Arithmetic of K3 Surfaces

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November 30–December 5, 2008

1 Overview of the Field

Understanding Diophantine equations is one of the fundamental goals of mathematics. Algebraic geometry has proved to be indispensable in the study of Diophantine problems. It is therefore no wonder that throughout history the geometric complexity of the Diophantine problems in focus has been increasing steadily. While the arithmetic of curves has been studied for a long time now, only fairly recently has there been substantial progress on that of higher-dimensional varieties. Naturally, this started with the easier varieties, such as rational and abelian varieties. K3 surfaces, where many basic problems are still wide open, are the next step in complexity.

In the last five years the rate of progress on the arithmetic of K3 surfaces has increased dramatically. However, before this workshop, not a single international meeting had been held to join the forces of the people involved. The purpose of this workshop was to combine the many lines of research in this new area. The big open problems can only be tackled by combining different strengths, both computational and theoretical. The fields of specialization of the participants include the following.

- Modularity of K3 surfaces
- Potential density of rational points
- Brauer-Manin obstructions
- Weak approximation
- Growth of the number of rational points of bounded height
- Computability of the Picard group
- Applications to curves
- Universal torsors
- K3 surfaces in positive characteristic

- Enriques surfaces

Quite a few of the participants were young researchers, both new postdocs and graduate students. For this reason, and because the participants all come from different backgrounds, the workshop started with several survey lectures on the topics mentioned above. The participants were then able to form small groups to focus on more specialized issues, which were complemented by more specialized talks as the workshop progressed.

The name “K3 surfaces” refers to the three algebraic geometers Kummer, Kähler and Kodaira, but also alludes to the mountain peak K2, which had recently been climbed for the first time when the name was given during the 1950s. By having this workshop at Banff, K3 surfaces are once again linked to the mountains.

2 Recent Developments and Open Problems

Our understanding of the arithmetic of K3 surfaces has expanded extremely rapidly in the past five years, and the workshop reflected the broad diversity of the advances in the field. At the workshop, there was a one hour session dedicated to describing some of the many open problems in the field. The session was extremely successful, lasting well past its appointed end and into dinnertime, and resulted in 26 open problems, many of which have related subquestions. The list of open problems was compiled by Bjorn Poonen, Anthony Várilly-Alvarado, and Bianca Viray and is reproduced below.

Unless otherwise specified, X is an algebraic K3 surface and k is a number field.

1. (McKinnon) Given X/k and $P \in X(k)$, determine if there is a non-constant map $f\colon \mathbb{P}^1_k \to X$ such that P is in the image.

   (Beauville) Same question over $\overline{k}$.

   (Colliot-Thélène) Related conjecture by Beilinson: $\mathrm{CH}_0(\overline{X})$ should be $\mathbb{Z}$.

2. Given X/k, compute the rank of $\operatorname{Pic} X$. Same question for $\operatorname{Pic} \overline{X}$.

   (Poonen) If one can determine $\operatorname{Pic} \overline{X}$, then one can determine $\operatorname{Pic} X$.

3. Given $L \in \operatorname{Pic} X$ and $m > 1$, determine if $L = mM$ for some $M \in \operatorname{Pic} X$. Given a nonzero L, determine an upper bound for $\{m : L \in m \operatorname{Pic} X\}$.

   (Baragar) Given $\Lambda \subset \operatorname{Pic} X$, decide if $\Lambda = \operatorname{Pic} X$. Determine if Λ is saturated in $\operatorname{Pic} X$. This is equivalent to determining if Λ is saturated in $H^2(X, \mathbb{Z})$.

4. (McKinnon) Given $L \in \operatorname{Pic} X$, decide if $L = [C]$ for some *integral* curve C.

5. (Poonen) Given $D \in \operatorname{Div} X$, can one compute $H^0(X, \mathcal{O}(D))$? The answer seems to be yes.

6. (Silverman) Suppose that X is defined over $\mathbb{C}$ and $\operatorname{Aut}(X)$ is infinite. Let C be an integral curve on X and $P \in X$. Suppose that $(\operatorname{Aut}(X) \cdot P) \cap C$ is infinite. Does this imply that there exists a non-trivial automorphism σ of infinite order mapping C to itself? This is an analogue of Mordell-Lang for K3 surfaces.

