$ and there exists $\gamma_1 > 0$ such that the ring solution exists for $\gamma > \gamma_1$.

Next he considered the stability of these solutions by analyzing their critical eigenvalues. He showed that there exists $\gamma_0 > 0$ such that the spot solution is stable if $\gamma < \gamma_0$ and unstable if $\gamma > \gamma_0$. For the ring solution, there exists $\gamma_2 > \gamma_1$ such that the ring solution is stable if $\gamma \in (\gamma_1, \gamma_2)$ and unstable if $\gamma > \gamma_2$. Finally he made a comparison between the diblock copolymer problem and the Cahn-Hilliard problem, which is obtained by setting $\gamma = 0$ in the definition of I.

Glotov studied the 'variable thickness' Ginzburg-Landau equations describing type-II superconducting thin films. The convergence of the order parameter was discussed in the literature in a paper by Chapman, Du, and Gunzburger. Glotov and his collaborators focussed their attention to the equation for the magnetic potential and obtained results on convergence of various quantities involved in the latter equation. They also showed that the limiting order parameter is a minimizer of the two-dimensional thin-film energy. The limiting problem, among other properties, has an advantage, from a computational point of view, of being restricted to a bounded domain. The regularity of the solutions to the three-dimensional problem presents another interesting question for us. Using regularity, they obtain uniform convergence of the three-dimensional minimizers. This in turn allows them to conclude, thanks to the description of the vortex structure for minimizers of the two-dimensional thin-film energy available from the work of Ding and Du, that the three-dimensional minimizers exhibit vortices and their degree is preserved as the thickness of the film tends to zero.

Alama and his collaborators consider the following variational problem arising from a two-dimensional model for rotating Bose–Einstein Condensates (BEC.)Let $a = a(r)$ be a real-analytic radially symmetric function in the plane, with the property that

$$\mathcal{A} = \{x \in \mathbf{R}^2 :\ a(|x|) > 0\}$$

is an *annulus,* and such that a vanishes linearly at each edge of the annulus $\mathcal{A}$. Examples include $a(r) = -b_0 + b_1 r^2 - b_2 r^4$ with appropriately chosen coefficients. Let $\Omega \in \mathbf{R}$,$x = (x_1, x_2) \in \mathbf{R}^2$, $x^\perp = (-x_2, x_1)$,and $\epsilon > 0$. They study minimizers $u \in H_0^1(\mathcal{A}; \mathbf{C})$ of the energy functional

$$E_\epsilon(u) = \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 - \Omega \mathbf{x}^\perp \cdot (iu, \nabla u) + \frac{1}{4\epsilon^2}(|u|^2 - a(x))^2 \right\}\, dx,$$

in the singular limit as $\epsilon \to 0$. In the context of BEC, u is the quantum wave-function, Ω is the angular speed of rotation, and $-a(r)$ gives a potential well imposed to"trap" the condensate (by means of lasers) in a bounded region of space. The choice of an annular trap here is meant to simulate certain current experiments for BEC.

Alama showed how the annular topology of the condensate domain affects the presence and location of vortices as a function of the angular speed Ω. His results concern both fixed rotation Ω (independent of ϵ and rotations which grow with ϵ. When Ω is fixed,

it is proved that minimizers converge to anon-zero (radially equivariant) solution away from the hole, while the hole itself plays the role of a "Giant Vortex" with degree increasing with Ω. Alama also considered angular velocities of the form

$$\Omega = \omega_0 |\ln \epsilon| + \omega_1 \ln |\ln \epsilon|,$$

with ω_0, ω_1 constant. He showed that there is a critical value of the coefficient $\omega_0 = \omega_0^*$ such that whenever $\omega_0 < \omega_0^*$, minimizers have no vorticity in the interior of the annulus $\mathcal{A}$, but when $\omega_0 = \omega_0^*$ and ω_1 is large enough, then vortices begin to appear inside $\mathcal{A}$. The location of these vortices is completely determined by the coefficient a: they lie on one or several concentric circles in $\mathcal{A}$ whose radii attain a given minimization problem involving $a(r)$. This method involves deriving sharp upper and lower bounds on the energy of minimizers via a vortex-ball construction as in the work of Sandier–Serfaty. In order to determine the number and location of the vortices for supercritical rotations they must take into account the effect of the Giant Vortex in constructing the upper and lower bounds.

