REPORT ON THE WORKSHOP: "DERIVED CATEGORY METHODS IN COMMUTATIVE ALGEBRA"

LARS WINTHER CHRISTENSEN AND HENRIK HOLM

This is our report on the Research in Teams Workshop "Derived Category Methods in Commutative Algebra" held at Banff International Research Station (BIRS), 1–8 June 2008.

Background

Derived category methods have proved to be very successful in ring theory, in particular in commutative algebra. Evidence is provided by [1,4–8,11,12, 15,16,19,20,23,24,27], to list some work of considerable importance.

Surprisingly, there is no accessible introduction or reference to the applications of derived category methods in commutative algebra, or in general ring theory for that matter. To be an effective practitioner of these methods, one must be well-versed in a series of research articles and lecture notes, including unpublished ones: [2,3,9,10,13,14,16–18,22,25,28,29]. To get an overview of their applications in commutative algebra, the list grows further. The purpose of the BIRS workshop was to make progress on a book manuscript—authored by L.W. Christensen, H.-B. Foxby, and H. Holm—that will remedy this deficiency.

As implied in the discussion above, the book has no direct competition. Many books cover applications of classical homological algebra in (commutative) ring theory, but only a few books address derived category methods and their applications in this field: In *Homological Algebra* [9] by Cartan and Eilenberg, resolutions of complexes and derived functors are briefly discussed in the final chapter; no applications are given. In Weibel's *An introduction to homological algebra* [30], derived categories are introduced in the final chapter; a few applications to ring theory are included as exercises. Derived categories are also covered in *Methods of Homological Algebra* [21] by Gelfand and Manin, but applications to ring theory are not. A very thorough construction of derived categories is given in *Categories and Sheaves* [26] by Kashiwara and Schapira. However, the aim of [26] is sheaf theory, so beyond the construction of derived categories, there is barely any overlap with this book. Finally, Christensen's *Gorenstein Dimensions* [10] has an appendix on derived category methods. It provides a rudimentary and incomplete survey of technical results without proofs. The fact that it has, nevertheless,

Date: 12 Aug 2008.

become a frequently cited reference betrays a significant gap in the existing literature.

History of the book project

As one of the pioneers in the applications of derived category methods in commutative algebra, Foxby has previously circulated two sets of lecture notes on the topic [13, 17].

In 2006 Christensen and Foxby started the current project. The book in progress offers a systematic development of hyperhomological algebra. This includes the construction of the derived category of a general (associative) ring and a careful study of the functors of importance in ring theory. To demonstrate the strength and utility of the theory, and to motivate the choice of topics, the book includes an extensive course in central homological aspects of commutative ring theory. This part includes many recent results, which were discovered by means of derived category methods, and gives valuable new insight into the theory of commutative rings and their modules.

Based on four peer reviews, Springer-Verlag offered to publish the book, and a contract was signed in late 2007.

For health related reasons, Foxby has been unable to work on the project for some time. To ensure timely completion of the book, Christensen and Foxby decided to add a third author, and in April 2008 Holm accepted to joint the project.

Aim and results of the workshop

The workshop at BIRS had two purposes. To introduce the new coauthor Holm to the project, and to complete a first rough draft of the manuscript—taking into account the extensive comments in the reports solicited by Springer-Verlag.

The first three days of the workshop were spent on a major reorganization of the manuscript based on the referees' suggestions and feedback from students and colleagues. This reorganization serves two purposes:

- To make the book more useful as a reference to derived category methods also for researchers in non-commutative algebra.
- To structure the applications to commutative algebra in a fashion more familiar to researchers in that field.

This process was an excellent way to introduce Holm to the scientific as well as the technical and administrative aspects of the book project.