   (Beauville) If C is an integral curve on $X/\mathbb{C}$ and σ is an automorphism of infinite order such that $\sigma C = C$, then $g(C) \leq 1$.

7. Let σ be an automorphism of infinite order of X and let $C \subset X$ be a curve. Assume that the set of periodic points of σ contained in C is infinite. Does this imply that C is periodic, i.e. $\sigma^n C = C$ for some n?

8. (Skorobogatov) What does the Bloch-Beilinson conjecture say explicitly for a K3 over $\mathbb{Q}$? What interesting numbers should appear in special values of L-functions?

9. (Bogomolov and Tschinkel) Given $X/\overline{\mathbb{F}}_p$, is there a rational curve through every $\overline{\mathbb{F}}_p$-point? Same for $\overline{\mathbb{Q}}$? There is a positive result due to Bogomolov and Tschinkel for the first question for Kummer surfaces over $\overline{\mathbb{F}}_p$.

   (McKinnon) Over $\overline{\mathbb{Q}}$, this is equivalent to the K3 being “rationally connected” over $\overline{\mathbb{Q}}$.

10. (Skorobogatov) Find a formula for

$$\#\frac{\operatorname{Br} A}{\operatorname{Br}_1 A} = \#\operatorname{im}(\operatorname{Br} A \to \operatorname{Br} \overline{A})$$

for an abelian surface A. This is finite (proved by Skorobogatov and Zarhin). Same question for K3s. Given a fixed A (or X), the size can be made arbitrarily large by making a suitable extension of the base field.

11. (Wittenberg) Is there a K3 surface X over a number field k such that

$$\frac{\operatorname{Br} X}{\operatorname{Br} k} = 0?$$

12. (McKinnon) Find X/k with geometric Picard rank 1 and an accumulating curve over k, i.e., 100% of k-points of X lie on the curve (asymptotically, ordered by height). There are examples of K3s with accumulating curves, but all have large geometric Picard rank.

13. (Silverman) Find an example of a K3 surface with an interesting automorphism of infinite order (e.g., excluding composition of involutions).

14. (Poonen) Describe the possibilities for $\operatorname{Aut}(X/\mathbb{C})$ as an abstract group. Sterk proved that $\operatorname{Aut}(X/\mathbb{C})$ is finitely generated.

    (Kumar) Can one compute generators for $\operatorname{Aut}(X)$? Is there an upper bound on the number of generators?

    (Poonen) Can one compute the relations?

    (Bender) Can one compute the minimum number of generators?

    (Silverman) Is there an upper bound on the number of generators of the subgroup of $\operatorname{Aut}(X)$ generated by all the involutions?

15. (Poonen) Given K3 surfaces X and Y over $\overline{\mathbb{Q}}$, can one decide if $X \cong Y$?

    (McKinnon) Given K3 surfaces X and Y over $\overline{\mathbb{Q}}$, can one decide if $\operatorname{Aut} X \cong \operatorname{Aut} Y$ as abstract groups?

16. Is there a K3 surface X over a number field k such that $X(\mathbb{A}_k) \neq \emptyset$ and X satisfies weak approximation? If so, find it.

    (Colliot-Thélène) Is there a K3 surface X over a number field k such that $X(k) \neq \emptyset$ and X satisfies weak weak approximation?

17. Is there a K3 surface X over a number field k with $X(k)$ nonempty and finite? Nonempty and not Zariski dense? What about over an arbitrary infinite field? Is there a K3 surface X having only finitely many points over its own function field?

18. (Poonen) Given $X/\mathbb{F}_p$ is $X(\mathbb{F}_p(t))$ finite?

    (Beauville) Not always.

    (Poonen) Can you compute $X(\mathbb{F}_p(t))$? Is $X(\mathbb{F}_p(t))/\operatorname{Aut}(X)$ finite?

19. Find a K3 surface over a number field with geometric Picard rank 1 for which one can either prove that the rational points are potentially dense or prove that they are not potentially dense.

