The Banff International Research Station's Workshop on Amenable Systems

The subject matter of the recent BIRS workshop on amenable systems could be roughly divided into four broad categories: classification of amenable C^*-algebras and related topics, C^*-algebras associated to directed graphs and related objects, commutative dynamical systems and C^*-algebras, and non-commutative dynamical systems. The state of research in amenable systems and the results presented at the workshop are discussed below under these headings.

§1. Classification of Amenable C*-algebras and Related Topics

Classification of amenable $C*$-algebras was only a dream some 16 years ago, a dream that started with Elliott's classification of AF-algebras. The Elliott program could be simply described as classification of amenable C^*-algebras by a K-theoretical invariant (the Elliott invariant). Today, the Elliott program of classification of amenable C^*-algebras has become a very successful and continuing story. To name a few break-through results in the program we mention: the Elliott-Gong theorem, which classified simple AH-algebra of real rank zero with local spectra of dimension at most three; the Kirchberg-Phillips theorem on classification of separable, amenable, purely infinite, simple, C^*-algebras which satisfy the Universal Coefficient Theorem; and the Elliott-Gong-Li classification theorem for simple AH-algebras with no dimension growth.

On the other hand, Villadesen's amazing construction of simple AH-algebras with higher stable rank opened a whole new horizon, as well as indicated new difficulties in the Elliott program. During the workshop, A. Toms exploited Villadesen's construction further. He reported that one can construct a class of simple AH-algebras whose isomorphic invariant set must include something other than the conventional Elliott invariant. This mystery injects new excitement into the Elliott program.

Z. Niu demonstrated possibilities of attacking general simple ASH-algebras which are not simple AH-algebras.

Interesting results on classification of non-simple C^*-algebras were also given in the work shop, for example, by Dadalart and Pasnicu.

Closely related to dynamical systems, H. Lin and N. C. Phillips reported that simple crossed products arising from minimal dynamical systems on finite dimensional compact metric spaces have zero tracial rank and therefore are classifiable if the ranges of their K_0-groups are dense in the affine functions on their tracial spaces. This result, together with Lin's work on amenable simple C^*-algebras with lower tracial rank, demonstrated that the above mentioned classification theorem of Elliott-Gong (as well as the result of Elliott-Gong-Li) can be applied to many naturally arising C^*-algebras, in particular, as the title of this work shop suggests, those C^*-algebras arising from amenable dynamical systems.

Other related topics were discussed during the work shop.

A C^*-algebra is said to be self-absorbing if $A \otimes A$ is isomorphic to itself. W. Winter reported that there are only a few such amenable simple C^*-algebras.

D. Kucerovski and P-W. Ng reported a number of absorbing theorems which are closely related to the classification of amenable C^*-algebras.

M. Dadarlat reported a new development regarding the Universal Coefficient Theorem. He revisited the topology on the Kasparov groups and showed that for two separable amenable C^*-algebras, KL-equivalence is the same as KK-equivalence. One may hope that all separable amenable C^*-algebras satisfy the UCT.

§2. C*-algebras Associated to Directed Graphs and Related Objects

A directed graph is a combinatorial object consisting of vertices and oriented edges joining pairs of vertices. We can represent such a graph by operators on a Hilbert space $\mathcal{H}$: the vertices are represented by mutually orthogonal closed subspaces, or more precisely the projections onto these subspaces, and the edges by operators between the appropriate subspaces. The graph algebra is, loosely speaking, the C^*-algebra generated by these operators.

When the graph is finite and highly connected, the graph algebras coincide with a family of C^*-algebras first studied by Cuntz and Krieger in 1980 [3]. The Cuntz-Krieger algebras were quickly recognised to be a rich supply of examples for operator algebraists, and also cropped up in unexpected places [13], [16]. In the past 10 years there has been a great deal of interest in graph C^*-algebras associated to infinite graphs, and these have arisen in new contexts: in non-abelian duality [12], [5], as deformations of commutative algebras [17],[7], in non-commutative geometry [4], [15], and as models for the clasification of simple C^*-algebras [8].

Graph algebras have an attractive structure theory, in which algebraic properties of the algebra are related to combinatorial properties of paths in the directed graph. The fundamental theorems of the subject are analogues of those proved by Cuntz and Krieger, and include uniqueness theorems and a description of the ideals in graph algebras. But we know so much more: just about any C^*-algebraic property a graph algebra might have can be determined by looking at the underlying graph.

