Random Fields and Stochastic Geometry

(09w5040)

Robert J. Adler (Technion), Jonathan E. Taylor (Stanford)

February 22–27, 2009

1 Overview of the Workshop

The main topics of this workshop lay at the point where Probability meets Geometry, specifically the study of the random geometry and topology generated by smooth random functions. While these problems have their roots in various applications of image and shape analysis in a wide variety of disciplines, their main mathematical content lies in probability theory. It is there that recent major advances led us to holding a workshop in this general area at this particular time.

Specifically, over the last few years, a minor revolution, which was born in Jonathan Taylor's McGill Ph.D. thesis, has been taking place regarding the way the sample path properties of smooth multi-parameter stochastic processes, or random fields, have been studied. While the setting is primarily in the world of Gaussian random fields, the basic results also extend out into the non-Gaussian world. The approach is based on treating parameter spaces, where possible, as Riemannian manifolds with metrics induced by the fields. These results involve an intruiging blend of probability and geometry, and although the revolution has, primarily, been taking place at the level of theoretical mathematics, it has already had an impact on applications of random field theory in applied settings as wide apart as astrophysics and medical imaging.

The interface between random field theory and stochastic geometry has been at the centre of this activity, which is why most of the participants in the workshop came from the areas of this interface. However, it turns out that this theory has also had a major impact on the problem of obtaining practical approximations for the tail of the distribution of the suprema of many random fields, an area of extremal theory that is rich in applications in applied probability and statistics. This area has seen independent developments of its own over the past few years, after a decade or two of relative stagnation. Consequently another group of participants in the workshop was made up of mathematicians from extreme value theory, the aim being to try to formulate a common framework for the extremal theory of Gaussian and related random fields that combines many of the recent approaches.

Yet another group of participants came from the area of theoretical physics. In that area Gaussian random field geometry has become a central component in topics as widely dispersed as the study of nodal domains in optical waves and quantum chaos, and in the study of the geometry of liquid crystals and superfluid helium.

As should be obvious by now, the primary aim of this workshop was to bring together researchers in the areas of random fields and stochastic geometry, to discuss and develop a new class of results at the interface of the two subjects. The secondary aim was to involve researchers in areas that have traditionally applied results from these areas, with the complementary purposes of aquainting them with the new theory, much of which was generated by specific issues arising in applications, and of identifying new areas of further theoretical development.

Both of these aims were successfully met, and the interactions between researchers from differing disciplines were excellent. Details follow.

2 Recent Developments

We are going to describe recent developments from a very myopic viewpoint - that of the two organsisers. This should not be taken as being too egocentric, for it *was* the motivation behind holding the workshop. Later it should become clear why this is the case.

Consider a smooth random field, f, a stochastic processe mapping a parameter space M to $\mathbb{R}^k$. In general, M could be taken to be piecewise smooth stratified manifold, so that examples include balls, cubes, spheres, and tori in $\mathbb{R}^N$, $N \geq 1$, as well as anything one can obtain by glueing these and other basic objects together.

The basic (random) geometrical objects to be studied are the *excursion sets, nodal domains,* or *super-level sets*, of the random field, defined as the random sets

$$A_D \;\equiv\; A_D(f;M) \;\stackrel{\Delta}{=}\; \{t \in M:\, f(t) \in D\} \;\equiv\; M \cap f^{-1}(D), \tag{1}$$

for D a nice subset of $\mathbb{R}^k$.

One wants to understand the structure of excursion sets, studying their geometry through numerical quantifiers. In essence, this geometry can be approached from two different, but complementary, directions. The first is via the tools of integral or differential geometry. The quantifiers are then concepts such as volume, surface area, measure of cross-sectional size, etc. As a collection, these go by many names, including Minkowski functionals and *Lipschitz-Killing curvatures*, which is what will be used below. As a consequence of the Gauss-Bonnet theorem, they also include the Euler, or Euler-Poincaré, characteristic.

The second approach is via the tools of algebraic topology. In this case the tools are homotopy-invariant concepts such as the number of connected components, Betti numbers, homologies, etc. The Euler characteristic appears among this collection of geometric quantifiers as well, as the archetypical topological invariant that can be computed via a curvature integral.