20. (Cantat) Suppose that $X(k)$ is Zariski dense. Must there exist a finite extension L/k and an embedding $L \subseteq \mathbb{C}$ such that $X(L)$ is analytically dense in $X(\mathbb{C})$?

    (Cantat) This is true for Abelian varieties.

21. (Cantat and Silverman) Suppose that $X(k)$ is Zariski dense and v is a place of k. Must there exist a finite extension L/k and a place w over v such that $X(L)$ is w-adically dense in $X(L_w)$? Must there exist a finite extension L/k such that $X(L)$ is w-adically dense in $X(L_w)$ for all w over v? Must there exist a finite extension L/k such that $X(L)$ is dense in $X(\mathbb{A}_L)$?

22. Is $X(k)$ always dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$, i.e., is the Brauer-Manin obstruction the only one to weak approximation? To the Hasse principle?

23. (Cantat) Let X be defined over $\mathbb{C}$. Does there exist $\mathbb{C}^2 \dashrightarrow X$ meromorphic and generically of maximal rank?

24. (Colliot-Thélène and Ojanguren) Let Y be an Enriques surface and X be the associated K3 double cover. Is the map
$$\frac{\mathrm{Br}\,\overline{Y}}{\mathrm{Br}\,k} \longrightarrow \frac{\mathrm{Br}\,\overline{X}}{\mathrm{Br}\,k}$$
always injective? If not, how can one determine if it is injective in any given example?

25. (Baragar) Find a K3 surface X over a number field k such there are infinitely many orbits of k-rational curves under the action of $\mathrm{Aut}\,X$.

26. In the following all K3 surfaces S are embedded in $\mathbb{P}^2 \times \mathbb{P}^2$ such that the projections do not contract any curves and $\mathcal{A}$ is a subset of the automorphism group. We are still assuming k is a number field. We define
$$S[\mathcal{A}] = \{P \in S : \mathcal{A}(P) \text{ is finite}\}.$$

   (a) K3 Uniform Boundedness Conjecture:
   There is a constant $c = c(k)$ such that for all K3 surfaces S/k,
   $$\#S[\mathcal{A}](k) \leq c.$$

   (b) K3 Manin-Mumford Conjecture:
   Let $C \subset S$ be a curve such that $\phi(C) \neq C$ for all $C \subseteq A$. Then $C \cap S[\mathcal{A}]$ is finite.

   (c) (Weak) K3 Lehmer Conjecture:
   Fix S/k. There are constants $c = c(S/k) > 0$ and $\delta = \delta(S/k)$ so that
   $$\hat{h}(P) \geq \frac{c}{[L:k]^{\delta}} \text{for all } L/k \text{ and } P \in S(L) \setminus S[\mathcal{A}].$$

   (d) K3 Lang Height Conjecture:
   There is a constant $c = c(k)$ such that for all K3 surfaces S/k,
   $$\hat{h}(P) \geq c \cdot h(S) \quad \text{for all } P \in S(k) \setminus S[\mathcal{A}].$$
   (Here $h(S)$ is the height of S as a point in the moduli space of K3 surfaces.)

   (e) K3 Serre Image-of-Galois Conjecture:
   For any subgroup $\mathcal{B} \subseteq \mathcal{A}$, let
   $$S_{\mathcal{B}} := \left\{P \in S(\overline{k}) : \mathcal{B} \text{ is the stabilizer of } P \text{ in } \mathcal{A}\right\},$$
   and define
   $$\rho_{\mathcal{B}} \colon \mathrm{Gal}(k(S_{\mathcal{B}})/k) \to \mathrm{SymGp}(S_{\mathcal{B}}).$$
   There is a constant $c = c(S/k)$ so that for all subgroups $\mathcal{B} \subset \mathcal{A}$ of finite index,
   $$(\mathrm{SymGp}\,(S_{\mathcal{B}}) : \ \mathrm{Image}\,(\rho_{\mathcal{B}})) < c.$$