Sternberg discussed various basic questions for the Ginzburg-Landau and Allen-Cahn equations that remain unanswered. One such question is on the monotonicity (i.e. increasing in one direction) of local minimizers for the Allen-Cahn energy in a convex domain. Another such question is the smoothness of the zero level set of solutions to Allen-Cahn energy.

Bates and his collaborators study a global minimizer of the van der Waals' free energy functional with nonlocal interaction. Short-range repulsive and long-range attractive interactions appear in, e.g., the van der Waals' forces (often modeled by the Lennard-Jones potential). For this model since there is no gradient term, the underlying space is not restricted to differentiable functions and critical points are possibly discontinuous functions. Indeed, monotone discontinuous heteroclinic critical points were constructed in previous work by the authors and others. For instance, they discovered families of critical points, discontinuous along arbitrarily prescribed interfaces, which are seemingly stable, since the formal second variation is positive. However, there is no variational sufficiency condition for minimizers, since if the functional is defined on the natural space L^2 or $u_0 + L^2$, it is only $C^{1,1}$ and the discontinuous solutions which usually form nonsmooth continua in L^2 are in general not local minimizers. Bates studied the case when the Fourier transform, $\hat{J} \le 1$, which assures that the energy functional is bounded below by 0. If $J \ge 0$, a monotone global minimizer can be constructed using monotone rearrangements. However, if J changes sign, monotonicity methods are not applicable, and in general the global minimizer will not be monotone. For this case ideas from concentration and convexification techniques are employed.

3 Bubbles, spikes and concentration

Many variational problems arises from geometry particularly in the study of the Yamabe problem, Kahler-Einstein manifolds, minimal surfaces, scalar curvature, harmonic maps, etc. A typical difficulty is the lack of compactness, i.e., some kind of bubble or singularity appears. This area is very active and has been a major source of new ideas in variational methods.

Let (M, g) be a compact smooth Riemannian manifold of dimension $n \ge 3$, and let

$$A_g := \frac{1}{n-2}(Ric_g - \frac{R_g}{2(n-1)}g)$$

denote the Schouten tensor of g, where Ric_g and R_g denote respectively the Ricci tensor and the scalar curvature of g. Let $\lambda(A_g) = (\lambda_1(A_g), \cdots, \lambda_n(A_g))$ denote the eigenvalues of A_g with respect to g. Let V be an open convex subset of R^n which is symmetric with respect to the coordinate axes. Assume that $\emptyset \ne \partial V$ is smooth and satisfies

$$\nu(\lambda) \in \{\mu \in R^n | \mu_i > 0, \forall 1 \le i \le n\}, \qquad \forall\ \lambda \in \partial V,$$

and

$$\nu(\lambda) \cdot \lambda > 0, \qquad \forall\ \lambda \in \partial V.$$

Let

$$\Gamma(V) := \{s\lambda \mid \lambda \in V,\ 0 < s < \infty\}$$

be the cone with vertex at the origin generated by V.

Conjecture. *Let (M^n, g), V and $\Gamma(V)$ be as above. Assume that*

$$\lambda(A_g) \in \Gamma(V), \qquad \text{on } M^n.$$

Then there exists a smooth positive function $u \in C^\infty(M^n)$ such that the conformal metric $\hat{g} = u^{\frac{4}{n-2}} g$ satisfies

$$\lambda(A_{\hat{g}}) \in \partial V, \qquad on\ M^n.$$

For $V = \{\lambda \in R^n \mid \sum_{i=1}^n \lambda_i > 1\}$, the conjecture is the Yamabe Conjecture in the positive case. Yanyan Li discussed recent joint work on this conjecture including some proofs of results on the existence and compactness of solutions as well as some Liouville type theorems.

Min Ji consider the famous Nirenberg problem: which positive function R can be the scalar curvature of some metric g which is pointwise conformal to g_0? Writing $g = e^u g_0$, the problem is equivalent to the solvability of the following PDE:

$$-\triangle_{g_0} u + 2 - R\, e^u = 0, \qquad \text{on} \quad S^2. \tag{1}$$

Several years ago Moser proved the solvability for R an even function and this was followed by much further research. Equation (1) can be reduced to a variational problem. The corresponding functional is bounded from below, however it has no minimum if R is not a constant. Most subsequent work involved attempts to look for minimax type of solutions. A.Chang–P.Yang and others made further important progress. Min Ji gave a nice general framework for getting solutions.