Higher-rank graphs are, as the name suggests, higher dimensional analogues of directed graphs. They were introduced by Kumjian and Pask [11], and have recently been attracting a good deal of attention. Uniqueness theorems have been proved, and though they are significantly more complicated than graph algebras, we are finding out more about them every day. Recently there have been some partial results on their K-theory [1], [6] and there are some recent results by Raeburn, Sims, and Pask which show that a large class of simple AT algebras can be realised as two dimensional graph algebras. The future may hold many more intriguing results.

Other generalisations of graph algebras that have been studied include the ultragraph C^*-algebras introduced by Tomforde [18] and the labelled graph C^*-algebras introduced by Bates and Pask [2]. An ultragraph is a generalisation of a directed graph in which the edges have a set-valued range. To form labelled graphs, the edges of a directed graph are given labels coming from some alphabet. At this time the basic uniqueness and simplicity results have been proved for these algebras, and theorems have been proved which show that some of their structural properties can be determined by looking at the underlying graph and its labelling. Katsura [9] has done a vast amount of work in describing C^*-algebras associated to topological graphs. There is also a substantial group of mathematicians working on non self-adjoint operator algebras associated to directed graphs (see for example, [10]. Other practitioners include Muhly, Solel and Hopenwasser). The results here are remarkable: the directed graph itself is the invariant for classification of these operator algebras.

At the workshop, Teresa Bates presented some preliminary results on labelled graph C^*-algebras, Toke Carlsen and Alex Kumjian discussed higher rank graph C^*-algebras, and Takeshi Katsura presented some results related to topological graph C^*-algebras. David Kribs presented some new results on weighted graph C^*-algebras.

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§3. Commutative Dynamical Systems and C*-algebras.

Already in Murray and von Neumann's first papers, the links between the theories of dynamical systems and operator algebras have been very important. Thanks to Connes' classification of injective von Neumann factors (the type III_1 case having been settled by Haagerup), Krieger's theorem, and the Connes-Feldman-Weiss characterization of amenable measurable actions; there is a bijective correspondence between amenable, ergodic, non-singular actions up to orbit equivalence, and injective von Neumann factors up to isomorphism.

In this report, we will present some of the known results in the interplay between topological dynamics and C*-algebras. Many of the new results were presented at this BIRS workshop. We begin by reviewing the transformation group C*-algebras of minimal homeomorphisms of compact metric spaces. For example, both the C*-crossed products associated to Cantor minimal systems and the irrational rotation algebras are AT-algebras (direct limits of circle algebras) with real rank zero and therefore belong to the class of algebras classifiable by K-theoretical invariants. The first result was proved by Putnam and the second one by Elliott and Evans. In a very recent preprint, H. Lin and N.C. Phillips proved the following remarkable result:

> *Let (X, ϕ) be a minimal dynamical system where X is an infinite compact metric space with finite covering dimension. Let $A = C^*(X, \phi)$ be the associated crossed product, and Aff$(T(A))$ be the space of real valued affine continuous functions on $T(A)$, the compact convex set of tracial states of A. If the natural map from $K_0(A)$ to Aff$(T(A))$ has dense range, then A is a simple unital AH algebra with rank zero and therefore is classifiable.*

In the smooth case, let us recall that Q. Lin and N.C. Phillips showed that the C*-crossed product associated to a minimal diffeomorphism of a compact smooth manifold is also classifiable, being a direct limit, with no dimension growth, of recursive subhomogeneous C*-algebras.

For a general minimal dynamical system (X, ϕ), no Krieger type theorem has yet been proved. Only for two classes of dynamical system have dynamical characterizations of isomorphism of the associated C*-crossed products been given. Before describing them, let us notice first of all that, due to an old result of Sierpinski, two (topologically) orbit equivalent minimal homeomorphisms on a connected compact metric space are flip conjugate.

For minimal homeomorphisms of the circle, the isomorphism of the C*-crossed product implies flip conjugacy (this follows from the following two facts: every minimal homeomorphism of S^1 is conjugate to an irrational rotation, and $C^*(S^1, R_\alpha) \cong C^*(S^1, R_\beta)$ iff α has the same image as $\pm\beta$ in $\mathbb{R}/\mathbb{Z}$).

For Cantor minimal systems, Giordano, Putnam, and Skau introduced the slightly technical notion of strong orbit equivalence (SOE) and proved that two Cantor minimal systems are SOE iff the associated C*-crossed products are isomorphic.