The random fields which are of interest can also be classified into two main classes, Gaussian (and Gaussian-related) fields, and non-Gaussian fields.

A little over 30 years ago, the first significant result in this area was developed in the PhD thesis of Robert Adler, who obtained an explicit formula for the expected value of a geometric characteristic closely related to the Euler characteristic of the excursion sets of stationary, real valued, Gaussian random fields on N-dimensional rectangles. Over the following two decades there were many advances on these basic results, the most important of which were associated with Keith Worsley and colleagues in the mid 1990's. See, for example, [8, 9, 10]. These results extended the basic theory from Gaussian to some non-Gaussian fields and looked at parameter sets M somewhat more general than cubes.

While this activity significantly extended the earlier results, the underlying methodologies were essentially unchanged. All of this, however, changed with the 2001 PhD thesis of Jonathan Taylor [4] which lead to a completely new way of handling Gaussian and "Gaussian related" random fields, over general stratified manifolds M and taking values in $\mathbb{R}^k$ for $k \geq 1$. The basic results of the thesis have since been generalised to what is now known as a *Gaussian kinematic formula*, or *GKF*. Full details can be found in [1, 5] but a quick description is as follows:

Note firstly that if M is a stratified manifold then it can be (non-uniquely) decomposed into the disjoint union of j-dimensional manifolds $\partial_j M$, $0 \leq j \leq N$, so that we can write $M \;=\; \bigcup_{j=0}^N \partial_j M$. Assuming that we have a C^2 Riemannian metric g defined on M, one can define geometric quantities, known, with various orderings and normalisations, as Quermassintegrales, Minkowski or Steiner functionals, integral curvatures, intrinsic volumes, and *Lipschitz-Killing curvatures*. There are a number of ways to define the Lipschitz-Killing curvatures of a set but, for Riemannian manifolds M with metric g, and without boundary, they can be defined by

$$\mathcal{L}_j(M) \;=\; \frac{1}{(2\pi)^{(N-j)/2}((N-j)/2)!} \int_M \mathrm{Tr}^M(-R)^{(N-j)/2}\,\mathrm{Vol}_g, \tag{2}$$

when $N-j \geq 0$ is even, and 0 when $N-j \geq 0$ is odd. Here Vol_g is the volume form of (M,g), R is the curvature tensor and Tr^M the trace operator on the algebra of double forms on M. When M is a stratified manifold then (2) has many more terms on the right hand side, in which powers of second fundamental forms of the $\partial_j M$ appear along with complementary powers of R, and Tr^M is replaced by $\mathrm{Tr}^{\partial_j M}$. However, the underlying principle is the same, in that they are defined as integrals of local measures of curvature.

The Lipschitz-Killing curvatures have considerable geometric significance. For example, if M is imbedded in $\mathbb{R}^{N'}$ with $N' \geq N$ and the metric g is the usual Euclidean one, and $\mathcal{H}_{N'}$ is Hausdorff measure in $\mathbb{R}^{N'}$, then Weyl's tube formula states that, for small enough ρ,

$$\mathcal{H}_{N'}\left(\mathrm{Tube}(M,\rho)\right) \;=\; \sum_{i=N'-N}^{N'} \rho^{N'-i}\omega_{N'-i}\mathcal{L}_i(M), \tag{3}$$

where $\mathcal{L}_j(M) \stackrel{\Delta}{=} 0$ for all $j > N$,

$$\mathrm{Tube}(M,\rho) = \{t \in \mathbb{R}^{N'} :\ \|t-s\| \leq \rho \text{ for some } s \in M\},$$

and ω_j is the volume of the j-dimensional unit ball. Treating (3) as a power series in ρ actually defines the Lipschitz-Killing curvatures in this case.

If M is a compact set of full dimension in $\mathbb{R}^N$, then $\mathcal{L}_N(M)$ is its N-dimensional volume, $\mathcal{L}_{N-1}(M)$ is its $(N-1)$-dimensional surface area, etc. The final Lipschitz-Killing curvature, $\mathcal{L}_0(M)$ is the *Euler*, or *Euler-Poincaré, characteristic* of M, and it is the only one of the Lipschitz-Killing curvatures which is a homotopy invariant.