Wei and his coauthors consider the following nonlinear elliptic equation

$$\Delta u - \mu u + u^q = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{2}$$

where Ω is a bounded and smooth domain in R^N, $\mu > 0$ and $q = \frac{N+2}{N-2}$. Problem (2) has been studied by many authors in recent years . Wei mentioned the following results of Gui-Wei: Let $q < \frac{N+2}{N-2}$. Given arbitrary two positive integers K, l, there exists a $\mu_{k,l}$ such that for $\mu > \mu_{k,l}$, there exists a solution to (2) with $k-$interior spikes and $l-$boundary spikes. In his talk, Wei showed similar phenomena for the critical exponent case. Wei's first result concerns the case of μ large and $N \geq 7$. (This is joint work with C.-S. Lin.) They showed that at a positive nondegenerate local minimum point Q_0 of the mean curvature, (they may assume that $Q_0 = 0$), for any fixed integer $K \geq 2$, there exists a $\mu_K > 0$ such that for $\mu > \mu_K$, the above problem has a $K - bubble$ solution u_μ concentrating at the same point Q_0. More precisely, they show that u_μ has K local maximum points $Q_1^\mu, ..., Q_K^\mu \in \partial\Omega$ with the property that $u_\mu(Q_j^\mu) \sim \mu^{\frac{2}{N-2}}, Q_j^\mu \to Q_0, j = 1, ..., K$, and $\mu^{\frac{3-N}{N}}(Q_1^\mu, ..., Q_K^\mu)$ approach an optimal configuration of the following problem

($*$) *Find out the optimal configuration that minimizes the following functional:*

$$R[Q_1, ..., Q_K] = c_1 \sum_{i=1}^K \varphi(Q_j) + c_2 \sum_{i \neq j} \frac{1}{|Q_i - Q_j|^{N-2}}.$$

where $c_1, c_2 > 0$ are two generic constants and $\varphi(Q) = Q^T \mathbf{G} Q$ with $\mathbf{G} = (\nabla_{ij} H(Q_0))$. This result shows that the bubbling accumulations phenomenon can occur for $N \geq 7$. (When $N = 3$, it was proved by Y.Y. Li that no bubbling accumulations can occur.)

Wei's second result concerns μ and the lower dimension case $N = 4, 5, 6$. (This is joint work with O. Rey.) They show that for $N = 4, 5, 6$ and any positive integer K such that $K \neq 2$, there exists $\mu_K > 0$ such that for $0 < \mu < \mu_K$, the above problem has a nontrivial solution which blows up at some K interior points in Ω, as $\mu \to 0$. The locations of the blowing up points are related to the domain geometry. No assumption on the symmetry or the geometry or the topology of the domain is needed.

4 Sharp inequalities and symmetry of extreme functions

Inequalities are a crucial part of mathematics. Many inequalities have a variational formulation. It is very useful to get sharp inequalities by finding the extreme functions and their properties, which often involve symmetries and underlying invariance of the functionals. In the workshop, many participants discussed the relationship of sharp inequalities and symmetry.

Z.Q. Wang and his coauthors consider a family of weighted Hardy-Sobolev type inequalities due to Caffarelli, Kohn and Nirenberg: There is $S(a,b)>0$ such that for all $u \in C_0^\infty(\mathbf{R}^N)$,

the inequality

$$\int_{\mathbf{R}^N} |x|^{-2a}|\nabla u|^2 \, dx \geq S(a,b)\left(\int_{\mathbf{R}^N} |x|^{-bq}|u|^q \, dx\right)^{2/q} \tag{3}$$

holds for $N \geq 3$: $-\infty < a < \frac{N-2}{2}, 0 \leq b-a \leq 1$ and $q = \frac{2N}{N-2+2(b-a)}$. These inequalities extend to $D_0^{1,2}(\mathbf{R}^N) := \overline{C_0^\infty(\mathbf{R}^N)}^{||\cdot||}$ with respect to the norm $||u||_a^2 = \int_{\mathbf{R}^N} |x|^{-2a}|\nabla u|^2 dx$, and have the associated Lagrange equation $-div(|x|^{-2a}\nabla u) = |x|^{-bq}u^{q-1}$, which is a prototype of more general anisotropic type nonlinear elliptic PDEs with multiple singularities and degeneracies. Wang then discussed:

• **Symmetry and symmetry breaking of extremal functions.** Here due to the work of Aubin(1976), Talenti(1976), Lieb(1983), and Chou-Chu(1993), for $a \geq 0, a \leq b < a+1$, all extremal functions of the inequalities are radially symmetric. Some recent work have partially clarified the symmetry property of extremal functions for the remaining parameter region. More precisely:

Theorem *(Catrina-Wang, 2001) There is a function $h(a)$ defined for $a \leq 0$, satisfying $h(0)=0$, $a < h(a) < a+1$ for $a<0$, and $a+1-h(a) \to 0$ as $-a \to \infty$, such that for (a,b) satisfying $a<0$ and $a<b<h(a)$, the extremal functions for $S(a,b)$ are non-radial.*

A more precise result was given by Felli-Schneider(2003) who showed $h(a) = 1 + a - \frac{N}{2}(1 - \frac{N-2-2a}{\sqrt{(N-2-2a)^2+4(N-1)}})$.

As a more recent result we have:

Theorem *(Lin-Wang, 2004) For (a,b) satisfying $a<0$ and $a<b<h(a)$, any extremal function u to $S(a,b)$ is axially symmetric about a line through the origin. Moreover, up to a rotation, $u(x)$ only depends on the radius r and the angle θ_N between the x_N-axis and $\vec{ox}$, and on each sphere $\{x \in \mathbf{R}^N \mid |x| = r\}$, u is strictly decreasing as the angle θ_N increases.*

Next Wang spoke on:

• **Sharp versions of the improved Hardy inequalities.** When restricted to bounded domains, on the right hand side of (3) one can add additional terms leading to Hardy-Sobolev inequalities with remainder terms. The following is the improved weighted Hardy inequality which gives the sharp version of the improved Hardy inequality due to Brezis-Vazguez(1997) and Vazguez-Zuazua(2000), as well as generalizes their results to the weighted versions. These inequalities are useful tools for elliptic and parabolic equations having singular potentials.

Theorem *(Wang-Willem, 2003) Let $N \geq 1$, $a < \frac{N-2}{2}$, and $\Omega \subset\subset B_R(0)$ for some $R>0$. Then there exists $C = C(a,\Omega) > 0$ such that for all $u \in C_0^\infty(\Omega)$*

$$\int_\Omega |x|^{-2a}|\nabla u|^2 dx - \left(\frac{N-2-2a}{2}\right)^2 \int_\Omega |x|^{-2(a+1)}u^2 dx \geq C \int_\Omega \left(\ln \frac{R}{|x|}\right)^{-2} |x|^{-2a}|\nabla u|^2 dx.$$

When $0 \in \Omega$, the inequality is sharp in the sense that $\left(\ln \frac{R}{|x|}\right)^{-2}$ can not be replaced by $g(x) \ln \left(\frac{R}{|x|}\right)^{-2}$ with g satisfying $|g(x)| \to \infty$ as $|x| \to 0$.

• **Further questions.** i.) The symmetry of extremal functions for parameters $a \leq 0, h(a) \leq b < a+1$. ii.) Related issues for the L^p versions of the weighted Hardy-Sobolev inequalities.

Congming Li and his coauthors studied the well-known Hardy-Littlewood-Sobolev inequality:

$$\begin{aligned} &\int_{R^n}\int_{R^n} f(x)|x-y|^{\alpha-n}g(y)dxdy \\ &\leq C(n,s,\alpha)||f||_r||g||_s. \end{aligned}$$

Here $f \in L^r(R^n)$, $g \in L^s(R^n)$, $0 < \alpha < n$ and $\frac{1}{r}+\frac{1}{s} = \frac{n+\alpha}{n}$. They were mainly interested in the study of non-negative solutions to the associated Euler-Lagrange equations which can be transformed to the following system of integral equations in R^n:

$$\begin{cases} u(x) = \int_{R^n} |x-y|^{\alpha-n} v(y)^q dy \\ v(x) = \int_{R^n} |x-y|^{\alpha-n} u(y)^p dy \end{cases}$$

with $\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}$. First, under the natural integrability conditions $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$, They prove that all the solutions are radially symmetric and monotone decreasing about some point. In the special case $p = q$, they classified all the solutions which solved a open problem posed by E. Lieb.