3 Presentation Highlights

There were 18 talks at the workshop, on a variety of topics related to the arithmetic of K3 surfaces. Some of these explored algebro-geometric techniques that have applications to arithmetic. For example, Arnaud Beauville described the Chow ring, which is a fundamental object describing algebraic cycles on an algebraic variety such as a K3 surface. Unfortunately, the Chow ring is known to be very unwieldy, and so Beauville described a finitely generated subring of the Chow ring which is most relevant for arithmetic, and used it to describe a theorem of Huybrechts. Also in this vein was the talk by Tetsuji Shioda, who demonstrated the structure of the Mordell-Weil lattice of elliptic K3 surfaces in general, and the elliptic K3 surfaces of Inose-Kuwata in particular. Jaap Top, meanwhile, gave a talk which used the framework of his joint work with Bert van Geemen to describe a variety of techniques from algebraic geometry that they successfully applied to study the arithmetic of two families of K3 surfaces with Picard rank 19.

Another way in which algebraic geometry can shed light on the arithmetic of K3 surfaces is to provide other algebraic varieties with analogous properties to K3 surfaces. Ekaterina Amerik, for example, with Claire Voisin, has constructed a Calabi-Yau fourfold of Picard number one with a Zariski dense set of rational points. One of the open problems listed at the workshop was to find a K3 surface of Picard rank one that has a Zariski dense set of rational points. Amerik's result is an answer to a four-dimensional analogue of this question.

Chad Schoen, by contrast, worked with threefolds, but changed the characteristic of the underlying field from zero to $p > 0$, in order to describe Calabi-Yau threefolds with vanishing third Betti number. Such threefolds do not exist in characteristic zero, but in arithmetic considerations they can arise as specializations of Calabi-Yau varieties over number rings, which are of mixed characteristic.

One of the major themes in the arithmetic of K3 surfaces has to do with the distribution of rational points. Many of the fundamental questions about this distribution become more tractable if the K3 surface admits a self-map of some kind, and so many of the talks dealt with this special case. Arthur Baragar's talk, for example, described methods for computing the group of automorphisms of K3 surfaces. This computation is closely linked with the question of finding generators for the nef and effective cones, and involves techniques from hyperbolic geometry and fractals, as well as the usual algebraic geometry and arithmetic.

Serge Cantat also discussed automorphisms, by describing the dynamics of automorphisms on a K3 surface, and giving an overview of what is known about the distribution of periodic points and the closure of the set of periodic points. Joe Silverman's talk pertained to similar subject matter, focussing on the distribution of heights of points in orbits, specifically with reference to K3 surfaces embedded as hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Alessandra Sarti, by contrast, described automorphisms of prime (and therefore finite) order on K3 surfaces. Such surfaces can be described by moduli spaces, and Dr. Sarti's talk further described the fixed locus of such automorphisms, and contrasted the symplectic and non-symplectic cases.

Self-maps of K3 surfaces need not be defined everywhere, however, and Thomas Dedieu highlighted this fact in his talk, in which he described several examples of rational self-maps of K3 surfaces of degree greater than one. Connected to this are conjectures that a generic K3 surface does not admit such a map, and that the universal Severi variety is irreducible.

Another hot topic in the arithmetic of K3 surfaces is the Brauer group, and its role in the Brauer-Manin obstruction to the existence of rational points. Several of the talks made reference to the Brauer group. An overview of this area was given by Olivier Wittenberg, who described the Hasse principle and weak approximation, and how these do not necessarily apply to K3 surfaces because of the Brauer-Manin obstruction. A view of the computational side of the subject was given by Martin Bright, who gave both the theoretical background and a hands-on demonstration of how to compute the (algebraic) Brauer-Manin obstruction of a diagonal quartic surface in $\mathbb{P}^3$.

Dr. Bright's discussion of the algebraic part of the Brauer group was complemented by Evis Ieronymou's talk on the transcendental part, and the role it plays in the Brauer-Manin obstruction for diagonal quartic surfaces. In particular, Dr. Ieronymou described how to construct transcendental elements of the Brauer group, and the relation of these elements to the problem of weak approximation.

These talks were complemented by Yuri Zarhin's talk, in which he described his and Alexei Skorobogatov's finiteness results for Brauer groups. These results are derived from work relating to various conjectures of Tate and their analogues, relating to the algebraicity of Galois-invariant cohomology classes, homomorphisms of abelian varieties, and their Tate modules.