The Lipschitz-Killing curvatures can be used as quantifiers of the global geometry of excursion sets, and an amazing recent result gives their expectations, for the statement of which we need one more concept, that of Gaussian Minkowski functionals, which we define by adopting a definition analogous to defining Lipschitz-Killing curvatures via the Weyl formula (3). Taking D to be a nice (e.g. locally convex) set in $\mathbb{R}^k$, and γ to be standard Gauss measure on $\mathbb{R}^k$, then one has that, for small enough ρ, there exist numbers $\mathcal{M}_j(D)$, $j \geq 0$, known as the Gaussian Minkowski functionals of D, such that

$$\gamma(\mathrm{Tube}(D,\rho)) \;=\; \sum_{j=0}^{\infty} \frac{\rho^j}{j!}\mathcal{M}_j(D). \tag{4}$$

In many cases, the $\mathcal{M}_j$ are quite easy to compute. For example, if $M = [u,\infty) \subset \mathbb{R}$, then, since $\mathrm{Tube}([u,\infty),\rho) = [u-\rho,\infty)$, comparing a Taylor series expansion of $\gamma(\mathrm{Tube}(M,\rho))$ with (4) easily gives

$$\mathcal{M}_j^{\gamma}([u,\infty)) \;\;=\;\; (\sqrt{2\pi})^{-1} H_{j-1}(u) \mathrm{e}^{-u^2/2}, \tag{5}$$

where, for $n \geq 0$, H_n is the n-th Hermite polynomial and we define $H_{-1}(x) = (2\pi)^{1/2} e^{u^2/2}\Psi(x)$, where Ψ is the Gaussian tail probability $\Psi(u) \stackrel{\Delta}{=} (2\pi)^{-1/2}\int_x^{\infty} \exp(-u^2/2)\,du$.

With all the pieces in play, we can now describe the main new result in the study of smooth random fields:

THE GAUSSIAN KINEMATIC FORMULA (GKF): Let M be a nice N-dimensional stratified manifold, and D a nice stratified submanifold of $\mathbb{R}^k$. Let $f = (f^1,\dots,f^k)$ be a vector valued random process, the components of which are independent, identically distributed, real valued, smooth (C^2) centered, unit variance, Gaussian processes. Then

$$\mathbb{E}\left\{\mathcal{L}_i\left(M \cap f^{-1}(D)\right)\right\} \;=\; \sum_{j=0}^{N-i} \begin{bmatrix} i+j \\ j \end{bmatrix} (2\pi)^{-j/2}\mathcal{L}_{i+j}(M)\mathcal{M}_j^{\gamma}(D). \tag{6}$$

The $\mathcal{L}_j$, $j = 0,\dots,N$ are the Lipschitz-Killing measures on M considered as a Riemannian manifold with the Riemannian metric induced by the f^i. This metric is defined as

$$g_t(X,Y) \;\stackrel{\Delta}{=}\; \mathbb{E}\{(Xf_t^i)\,(Yf_t^i)\}, \tag{7}$$

for any i and for $X, Y \in T_tM$, the tangent space to M at $t \in M$. The combinatorial coefficients in (6) *are the standard 'flag coefficients' of integral geometry.*

A book could be written about the proof and consequences of (6), and, in fact, has been; cf. [1]. Here we note just four of its implications:

(i) Parameter spaces M and hitting sets D are very general. Furthermore, the usual assumptions of stationarity on f are not required.

(ii) The result allows generalisations to 'Gaussian related' random fields of the form $g = F(f)$, where f is as in the statement of the GKF and the transformation $F : \mathbb{R}^k \to \mathbb{R}^d$ is smooth. For such fields, excursion sets can be written in the form $A_D(g; M) = A_{F^{-1}(D)}(f, M)$ for $D \subset \mathbb{R}^d$, turning a non-Gaussian problem into a Gaussian one. A typical example of F would be $F(x) = \|x\|^2$, $x \in \mathbb{R}^k$, for which the corresponding g is a real valued χ^2 random field with k degrees of freedom.