Congming Li also presented some of his joint work on regularity, radial symmetry, and monotonicity of solutions to this and some related systems which include subcritical cases, super critical cases, and singular solutions in all cases; and obtain qualitative properties for these solutions.

Burchard used symmetrization to gain compactness and study related inequalities. Indeed, lack of compactness is a principal analytical difficulty in the study of functionals on unbounded domains. For symmetric functionals, the existence of minimizers can often be established by first restricting the problem to radially symmetric functions with the help of a rearrangement inequality, and then using the additional compactness properties of symmetric functions, as captured by the Strauss radial lemma [1977], to find a convergent minimizing sequence. This strategy was used in the determination of the sharp Sobolev constants by Talenti [1976], in the analysis of the sharp Hardy-Littlewood-Sobolev inequalities by Lieb [1983], and for the study of ground states for many functionals of Mathematical Physics.

Certain dynamical stability problems can also be reduced to the study of related variational problems. Here, it is the compactness of arbitrary minimizing sequences, not just the existence of minimizers, that plays the key role. In a series of famous papers, Lions [1984] introduced a general abstract *concentration compactness* principle which has lead to many applications. In order to apply this principle to a specific problem, some additional analysis is usually needed. In recent joint work with Y. Guo [2004], Burchard closely examines the role of translations for minimizing sequences of two classes of functionals that appear in many applications of the concentration compactness principle: convolution integrals of the form

$$\mathcal{I}(f) = \int_{R^n} \int_{R^n} f(x) K(|x-y|) f(y) \, dx dy$$

with some strictly decreasing, positive definite kernel K, and gradient integrals of the form

$$\mathcal{J}(g) = \int_{R^n} \Phi(|\nabla g(x)|) \, dx$$

with some strictly convex, increasing integrand Φ. Special cases are the Coulomb kernel in three dimensions, and the p-norm of the gradient. They show that the difference between a minimizing sequence and the corresponding sequence of symmetrized functions is characterized by appropriate translations. Besides the interest of their results in classical analysis, this characterization suggests a practical two-step procedure for establishing compactness on an unbounded domain. *Step 1.* Show convergence of all symmetric minimizing sequences. *Step 2.* Show convergence up to translations for general minimizing sequences, assuming that their symmetrizations converge. The first step implies the existence of minimizers; it is also a necessary ingredient in the proof that these minimizers are dynamically stable under symmetric perturbations. They focus on the second step, which implies dynamical stability under more general perturbations. They discuss applications to symmetric galaxy configurations appearing in recent work of Guo and Rein [1999-2001], and to functionals with additional scaling symmetries.

Technically, their results are inspired by *asymmetry* inequalities, which estimate the difference between a function or a body and a symmetric one by a related geometric quantity. The most powerful result in that direction, due to Hall [1992], states that a body whose surface area is close to the surface area of a ball of the same volume is in fact close (in symmetric difference) to a suitable translate of the ball. They expect that asymmetry inequalities should hold for large classes of symmetric functionals, including the Coulomb electrostatic energy. They hope that their approach can give another perspective on concentration compactness for symmetric functionals.

On another front, McKenna discussed the symmetry of approximate solutions. Over the past quarter century, one field of intense research activity has been the study of what symmetry properties the solution of a nonlinear elliptic boundary value problem can inherit from the domain on which it is being solved. A classic paper is that of Gidas-Ni-Nirenberg, in which a typical result of the type we have in mind is: a positive solution of the boundary value problem

$$\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{4}$$

must be radially symmetric if Ω is a ball. More recently, a related area has been attracting growing attention, namely how does one approximate solutions of this type of nonlinear boundary value problem? Typically, the work in this area relies on a suitable discretization of (4), (most commonly by finite-differences), and then uses theoretical ideas from nonlinear analysis such as monotonicity methods, mountain pass algorithms, or linking methods, to develop an approximate or exact solution to the discretized problem.