The balance of the workshop was spent on discussions and "prototyping" aimed at merging contributions from the different authors into a coherent text. This includes

- Laying down principles for indexing and cross-referencing
- Standardizing formulations of mathematical statements
- Homogenizing levels of abstraction between chapters
- Homogenizing levels of details in proofs

As the workshop participants live on different continents, and in different time zones, face-to-face meetings as provided by this workshop are of utmost importance for solving scientific as well as editorial problems. We thank BIRS sincerely for providing us with this opportunity.

References

1. Leovigildo Alonso Tarrío, Ana Jeremías López, and Joseph Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 1, 1–39. MR 1422312
2. Dmitri Apassov, *Annihilating complexes of modules*, Math. Scand. **84** (1999), no. 1, 11–22. MR 1681740
3. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155. MR 1117631
4. ———, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), no. 5, 1007–1047. MR 1183530
5. ———, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270. MR 1455856
6. ———, *Cohen-Macaulay properties of ring homomorphisms*, Adv. Math. **133** (1998), no. 1, 54–95. MR 1492786
7. Luchezar L. Avramov, Srikanth Iyengar, and Claudia Miller, *Homology over local homomorphisms*, Amer. J. Math. **128** (2006), no. 1, 23–90.
8. Nicolas Bourbaki, *Éléments de mathématique*, Masson, Paris, 1980, Algèbre. Chapitre 10. Algèbre homologique. [Algebra. Chapter 10. Homological algebra]. MR 610795
9. Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, With an appendix by David A. Buchsbaum, Reprint of the 1956 original. MR 1731415
10. Lars Winther Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, vol. 1747, Springer-Verlag, Berlin, 2000. MR 1799866
11. ———, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1839–1883. MR 1813596
12. Lars Winther Christensen, Anders Frankild, and Henrik Holm, *On Gorenstein projective, injective and flat dimensions—A functorial description with applications*, J. Algebra **302** (2006), 231–279. MR 2236602
13. Hans-Bjørn Foxby, *Hyperhomological algebra & commutative rings*, lecture notes, 1998.
14. ———, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), no. 1, 5–19. MR 56 #5584
15. ———, *On the μ^i in a minimal injective resolution. II*, Math. Scand. **41** (1977), no. 1, 19–44.
16. ———, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172. MR 83c:13008
17. ———, *A homological theory for complexes of modules*, Preprint Series 1981 nos. 19a&b, Matematisk Institut, Københavns Universitet, 1981.
18. Hans-Bjørn Foxby and Srikanth Iyengar, *Depth and amplitude for unbounded complexes*, Commutative algebra (Grenoble/Lyon, 2001) (Providence, RI), Contemp. Math., vol. 331, Amer. Math. Soc., 2003, pp. 119–137. MR 2013162
19. Anders Frankild, *Quasi Cohen-Macaulay properties of local homomorphisms*, J. Algebra **235** (2001), no. 1, 214–242. MR 1807663
20. ———, *Vanishing of local homology*, Math. Z. **244** (2003), no. 3, 615–630. MR 1992028

21. Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR 2003m:18001
22. Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR 0222093
23. Birger Iversen, *Amplitude inequalities for complexes*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 547–558. MR 0568903
24. Srikanth Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230** (1999), no. 3, 545–567. MR 1680036
25. Peter Jørgensen, *Non-commutative graded homological identities*, J. London Math. Soc. (2) **57** (1998), no. 2, 336–350.
26. Masaki Kashiwara and Pierre Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 332, Springer-Verlag, Berlin, 2006. MR 2182076
27. Paul Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 103–106. MR 0399075
28. ———, *Homological invariants of modules over commutative rings*, Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], vol. 72, Presses de l'Université de Montréal, Montreal, Quebec, 1980. MR 569936
29. Nicolas Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.
30. Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324

Department of Mathematics and Statistics, Texas Tech University, Lubbock TX 79409, U.S.A.

E-mail address: lars.w.christensen@ttu.edu

Department of Natural Sciences, University of Copenhagen, Thorvaldsensvej 40, DK-1871 Frederiksberg C, Denmark

E-mail address: hholm@life.ku.dk