Using the Bratteli-Vershik model of Cantor minimal systems created by Herman, Putnam and Skau, H. Dahl has characterized the (finite dimensional) Choquet simplices of probability measures on the Cantor set which are the set of invariant measures of a Cantor minimal system. This generalizes a result of E. Akin.

Recently H. Lin has proposed the study of different versions of approximate conjugacy for minimal dynamical systems. The first results appear in three preprints by Lin, Lin and Matui, and Matui. For Cantor minimal systems, the approximate conjugate relation is closely related to orbit equivalence and strong orbit equivalence.

For minimal actions of groups other than $\mathbb{Z}$, the situation is more complicated. Itzá-Ortiz has recently established a correspondence between the group of the eigenvalues of a minimal suspension dynamical flow (whose ceiling function is not necessarily constant) and a multiplicative subgroup of the K_0-group associated to the base transformation of this flow. For minimal actions of $\mathbb{Z}^n$, it is not yet known if the corresponding C*-crossed-product is classifiable in the Elliott sense. The best result up to now has been that obtained by N.C. Phillips, who showed that the C*-crossed-product associated to a minimal, free $\mathbb{Z}^n$-action on the Cantor set has stable rank one, real rank zero, and cancellation of projections, and that the order on its K_0-group is determined by traces.

On the dynamical side, Giordano, Putnam, and Skau studied the so-called affable equivalence relations and proved that a "small extension" of an AF-equivalence relation is still (orbit equivalent to) an AF-equivalence relation. This gives a new topological dynamic proof that any Cantor minimal system is orbit equivalent to an AF-equivalence relation. In a recent preprint, they introduce a cohomological condition on minimal $\mathbb{Z}_2$-actions on the Cantor set, give two large classes of actions satisfying it and show that such minimal $\mathbb{Z}_2$-actions are orbit equivalent to AF-equivalence relations, using the extension result mentioned above.

§4. Noncommutative Dynamical Systems

The classification program for C*-algebras has had the most success with purely infinite, simple C*-algebras (see, for example, [5] and [8]), with simple C*-algebras with tracial rank zero as introduced in [6] (see, for example, [7]), and especially with various classes of C*-algebras obtained as direct limits of special kinds of type I C*-algebras (see the discussion in Section 1. of this report). The classification program is currently interacting with noncommutative dynamics in two important ways. First, C*-algebraists tend to be more interested in crossed product C*-algebras than in most of the classes just mentioned. Work on the classification of crossed products has generally taken the form of proving that certain crossed products belong to one of the classes already covered by other classification theorems, or, less satisfactorily, at least proving structural properties of crossed products which suggest that they should belong to one of these classes. Work on crossed products by groups acting on compact spaces is discussed in Section 3. of this report, but some results for actions on noncommutative C*-algebras were presented at the workshop. Secondly, having classified algebras, it is natural to try to classify group actions on algebras.

Recent work on classifiability of crossed products of noncommutative C*-algebras has relied on the tracial Rokhlin property. This property is a weakening of the Rokhlin property [3] that Izumi uses in his classification work for automorphisms. The Rokhlin property is a rather rigid condition: K-theoretic obstructions (some obvious, some less so; see [4]) show that many purely infinite simple C*-algebras admit no actions of finite groups with the Rokhlin property. The tracial Rokhlin property for actions of finite cyclic groups first appeared in [9], where it was proved that if A is a simple separable unital C*-algebra with tracial rank zero, and if $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a finite cyclic group with the tracial Rokhlin property, then $C^*(G, A, \alpha)$ again has tracial rank zero. The applications there were to C*-algebras on which no nontrivial action of a finite group can have the full Rokhlin property.

Hiroyuki Osaka talked about actions of $\mathbb{Z}$ with the tracial Rokhlin property. For $\mathbb{Z}$, there are no known K-theoretic obstructions which prevent an action from having the Rokhlin property while allowing it to have the tracial Rokhlin property. However, there are a number of interesting actions of $\mathbb{Z}$ which are known to have the tracial Rokhlin property but not known to have the Rokhlin property. Osaka described two results, strongly suggestive but still incomplete. Let A be a simple separable stably finite unital C*-algebra, and let $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$ be an action with the tracial Rokhlin property. If A has real rank zero and stable rank one, and if the order on projections over A is determined by traces, then $C^*(G, \mathbb{Z}, \alpha)$ again has these properties. If A has tracial rank zero, and if α_n is approximately inner for some nonzero n, then $C^*(\mathbb{Z}, A, \alpha)$ again has tracial rank zero.