McKenna addressed the so-far-neglected question: if the partial differential equation (4) has inherited certain symmetry properties from the domain, to what extent does the discretized problem also inherit these symmetry properties?

This leads to the study of the most natural discretization of (4), namely,

$$\begin{aligned} &u_{i+1} - 2u_i + u_{i-1} = h^2 f(u_i), \quad u_i > 0, \quad i = -(N-1)\ldots N-1, \\ &u_{-N} = u_N = 0, \end{aligned} \tag{5}$$

where $h = L/N > 0$ is the mesh-size of an equidistant mesh on $[-L, L]$. Suppose that $f : [0, \infty) \to \mathbf{R}$ is a given function. A solution of (5) is represented as a vector $u = (u_{-N}, \ldots, u_N) \in \mathbf{R}^{2N+1}$ with $\|u\|_\infty = \max_{i=-N\ldots N} |u_i|$. The first natural conjecture would be that the discrete approximate solution u_i would have a maximum at $j = 0$, and be symmetric about 0 in the sense that $u_{-j} = u_j$. This would exactly reflect the symmetry properties of the analogous continuous problem. This conjecture is false. Roughly speaking McKenna's result states *as $h \to 0$, the solution becomes more and more symmetric about the origin and the maximum $\to$ towards the origin.* Thus, the correct result is that for a sufficiently small space step, the solution will be "approximately" symmetric about the origin. They hope to prove an analogous result in the partial differential equation setting.

5 Hamiltonian systems and mathematical physics

Let $V \in C^2(\mathbf{R}^n, \mathbf{R})$ and $h > 0$ such that $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$ is bounded, open and connected. Consider the following given energy problem of the second order Hamiltonian system:

$$\ddot{q}(t) + V'(q(t)) = 0, \quad \text{for } q(t) \in \Omega,) \tag{6}$$

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h, \qquad \forall t \in \mathbf{R},) \tag{7}$$

$$\dot{q}(0) = \dot{q}(\frac{\tau}{2}) = 0,) \tag{8}$$

$$q(\frac{\tau}{2} + t) = q(\frac{\tau}{2} - t), \qquad q(t + \tau) = q(t), \quad \forall t \in \mathbf{R}.) \tag{9}$$

A solution (τ, q) of (6)-(9) is called a *brake orbit* on Ω. Two orbits q and $p : \mathbf{R} \to \mathbf{R}^m$ are said to be *geometrically distinct*, if $q(\mathbf{R}) \neq p(\mathbf{R})$. Denote by $\mathcal{J}(\Omega)$ and $\tilde{\mathcal{J}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in Ω respectively.

In 1948, H. Seifert proved ${}^{\#}\mathcal{J}(\Omega) \geq 1$ provided V is analytic, Ω is homeomorphic to the unit ball in $\mathbf{R}^n$, and $V'(q) \neq 0$ for $q \in \partial\Omega$. Then he conjectured that ${}^{\#}\tilde{\mathcal{J}}(\Omega) \geq n$ holds under the same conditions. Since then many studies have been carried out for brake orbits. Specially in 1983-1984, K. Hayashi, H. Gluck-W. Ziller, and V. Benci proved independently that ${}^{\#}\mathcal{J}(\Omega) \geq 1$, if V is C^1, $\bar{\Omega} = \{V \leq h\}$ is compact, and $V'(q) \neq 0$ for all $q \in \partial\Omega$. In 1987, P. Rabinowitz proved the corresponding result for first order Hamiltonian systems. For multiplicity results concerning Seifert's conjecture, there are only the papers of E. van Groesen in 1985, A. Szulkin in 1989, and A. Ambrosetti-V. Benci-Y. Long in 1993, in which $^{\#}\tilde{\mathcal{J}}(\Omega) \geq n$ was proved under various pinching conditions on the hypersurface $\partial\Omega$.