Another major theme in the arithmetic of K3 surfaces is the question of modularity. Modularity of elliptic curves played a famously important role in the proof of Fermat's Last Theorem, and is closely related to certain two-dimensional representations of the absolute Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. To find a suitable two-dimensional representation of G connected to a K3 surface, one must use the transcendental part of the lattice $\mathrm{H}^2(S, \mathbb{Z})$. In order for this to have dimension two, the Picard lattice must have the maximal dimension, 20. Thus, the discussion of the modularity of K3 surfaces begins with those of Picard rank 20, namely, the singular K3 surfaces.

Matthias Schütt explained in his talk that all singular K3 surfaces have been known to be modular for some time. His innovation, in joint work with Noam Elkies, is that every newform of weight three with rational coefficients can actually be associated to a K3 surfaces defined over $\mathbb{Q}$. Noriko Yui's talk went further, and described her joint work with Matthias Schütt and Ron Livné in which they classify complex K3 surfaces with a non-symplectic group acting trivially on algebraic cycles, show that they are all of CM type, and prove that they are all modular.

Michael Stoll's talk was on a slightly more computational bent, and showed how the arithmetic of K3 surfaces can be used to find points of extremely large height on curves of genus two. Such curves can be embedded in their Jacobians, which are two-dimensional and so their associated Kummer variety is a K3 surface. Dr. Stoll described how this Kummer surface can be used to find generators of the Jacobian of logarithmic height nearly 100.

Ronald van Luijk's talk presented an analogue of the Batyrev-Manin conjectures for K3 surfaces. The original conjectures predict the number of rational points of bounded height on rational surfaces, and do not generalize in a straightforward way to K3 surfaces. However, Dr. van Luijk's talk gave some precise conjectures about the number of points of bounded height on K3 surfaces and certain open subsets of them, and he presented considerable numerical data to support his conjectures.

4 Website

For preparation for the conference and for future reference, a website

`www.math.leidenuniv.nl/~rvl/K3Banff`

was set up where all speakers suggested literature related to the subject of their talks. All other participants were also invited to present their papers. The website will continue to exist.

5 Outcome of the Meeting

It is too early to say what the final outcome of the meeting will be, ultimately, but it is clear that the workshop was a success. Participants in the workshop had many good things to say:

It was really a nice workshop! Even the weather, quite chilly, helped us to stay inside and share more time together. I liked most talks. The nice set of open problems formulated in the problem session seems to play for a few years an important guiding role in this topic of research. It was also a rare opportunity for me, an algebraic geometer, to meet experts from friendly neiboring areas such as number theory and complex dynamics. – Jonghae Keum

It was a great conference. I learned a tremendous amount about the geometry of K3 surfaces that I hadn't known, being primarily a number theorist myself. It was an amazing group of people and I had a lot of helpful math conversations with people at meals and in the common room. And even the cold temperatures were okay (-8 Fahrenheit one morning). – Joe Silverman

I enjoyed the workshop very much. It was a good mix of mathematicians, young and senior, geometry and arithmetic, computational and theoretical. You should try to apply for another one in two years time, again at BIRS. By then the campus reconstruction will be completed. – Noriko Yui

Perhaps the best thing I can say about BIRS is that even the stunning natural beauty or the impressive resources of the musical department of the Banff Centre could not distract me from focussing on our subject and on taking the opportunity to exchange ideas with the other participants. This was my first visit to BIRS and I sincerely hope it will not the be last. – Andreas Bender

I really enjoyed to be at BIRS, it is a very good place to work, find new ideas and discuss with many experts on the field, and of course the location is also great. – Alessandra Sarti

Thank you very much for organizing such a successful conference! I certainly enjoyed the conference very much, and I am sure many others did, too. This was my first conference in Banff, and it is difficult for me to distinguish whether this particular conference was well organized, or the environmental BIRS provides is very good in general. I think both contribute. The best point of this conference, "Arithmetic of K3 surfaces" was that the theme was well focused, and most speakers gave a talk on the topics surrounding this theme, even though most speakers do research in a wider area of mathematics. It was amazing that so many experts in this area were assembled in this conference. It was also nice to see that young active mathematicians are pushing the frontier of the subject very far. The web site you set up helped very much, and will be useful in the future too. I hope one day that many of us gather once again to discuss on the progress in the subject inspired by this conference. – Masato Kuwata