(iii) It has long been known that the mean Euler characteristic of Gaussian excursion sets is a good approximation to exceedence probabilities at high levels, with the modern, geometric approach to this starting in 1993 with an important paper by Sun [2]. Recently it was shown in [7] that, for real valued Gaussian f,
$$\liminf_{u\to\infty} -u^{-2} \log \left| \mathbb{P}\left\{ \sup_{t\in M} f(t) \ge u \right\} - \mathbb{E}\left\{ \mathcal{L}_0\left(A_u(f;M)\right)\right\} \right| \;\ge\; (1+\alpha)/2, \qquad (8)$$
for an identifiable α and with equality in certain circumstances. The existence of a result like (6), which allows one to actually compute the expectation here, is what makes this approximation useful in practice.

(iv) The form of (6) is reminiscent of the kinematic fundamental formula of integral geometry. The similiarity is not coincidental, and it from this that the name 'GKF' for the result (6) comes. The GKF actually has purely geometric implications, which became one of the main sub-themes of the workshop.

3 Presentation Highlights

The workshop was built around three types of sessions. Each morning three formal, one hour talks were given, on topics that were pre-chosen to cover as wide a range of possible problems and viewpoints on the random geometry/random fields theme, from both theoretical and applied aspects. The early afternoons were devoted to 2–3 working groups, that either went into further depth for the topics discussed in the morning, or, more generally, concentrated on tangential issues that arose from these. Typically, they were of the "can you explain more about...", "how do we extend this to...", "how do we apply this to the problem of...", "can a corresponding result be developed to cover the issue of..." nature.

In three late afternoons a number of shorter talks were presented, related to the various themes of the main sessions. While it was, of course, harder for the speakers in these sessions to give full credit to their work in the short time available, these sessions nevertheless played an important role in terms of letting all participants know what other participants knew how to do, and were important in developing group spirit and establishing contacts for future collaborations.

3.1 Formal talks

The opening talk of the workshop was given by Jonathan Taylor, who presented, in some detail, and with a hint as to applications, the Gaussian Kinematic Formula described above.

Taylor's talk was followed by his ex-student, Nick Chamandy, who spoke on *Saddlepoint approximations for asymptotically Gaussian random fields, with applications*. In his lecture, Chamandy explained why the Gaussian assumption underlying the GKF is untenable in many applications, and presented a correction to the expected Euler characteristic formula (equation (6) with $i = 0$) for these cases involving a saddlepoint approximation.

For asymptotically Gaussian fields of a particular type, and when the threshold is allowed to grow with the sample size, this 'tilted' expression has a smaller relative asymptotic error than the Gaussian one. Practical implications were discussed, and inference under the Gaussian and tilted regimes was contrasted using real data and simulation. Finally, multivariate extensions of the tilted theory were briefly discussed, as well as applications from brain mapping, astronomy, cognitive psychology and web search eye-tracking.

The third speaker of the first morning was Jean-Marc Azaïs, who spoke on *Some applications of an implicit formula for the maximum of a Gaussian random field*. This work, much of it joint with Mario Wschebor, attacks the exceedence probability in (8) from a completely different angle than does the GKF. By looking, among the local maxima of a random field, to those that are actually a global maxima, they obtain an implicit formula for the density of the maximum of a smooth random field on a well behaved parameter space. This formula can be used in at least two ways. One is to study the regularity of the density. When the parameter is one-dimensional an induction proves that under certain conditions the distribution can very regular. When the parameter is multidimensional results are more limited but in both cases they obtain results that go beyond the classical ones of Tsirelson. The other direction is to give bounds to the density to obtain non-asymptotic bounds for the distribution of the maximum as well as an expansion to the second order.

On Tuesday morning, the workshop moved from theory (albeit application driven theory) to statistical methodology and applications.