Yiming Long and his students study the multiplicity of brake orbits without any pinching conditions. Their main result is the following:

Theorem. *For $n \geq 2$ and $V \in C^2(\mathbf{R}^n, \mathbf{R})$, suppose $V(0) = 0$, $V(q) \geq 0$, $V(-q) = V(q)$, and $V''(q)$ is positive definite for all $q \in \mathbf{R}^n \backslash \{0\}$. Then for any given $h > 0$ and $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$, there holds*

$$^{\#}\tilde{\mathcal{J}}(\Omega) \geq 2. \tag{10}$$

Bolotin studied another class of problems for (6). Consider the 3-body problem in R^2 where we have a sun of mass 1, Jupiter of mass ε and an asteroid of negligible mass. Let $u(t)$ be the elliptic T-periodic orbit of the Jupiter. The motion of the asteroid is described by a Lagrangian system (L_ε) with

$$L_\varepsilon(q, \dot{q}, t) = |\dot{q}|^2/2 + |q + \varepsilon u(t)|^{-1} + \varepsilon |q - u(t)|^{-1}.$$

The system (L_ε) is a singular perturbation of the Kepler problem (L_0).

Fix $m, n \in \mathbf{N}$. Let Π be the set of chains $c = (c_i)_{i=1}^n$ of collision curves $c_i : [t_{i-1}, t_i] \to \mathbf{R}^2 \setminus \{0\}$ such that $c_i(t_{i-1}) = u(t_{i-1})$, $c_i(t_i) = u(t_i)$. The time moments $t_0 < \ldots < t_{n-1}$ are independent variables and $t_n = t_0 + mT$. Thus Π is an open set in $W_0^{1,2}([0,1], \mathbf{R}^{2n}) \times \mathbf{R}^n$. Critical points of the action functional

$$I(c) = \sum I(c_i), \qquad I(c_i) = \int L_0(c_i(t), \dot{c}_i(t), t)\, dt$$

are chains of collision orbits of system (L_0) such that the relative Hamiltonian $h = H_0 - \dot{q} \cdot \dot{u}(t)$ does not change at collisions: $h_i^+ = h_i^- = h_i$. We say that $c = (c_i)_{i=1}^n$ is a *nondegenerate collision chain* if it is a nondegenerate critical point of I and at each collision the direction of the relative velocity $v = \dot{q} - \dot{u}(t)$ changes: $v_i^+ \nparallel v_i^-$. Bolotin's main result is:

Theorem 1. *For any nondegenerate periodic collision chain $c = (c_i)_{i=1}^n$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a unique mT-periodic solution of system (L_ε) which is $O(\varepsilon)$-close to $c_i(t)$ for $t_{i-1} \leq t \leq t_i$.*

Such shadowing periodic orbits were called periodic solutions of the second kind by Poincaré. However, Poincaré didn't prove their existence. A similar result holds for infinite collision chains. Take N open bounded sets $U_k \subset \mathbf{R}^2$ such that for each $(t_1, t_2) \in U_k$ there exists a collision orbit $c : [t_1, t_2] \to \mathbf{R}^2$ of (L_0) with $c(t_1) = u(t_1)$, $c(t_2) = u(t_2)$, smoothly depending on (t_1, t_2). In particular $t_1 < t_2$ are not conjugate along c. Then $I(c) = S_k(t_1, t_2)$ is a smooth function on U_k. Sequences $\varkappa = (k_i)_{i\in\mathbf{Z}}$ and $\tau = (t_i)_{i\in\mathbf{Z}}$ such that $(t_{i-1} - Tm_i, t_i - Tm_i) \in U_{k_i}$, $m_i \in \mathbf{Z}$, define a collision chain $c = (c_i)_{i\in\mathbf{Z}}$. Set

$$A_\varkappa(\tau) = \sum I(c_i) = \sum S_{k_i}(t_{i-1}, t_i).$$

The functional is formal but its derivative $A'_\varkappa(\tau) \in l_\infty$ is well defined. A collision chain $c = (c_i)_{i\in\mathbf{Z}}$ corresponding to the critical point τ is called nondegenerate if the second derivative $A''_\varkappa(\tau) : l_\infty \to l_\infty$ has a bounded inverse and the changing direction condition is uniform in i. Then for small $\varepsilon \in (0, \varepsilon_0)$ there exists an orbit of (L_ε) shadowing the chain c.