I truly apreciated this conference in BIRS, and I want to thank you for this. This was the occasion for me to learn more on certain areas, with which I am not very familiar. I really benefited from meeting some of the great specialists in number theory, and the very friendly atmoshpere of this conference eased the contacts between all people present. I can say that this conference will have a positive impact on my forthcoming research. – Thomas Dedieu

During the workshop, the BIRS lounge was constantly abuzz with mathematical conversation, and the discussion was lively and productive. BIRS is a fantastic place to do and discuss mathematics, and our workshop took full advantage.

References

[1] E. Amerik and C. Voisin, Potential density of rational points on the variety of lines of a cubic fourfold, to appear in Duke Math. J.

[2] E. Amerik, A computation of invariants of a rational self-map, to appear in Ann. Fac. Sci. Toulouse.

[3] M. Artebani and A. Sarti, Non-symplectic automorphisms of order 3 on K3 surfaces, preprint.

[4] A. Baragar, Orbits of points on certain K3 surfaces, to appear.

[5] A. Baragar, The ample cone for a K3 surface, to appear.

[6] A. Baragar and D. McKinnon, K3 surfaces, rational curves, and rational points, to appear.

[7] A. Baragar, Orbits of curves on certain K3 surfaces, *Compositio Math*, **137** (2003), no. 2, 115–134.

[8] A. Baragar, Rational Points on K3 Surfaces in PxPxP, *Math. Ann.*, **305** (1996), 541–558.

[9] A. Beauville and C. Voisin, On the Chow ring of a K3 surface, *J. of Algebraic Geometry*, **13** (2004), 417–426.

[10] A. Beauville, Counting rational curves on K3 surfaces, preprint.

[11] G. Call and J. Silverman, Computing the canonical height on K3 surfaces. *Math. Comp.* **65** (1996), no. 213, 259–290.

[12] S. Cantat, Deux exemples concernant une conjecture de Serge Lang, *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000), no. 7, 581–586.

[13] S. Cantat, Dynamique du groupe d'automorphismes des surfaces K3, *Transform. Groups* **6** (2001), no. 3, 201–214.

[14] S. Cantat, Dynamique des automorphismes des surfaces K3, *Acta Math.* **187** (2001), no. 1, 1–57. (see also *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999), no. 10, 901–906)

[15] S. Cantat, Caratérisation des exemples de Lattès et de Kummer, to appear in *Compositio Mathematica*, **144** (2008), no. 5.

[16] S. Cynk and M. Schütt, Generalised Kummer constructions and Weil restrictions, preprint.

[17] T. Dedieu, Severi varieties and self-rational maps of K3 surfaces, preprint.

[18] A. Garbagnati and A. Sarti, Elliptic fibrations and symplectic automorphisms on K3 surfaces, preprint.

[19] A. Garbagnati and A. Sarti, Symplectic automorphisms of prime order on K3 surfaces, preprint.

[20] D. Harari and A. Skorobogatov, Nonabelian descent and the arithmetic of Enriques surfaces, *Int. Math. Res. Not.*, **52** (2005), 3203–3228.

[21] K. Hulek and R. Kloosterman and M. Schütt, Modularity of Calabi-Yau varieties. In *Global Aspects of Complex Geometry (Catanese et al, eds)*, Springer (2006).

[22] M. Kuwata and T. Shioda, Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface. In *Algebraic Geometry in East Asia–Hanoi 2005*, Advanced Studies in Pure Mathematics **50** (2008), 177–215.

[23] D. McKinnon, Counting Rational Points on K3 Surfaces, *J. Number Theory*, **84** (2000), no. 1, 49–62.

[24] M. Schütt, K3 surfaces with non-symplectic automorphism of 2-power order, preprint.

[25] M. Schütt, K3 surfaces of Picard rank 20 over $\mathbb{Q}$, preprint.

[26] M. Schütt and A. Schweizer, On the uniqueness of K3 surfaces with maximal singular fibre, preprint.