Armin Schwartzman presented a talk entitled *Inference for eigenvalues and eigenvectors of Gaussian symmetric matrices*. He discussed maximum likelihood estimators (MLEs) and log-likelihood ratio (LLR) tests for the eigenvalues and eigenvectors of Gaussian random symmetric matrices of arbitrary dimension, where the observations are independent repeated samples from one or two populations. These inference problems are relevant in the analysis of Diffusion Tensor Imaging (DTI) data, where the observations are 3-by-3 symmetric positive definite matrices. The parameter sets involved in the inference problems for eigenvalues and eigenvectors are subsets of Euclidean space that are either affine subspaces, embedded submanifolds that are invariant under orthogonal transformations or polyhedral convex cones. He showed that for a class of sets that includes those just described, the MLEs of the mean parameter do not depend on the covariance parameters if and only if the covariance structure is orthogonally invariant. Closed-form expressions for the MLEs and the associated LLRs were derived for this covariance structure.

The second talk of the morning was given by Domenico Marinucci, who spoke on *Statistics and probability challenges in the analysis of cosmic microwave background radiation*, joint work with P. Baldi, D. Geller, G. Kerkyacharian, Xiaohong Lan, G. Peccati and D. Picard. The talk centered around the Cosmic Microwave Background radiation (CMB) data, which can be viewed as a relic radiation providing a snapshot of the Universe approximately 13.7 billion years ago. Although CMB was detected in the mid sixties, it was only very recently that the first full-sky maps of observations were produced, leading to the 2006 Nobel prize for Physics to Smoot and Mather. In the last few years, the analysis of CMB data from satellite missions has attracted an enormous amount of attention from theoretical and experimental physicists. Indeed CMB is considered a goldmine of information for a number of fundamental issues in theoretical physics and cosmology, such as the large scale structure of the universe, models for big bang dynamics and the existence and nature of dark matter and dark energy.

From the mathematical point of view, CMB can be viewed as the realization of a mean-square continuous, isotropic random field on the sphere. The aim of this talk was to survey several probability and statistical issues which arise in connection with CMB. Most of these problems are still open. The focus was on the characterizations of Fourier coefficients under isotropy and their relationships with group representation theory; on high resolution asymptotics for Gaussian and Gaussian subordinated random fields; on the construction of wavelets and tight frames systems for spherical random fields; and on probability and statistical issues relating to tensor, rather than scalar-valued, random fields.

This was the first talk of the workshop that brought the participants close to real data and the statistical problems involved in applying the theoretical, geometrical, results of random field theory.

The final talk of Tuesday morning was to have been delivered by Keith Worsley, on *The statistical analysis of fMRI data*. At that stage Keith was quite ill, but since it was felt that his talk, which was meant to describe some of the flavor of the applications of random fields in neuroimaging, was too important to be left out of the program, Jonathan Taylor gave a similar talk based on some of Keith's slides from an earlier talk.

In the lecture, a method was proposed for the statistical analysis of fMRI data that seeks a compromise between efficiency, generality, validity, simplicity and execution speed. The main differences between this analysis and previous ones are: a simple bias reduction and regularization for voxel-wise autoregressive model parameters; the combination of effects and their estimated standard deviations across different scans/sessions/subjects via a hierarchical random effects analysis using the EM algorithm; overcoming the problem of a small number of scans/session/subjects using a regularized variance ratio to increase the degrees of freedom. In addition, Taylor explained how one used the GKF and, in particular, the exceedence result (8) to carry out significance tests for signal detection in fMRI images.

On Wednesday morning the workshop started with a talk in the spirit of classical stochastic (integral) geometry, when Rolf Schneider gave a talk on *Poisson hyperplanes*. Beginning with the early work of Miles and Matheron, Poisson processes in the space of hyperplanes in d-dimensional Euclidean space have been a prominent subject of stochastic geometry. After a brief introduction, Schneider described some more recent results concerning the tessellations generated by Poisson hyperplane processes and, in particular in the homogeneous case, their 'average cells' and k-dimensional faces. Various types of average faces were be considered, distinguished by different weightings, and formally defined with the help of suitable Palm distributions.

In a first class of problems, he described establish sharp bounds for the expected vertex numbers of weighted typical cells. The extremal cases characterize particularly simple hyperplane processes, like isotropic ones or those with only d directions of the hyperplanes. A second class of problems dealt with the phenomenon that average cells of large size (with different interpretations) may approximate certain definite shapes, with high probability.

