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Higher Resonance Varieties

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The purpose of this meeting was to describe some qualitative properties of the higher resonance varieties of hyperplane arrangements and related topological spaces. Our starting point was the Bernstein-Gel'fand-Gel'fand (BGG) correspondence, an equivalence of bounded derived categories of graded modules over a polynomial algebra and an exterior algebra, respectively. By using Eisenbud, Fløystad and Schreyer's [3] explicit formulation of the BGG correspondence, we were able to generalize some previous work of Schenck and Suciu [6] from the first to the higher resonance varieties: definitions follow below.

Various interrelated spectral sequences appear when the BGG correspondence is applied. In particular, these give convenient language to talk about syzygies in minimal, graded free resolutions. On the other hand, work of Jan-Erik Roos [5] and Maurice Auslander [2] gives a filtration of a module over a regular ring with supports that decrease in dimension, via a suitable Grothendieck spectral sequence. We considered the interplay between this filtration (in the case of polynomial algebras) and the BGG correspondence. We obtained some specific results that, in particular, relate the growth of Betti numbers in free resolutions of cohomology algebras (as modules over the exterior algebra) to the dimensions of components of resonance varieties.

More precisely, a hyperplane arrangement $\mathcal{A}$ is a finite collection of n hyperplanes in some fixed (usually complex) vector space V of dimension ℓ. The complement $X = V - \mathcal{A}$ can be viewed as the intersection of a torus $(\mathbb{C}^*)^n$ with a linear space: accordingly, for any field $\mathbb{k}$, there is a map onto $A = H^*(X, \mathbb{k})$ from the cohomology ring of the torus, the exterior algebra $E = \Lambda(\mathbb{k}^n)$. A result of Eisenbud, Popescu and Yuzvinsky [4] shows that the cohomology algebra A, regarded as a graded E-module, has a minimal injective resolution which is linear, in the sense that the differentials in the resolution can be expressed as matrices with degree-1 entries. Interpreted via the BGG correspondence, there is a certain "dual" module $F(X)$, over the polynomial algebra $S = \mathrm{Sym}((\mathbb{k}^n)^*)$, which also possesses a linear resolution

$$0 \leftarrow F(X) \leftarrow H^\ell(X,\mathbb{k}) \otimes_{\mathbb{k}} S \leftarrow H^{\ell-1}(X,\mathbb{k}) \otimes_{\mathbb{k}} S \leftarrow \cdots \leftarrow H^0(X,\mathbb{k}) \otimes_{\mathbb{k}} S \leftarrow 0 \tag{1}$$

where the differential is given by multiplication by a tautological element of degree $(1,1)$.

By definition, the kth resonance variety (over $\mathbb{k}$) of a space X having the homotopy type of a finite CW-complex is given pointwise by

$$R^k(X) = \{a \in A^1 \colon H^k(A^\cdot, a) \neq 0\},$$

where $(A^{\cdot}, a)$ denotes the cochain complex obtained from the cohomology algebra A of X with differential given by multiplication by the 1-cocycle a. Under certain hypotheses, the resonance varieties parameterize (infinitesimal versions of) one-dimensional local systems on X with non-vanishing cohomology. If the exterior algebra acts on A, as it does in the cases we consider, these are also the rank varieties of A, as introduced by Avramov, Aramova and Herzog in [1].

For spaces X possessing a linear complex of the form (1) (such as arrangement complements or the classifying spaces of certain right-angled Artin groups), one can extract information about the resonance varieties by specializing the complex (1). This was first noticed in [6], where Schenck and Suciu prove that, for arrangements, the first resonance variety is given by

$$R^1(X) = V(\mathrm{ann}(\mathrm{Ext}_S^{\ell-1}(F(X), S))).$$

More generally, we noted that the higher resonance variety

$$R^k(X) = \bigcup_{p \leq k} V(\mathrm{ann}(\mathrm{Ext}_S^{\ell-p}(F(X), S))).$$

for $k \geq 1$. Since the modules $\mathrm{Ext}_S^{\cdot}(F(X), S)$ are, roughly speaking, BGG-dual to a minimal free resolution of A as an E-module, we obtain a link between syzygies of A and the geometry of higher resonance varieties. Going somewhat further, we see that for certain classes of arrangements, some syzygies can be understood in terms of the combinatorics of the hyperplane arrangement. Correspodningly, this leads to an explicit description of the resonance varieties in those cases. A preprint is in preparation.

References

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