[27] M. Schütt, CM newforms with rational coefficients, to appear in *Ramanujan Journal*.

[28] M. Schütt, Arithmetic of K3 surfaces, To appear in *Jahresbericht der DMV*.

[29] M. Schütt, Arithmetic of a singular K3 surface, to appear in *Michigan Mathematical Journal*

[30] M. Schütt, Fields of definition of singular K3 surfaces, *Communications in Number Theory and Physics*, **1**, (2007), no. 2, 307–321.

[31] M. Schütt and T. Shioda, An interesting elliptic surface over an elliptic curve, *Proc. Jap. Acad.* **83** (2007), no. 3, 40–45.

[32] M. Schütt, Elliptic fibrations of some extremal K3 surfaces, *Rocky Mountain Journal of Mathematics* **37** (2007), no. 2, 609–652.

[33] M. Schütt, The maximal singular fibres of elliptic K3 surfaces, *Archiv der Mathematik* **79** (2006), no. 4, 309–319.

[34] M. Schütt and J. Top, Arithmetic of the [19,1,1,1,1,1] fibration, *Commentarii Mathematici Universitatis Sancti Pauli*, **55** (2006), no. 1, 9–16.

[35] T. Shioda, K3 surfaces and sphere packings, to appear.

[36] T. Shioda, A note on K3 surfaces and sphere packings, *Proc. Japan Acad.* **76A** (2000), 68–72.

[37] T. Shioda, The Mordell-Weil lattice of $y^2 = x^3 + t^5 - 1/t^5 - 11$, *Comment. Math. Univ. St. Pauli* **56** (2007), 45–70.

[38] T. Shioda, Correspondence of elliptic curves and Mordell-Weil lattices of certain elliptic K3 surfaces. In *Algebraic Cycles and Motives*, vol. 2 (2007), Cambridge Univ. Press, 319–339.

[39] T. Shioda, On the Mordell-Weil lattices, *Comment. Math. Univ. St. Pauli* **39** (1990), 211–240.

[40] J. Silverman, Rational points on K3 surfaces: a new canonical height, *Invent. Math.* **105** (1991), no. 2, 347–373.

[41] A. Skorobogatov and P. Swinnerton-Dyer, 2-descent on elliptic curves and rational points on certain Kummer surfaces. *Adv. Math.* **198** (2005), no. 2, 448–483.

[42] A. Skorobogatov and Yu. Zarhin, A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces, *J. Algebraic Geometry*, **17** (2008) 481–502.

[43] P. Swinnerton-Dyer, Arithmetic of diagonal quartic surfaces II, *Proc. London Math. Soc.* **80** (2000), no. 3, 513–544.

[44] J. Top and B. van Geemen, An isogeny of K3 surfaces, *Bull. London Math. Soc.*, **38** (2006), 209–223.

[45] R. van Luijk, K3 surfaces with Picard number one and infinitely many rational points, *Algebra and Number Theory*, **1** (2007), no. 1, 1–15.

[46] R. van Luijk, Quartic K3 surfaces without nontrivial automorphisms, *Mathematical Research Letters*, **13** (2006), no. 3, 423–439.

[47] O. Wittenberg, Transcendental Brauer-Manin obstruction on a pencil of elliptic curves. In *Arithmetic of higher-dimensional varieties, Palo Alto, CA, 2002 (B. Poonen and Yu. Tschinkel, eds.)*, Progress in Mathematics 226 (2004), Birkhäuser, 259–267.

[48] O. Wittenberg, *Intersections de deux quadriques et pinceaux de courbes de genre 1* (Chapter 1), LNM 1901, Springer-Verlag, 2007.

[49] Yu. Zarhin, The Brauer group of an Abelian variety over a finite field, *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), 211–243; English translation: *Math. USSR Izv.*, **20** (1983), 203–234.

[50] Yu. Zarhin, Hodge groups of K3 surfaces, *Journal für die reine und angewandte Mathematik*, **341** (1983), 193–220.

[51] Yu. Zarhin, A finiteness theorem for unpolarized Abelian varieties over number fields with prescribed places of bad reduction, *Inv. Math.*, **79** (1985), 309–321.