Rick Vitale, in a talk entitled *Gaussian proceses and convex bodies*, then turned to the long and fruitful interaction between Gaussian processes and the geometry of Banach spaces. He surveyed a number these connections, including the GB/GC (Gaussian bounded/continuous) classification and Itô-Nisio singularities. He also discussed Gaussian representation of intrinsic volumes, Gaussian samples and random projection of regular simplices, the Wills functional in a Gaussian context, deviation inequalities, and metrization of convex bodies.

Gennady Samorodnitsky then took a new direction, completely leaving the Gaussian or Gaussian related random fields that had been at the core of almost all talks so far, and spoke on the *Geometric characteristics of the excursion sets over high levels of non-Gaussian infinitely divisible random fields*, joint work with Adler and Taylor. He considered smooth, real valued, infinitely divisible random fields over the cube $[-1,1]^N$, with regularly varying Lévy measure. For such random fields he discussed the geometric characteristics of the excursion set $A_{[u,\infty)}(f,[-1,1]^N)$, for high levels u.

For a large class of random fields he computed the asymptotic (as $u \to \infty$), conditional on A_u being non-empty, joint distribution of the of the numbers of critical points over the level u of all types. This allows one, for example, to obtain the asymptotic conditional distribution of the Euler characteristic of the excursion set.

In a significant departure from the Gaussian situation, the excursion set over a high level for the smooth random fields Samorodnitsky considered can have very complicated geometry. In the Gaussian case a high level excursion set, unless it is empty, is nearly certain to be "a ball-like" and so have Euler characteristic equal to one. In contrast, the Euler characteristics of the excursion sets in the infinitely divisible case can have highly non-degenerate conditional distributions.

Thursday morning the workshop concentrated on random fields and stochastic geometry as they appear in a number of problems in theoretical physics.

The first topic was treated by Steve Zelditch, who spoke on *Large deviations for zeros of random polynomials and generalizations to Riemann surfaces*. Most people are familiar with results about zeros of random polynomials concentrating on the unit circle in the complex plane, but most do not realise that concentration on the circle, rather than some other set, is actually an artifact of the way the random coefficients of the polynomials are chosen.

Zelditch described joint work with Ofer Zeitouni in which they defined quite general inner products on the space P_N of polynomials of degree N using a weight $e^{-N\phi}$ and a positive measure ν on the Riemann sphere. The inner product induces a Gaussian measure on P_N and on the possible configurations of the N zeros. The configurations are then summarised by the empirical measure of zeros. As $N \to \infty$, these measures tend almost surely to the weighted equilibrium measure for (ϕ, ν). Moreover, they satisfy a large deviations principle with a good rate functional. The result generalizes to any compact Riemann surface.

Following Zelditch, Mark Dennis discussed *Complex nodes in Gaussian random waves: Optics, quantum* *waves and cosmology*. Noting that Gaussian random fields are a natural model for randomly interfering wave fields in physics, he pointed out that they are an important tool in, for example, statistical optics, quantum chaos, and cosmology. Geometric and topological aspects of these random fields – such as the configuration of their nodal sets – provides a handle to understand the morphology of these complicated physical fields. He described analytic and numerical calculations of the distribution and correlations of nodes in Gaussian random functions modelling wave fields from three physical examples. The first was quantum chaos. Here the complex quantum waves in a 2-dimensional chaotic enclosure are modelled by a Gaussian random solution of the Helmholtz equation, amd the nodal points in this case are vortices of the probability current flow.

As in Marinucci's talk, Dennis also noted that in cosmology the coherent state representation of the cosmic microwave temperature fluctuations also can be modelled by Gaussian random fields. In his talk, however, Dennis related this to random polynomials with roots on the Riemann sphere corresponding to multipole vectors.

Finally, optical speckle patterns in statistical optins occur as the mottled pattern in laser light reflected from a rough surface, and are modelled by complex Gaussian random waves propagating in three-dimensional space. The nodes are a dense tangle of lines in three dimensions, which has fractal and scaling properties revealed by numerical calculations.