If S_k satisfies the twist condition $D^2_{t_1 t_2} S_k \neq 0$, critical points of $A_\varkappa$ correspond to orbits of compositions $f_{k_n} \circ \ldots \circ f_{k_0}$ of symplectic maps $f_{k_i} : (t_{i-1}, h_{i-1}) \to (t_i, h_i) \in (\mathbf{R}/T\mathbf{Z}) \times \mathbf{R}$ with generating functions S_{k_i}. Such random dynamical systems have rich hyperbolic dynamics even if every map f_k is integrable. This makes it possible to construct many nondegenerate collision chains and hence periodic and chaotic shadowing orbits for system (L_ε).

Turning from classical to quantum mechanics, the Dirac-Fock equations are the Euler-Lagrange equations corresponding to the Dirac-Fock energy functional in a "sphere" of $L^2(R^3, C^4)^N$, N being a positive integer. This model corresponds, in an approximate way, to the search of stationary states for relativistic atoms and molecules. The Dirac operator being unbounded, both from above and from below, the corresponding energy functional is highly indefinite. However, this model "should contain" a notion of ground state if it is to describe a physical situation in which "minimal energy" solutions should exist and correspond to "most probable" configurations for the physical system. Moreover, the nonrelativistic limit of these equations (in the high light speed limit) can be shown to be the Hartree-Fock equations, for which ground state solutions exist under reasonable conditions (the Hartree-Fock energy is bounded from below). Maria Esteban first described the Dirac-Fock equations, and how taking the nonrelativistic limit leads us to the Hartree-Fock equations. Then, she concentrated in showing how for high light speeds, various different variational problems are equivalent to the one that she uses to show existence of solutions. From this, one can obtain a physically relevant notion of ground state solution for this model. By doing so, what is really shown is that the critical points that are physically relevant all lie in a subset of the "sphere" defined by a nonlinear constraint. What is indirectly shown is that the Dirac-Fock energy is bounded from below in that set while it is not in the whole "sphere", i.e. that its minimum is reached there and that the minimizers are critical points of the energy which correspond to the solutions that previously found by using an unconstrained variational argument.

Eric Séré also presented his work on existence of a stable polarized vacuum in the Bogoliubov-Dirac-Fock approximation. According to Dirac's ideas, the vacuum consists of infinitely many virtual electrons which completely fill up the negative part of the spectrum of the free Dirac operator D^0 (this model is called the "Dirac sea"). In the presence of an external field, these virtual particles react and the vacuum becomes polarized. In this work, Séré and his coauthors consider a nonlinear model of the vacuum derived from QED, called the Bogoliubov-Dirac-Fock model (BDF). In this model, the vacuum is represented by a bounded self-adjoint operator Γ on $L^2(R^3)$. An energy of this vacuum is defined. A stable vacuum is a minimizer of this BDF energy functional, under some convex constraints. Séré showed the existence of a minimizer of the BDF energy in the presence of an external electrostatic field and proved that this minimizer is a projector, which solves a self-consistent equation of Hartree-Fock type. This minimizer is interpreted as the polarized Dirac sea.

6 Other aspects and applications of variational methods

In addition to the topics discussed above, some new methods and applications related to variational problems were presented in the workshop.

Nassif Ghoussoub developed a theory of anti-self dual Lagrangians and new variational formulations of boundary value problems and evolution equations. Its antecedents are old work of Brezis and Ekeland. His theory of anti-self dual Lagrangians allows for surprising variational formulations and resolutions for many boundary value and initial value problems which normally cannot be obtained as Euler-Lagrange equations of action functionals. Examples include non-potential operator equations (like nonlinear transport and others involving first order differential operators), as well as certain dissipative evolution equations (like the heat equation, porous media, other gradient flows and the Navier-Stokes equations).

Ivar Ekeland and Louis Nirenberg studied a very interesting variational problem from economics. When computing conditional expectations by Monte-Carlo methods, one tries to minimize the mean variance of the error. Applying Malliavin calculus to the problem, one is led to a novel type of Sobolev space, consisting of all functions on the positive orthant of R^n, such that every derivative not containing terms in $(d^p)/(dx_i)^p$ with $p = 2$ or more is square integrable. The last derivative with this property is $d^n/(dx_1)...(dx_n)$. They show that this is a bona fide Sobolev space, and They consider the problem of minimizing a quadratic form on that space under boundary conditions. they show existence, uniqueness and regularity of the minimizer.