As seen above, in some ways crossed products by finite groups are more accessible than crossed products by $\mathbb{Z}$. Their K-theory, however, is much harder to compute. For example, there is an action of $\mathbb{Z}/2\mathbb{Z}$ on a contractible C*-algebra such that the K-theory of the crossed product is nonzero, which rules out anything resembling the Pimsner-Voiculescu exact sequences for crossed products by $\mathbb{Z}$ and by free groups. There are standard actions of $\mathbb{Z}/n\mathbb{Z}$ on the irrational rotation algebras A_θ, for $n = 2, 3, 4, 6$, which are among the actions of finite groups which have attracted the most attention. Computations of K-theory in the rational case (when the crossed products are type I and can be described explicitly) have led to the conjecture that, in the irrational case, all the crossed products are AF algebras. This has been known for some time for $\mathbb{Z}/2\mathbb{Z}$ (the proof relies on a fortuitous coincidence), and has been proved by Walters for $\mathbb{Z}/4\mathbb{Z}$ and "most" θ. It is shown in [9] that, for θ irrational, all the crossed products are AH algebras with slow dimension growth and real rank zero. Thus, the remaining step is to compute the K-theory. Julian Buck talked about work in this direction with Walters for $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ (where the least is known). It depends on cyclic cohomology in an essential way.

In the second direction, Masaki Izumi has previously proved some classification results for actions of finite groups with the Rokhlin property on Kirchberg algebras [3], [4]. In his talk at the conference, he described results for quasifree actions of finite groups on $\mathcal{O}_\infty$. These actions do not have the Rokhlin property; in fact, as follows from Izumi's earlier work, there are no nontrivial actions of finite groups on $\mathcal{O}_\infty$ which have the Rokhlin property. However, Izumi proved that quasifree actions are locally representable, which in a certain sense is dual to the Rokhlin property. (For an action α of a finite *abelian* group, α is locally representable if and only if $\widehat{\alpha}$ has the Rokhlin property.) One should note that this theory is really only just beginning; as with the classification of C*-algebras, the purely infinite simple case is the place to start.

Andrew Dean talked about classification for actions on AF algebras which are explicitly given as direct limit actions, but where the group is not compact. Such actions ("locally representable" in a sense stronger than that used by Izumi) for compact groups were considered long ago by Handelman and Rossmann [1], [2], as well as others. While keeping the direct limit structure (in contrast to Izumi), Dean has obtained results for certain specific kinds of actions of noncompact groups. In previous work, he has considered actions of $\mathbb{R}$, and in his talk at this conference he examined actions of two relatively elementary groups which have infinite dimensional irreducible representations, and in particular are neither compact nor abelian, namely $\mathrm{SL}_2(\mathbb{R})$ and the group of Euclidean motions of the plane. The direct limits are set up so as to allow these representations to appear in at least a limited way, and thus allow infinite dimensional algebras (copies of the compact operators) in the system. They can only appear in a limited way because the partial maps must have finite multiplicity; otherwise, the direct limit will not be AF.

Two talks at the conference described work on noncommutative dynamics farther afield from the classification program. Rui Okayasu presented work relating the entropy of certain subshifts to the values of a a numerical invariant introduced some time ago by Voiculescu for the purpose of measuring the obstruction to the existence of a quasi-central approximate identity relative to the Macaev ideal for a finite set of operators. Specifically, the set of operators should be the creation operators which appear in Matsumoto's construction of the C*-algebra of the shift. Okayasu has also computed this invariant for the images of generating sets of certain groups under the regular representation.

Ilan Hirshberg talked about finding certain kinds of representations of C*-correspondences (bimodules which are Hilbert modules on one side). A C*-correspondence can be thought of as a generalization of an automorphism of the algebra (also, simultaneously, as a generalization of some other things), and some of the associated C*-algebras (Cuntz-Pimsner algebras [10]) have attracted considerable interest recently. These algebras generalize not only crossed products but also Cuntz-Krieger algebras and graph algebras. From the point of view of dynamics, a representations of a C*-correspondence is a generalization of a covariant representation of $(\mathbb{Z}, A)$. Hirshberg's situation was of course much more complicated than just finding covariant representations.

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