The last "physics" talk of the morning was delivered by Igor Wigman on *Nodal lines for random eigenfunctions of the Laplacian on the torus and the sphere.* He described results related to the length of nodal lines for eigenfunctions of the Laplacian corresponding to large eigenvalues. In case of the torus or the sphere, the eigenspaces are highly degenerate, so that one may endow the eigenspaces with a Gaussian probability measure. He then described the distribution of the length of nodal lines of random eigenfunction in the corresponding ensemble.

Firstly, using a standard technique (essentially as simple special case of the GKF) he computed an exact expression for the expected value of the length. The main result, however, described the variance rather than the mean, a topic for which more specialised techniques are needed.

On Friday, the workshop returned to applications of various kinds. Peter Buergisser spoke on the *Stochastic analysis of numerical algorithms*, the first and only talk to link the workshop to the fascinating – and at first seemingly distant – area of computer science.

Iterative numerical algorithms are often easy to state, but difficult to analyze on random input data. Important examples include the simplex method, interior-point methods of linear and convex optimization, as well as homotopy methods for solving systems of polynomial equations. The probabilistic analysis of such algorithms, which Buergisser described, leads to fascinating problems of stochastic geometry, that can be systematically studied with tools from integral geometry. A key concept here is the notion of condition number, which has a clear geometric meaning. Besides being important in algorithmics, these studies have also triggered new investigations on the zero sets of systems of random polynomials and have led to developments of independent mathematical interest, such as the curvatures and Euler characteristic of random (semi) algebraic sets.

Returning to the anatomical applications motivating the talk from Keith Worsley's slides, Sarang Joshi gave an energetic talk on the *Statistics of shape: Simple statistics on interesting spaces.*

A primary goal of computational anatomy is the statistical analysis of anatomical variability. A natural question that arises is how does one define the image of an "average anatomy". Such an "average" must represent the intrinsic geometric anatomical variability present. Large Deformation Diffeomorphic transformations have been shown to accommodate the geometric variability but performing statistics of diffeomorphic transformations remains a challenge.

In his lecture Joshi further extended this notion of averaging for studying change of anatomy on average from a cross sectional study of growth. In particular, noting that regression analysis is a powerful tool for the study of changes in a dependent variable as a function of an independent regressor variable, he applied it to the study of anatomical growth and shape change. When the underlying process can be modeled by parameters in a Euclidean space, classical regression techniques are applicable and have been studied extensively. However, recent work suggests that attempts to describe anatomical shapes using flat Euclidean spaces undermines our ability to represent natural biological variability. Joshi therefore developed and described a method for regression analysis of general, manifold-valued data. Furthermore he extended the notion of robust estimation to manifold valued data. Using this approach, he also presented a robust brain atlas estimation technique based on the geometric median in the space of deformable images.

In the final talk of the workshop, Evgeny Spodarev described a method for the *Efficient simulation of stable random fields*, giving an overview of the simulation methods for Gaussian, alpha- and max-stable random fields. New efficient algorithms for the simulation of alpha-stable random fields were discussed together with their error bounds and performance issues.

3.2 Informal meetings

As described above, two hours each afternoon were set aside for informal meetings. In general, there were two large groups centered about specific topics, and smaller break-off groups that met either in the small lecture rooms or, as often as not, around coffee.

There were too many topics discussed in these informal meetings to decribe here, but they, as is usually the case in Banff meetings, were one of the great successes of the workshop, particularly since the workshop in question brought together people from such different backgrounds. However, among the sessions there were discussions on:

1. The GKF and its applications, including possible extensions to complex valued random fields over complex manifolds.

2. The structure of tensor valued random fields and their application to DTI brain imaging and cosmology.

3. The statistical estimation of the $\mathcal{L}_j(M)$ in the GKF, so that the result can be applied without the user having to know anything about differential geometry.

4. Non-Gaussian random fields including non-parametric statistics and applications in application.

5. Random Riemannian metrics.

3.3 Short talks

As described above, there were also a number of short talks in the later afternoon. It is worth noting, as an indication of the “group spirit’ developed in the workshop, that these talks were as well attended as the longer talks of the mornings! The speakers and topics were:
Hug, *Steiner formula and contact distributions.*
Khoshnevisan, *An asymptotic theory for randomly forced discrete heat equations*
Kuriki, *Multiplicity adjustments in detecting reproductive barriers caused by loci interactions.*
Nardi, *Maxima of empirical (asymptotically Gaussian) random fields*
Renaud, *Correlated random field: the application to detection of features in electro-encephalogram (EEG) data*
Sun, *Joint behavior of average and supremum of a Gaussian random field - challenges and applications to joint control charts.*
Takemura, *Properties of Ehrhart quasi-polynomials for hyperplane arrangements with integral coefficients.*
Teguia, *Random fields and stochastic microlensing*.
Vadlamani, *Gaussian Minkowski functionals in infinite dimensional Gaussian spaces.*
Weil, *Boolean models and translative integral formulas.*
Wschebor, *Random systems of polynomial equations and Rice formula.*
Xiao, *Some fractal properties of Gaussian random fields.*
Younes, *Applications of parallel transport on Riemannian manifolds of shapes for the analysis of anatomical time series.*

4 Keith Worsley

If this workshop had many high points, it also had one low, very low, point as well. On Friday morning we received the terrible news that Keith Worsley, who had been unable to come to Banff, had passed away. This is not the place to write an obituary, however we cannot but note that not only was Keith a close personal friend of both organisers, and many other participants, but he was a one of the major contributors to the development of the geometric aspects of random field theory, and a giant in developing its application to brain imaging. Obituaries can be found at [6] and [3].

5 Scientific Progress Made and Outcome of the Meeting

The main aim of this workshop was more of the consolidation and communication of existing knowledge in a number of closely related areas rather than making immediate breakthroughs in either mathematics or its applications.

In view of this, the workshop was a resounding success. It brought together researchers from classic probability and statistics, from integral and Banach space geometry, from medical imaging and cosmology, and from theoretical physics. The common thread was that all were interested in the random geometry generated by random fields. Sometimes this was the geometry of the GKF of (6), although not all particpants realised this before they came to Banff. Sometimes it was in the zeros of random polynomials, where the problems were similar but the language was very different. (For example, while random process people talk about spectral moments as measures of smoothness for random fields, the random polynomial people talk about complexity parameters. Despite the similarity of the mathematics, it took, in retrospect, a surprisingly large amount of time until the various groups understood one another.)

By the time Friday lunchtime arrived, a community of researchers with common language and aims had been formed where, before, there were only a number of smaller, isolated groups. This is what the workshop was meant to achieve, and it did it well.

One measure of the success of the workshop can actually be measured by that of another. In August 2009, the two organisers of this workshop joined with Steve Zelditch (a Banff participant with whom they had previously had only minimal contact and very little language-overlap) and with Shmuel Weinberger, a leading Chicago topologist, to run another workshop, this time under the asupices of the *American Institute of Mathematics* and on the topic *Topological Complexity of Random Sets*. This workshop tackled the thorny problem briefly alluded to in Section 2 above, of looking at the algebraic topology (rather than integral geometry or differential topology) of random sets generated by stochastic processes. The lists of participants at the AIM and Banff meetings had a considerable overlap, an outgrowth of the ‘community spirit’ developed at Banff. AIM, as opposed to Banff, *was* a workshop at which significant mathematical advances were made, but that is another story for another report. Let it suffice to say that from the cohesion developed in Banff and the developments made 8 months later at AIM, it is clear that the topics of stochastic geometry, random fields, and a newly developing field of stochastic topology are heading for a period of significant growth and development in the next few years.

6 List of Participants

Adler, Robert (Technion - Israel Institute of Technology)
Anderes, Ethan (University of Carlifornia, Berkeley)
Azais, Jean-Marc (Universit de Toulouse)
Bartz, Kevin (Harvard University)
Burgisser, Peter (Univerity of Paderborn)
Chamandy, Nicholas (Google)
Dennis, Mark (University of Bristol)
Farzan, Rohani (McGill University)
Feldman, Raya (University of California at Santa Barbara)
Hug, Daniel (Universitat Duisburg-Essen)
Jakobson, Dmitry (McGill University)
Joshi, Sarang (University of Utah)
Khoshnevisan, Davar (University of Utah)

Kou, Samuel (Harvard University)
Kuriki, Satoshi (Institute of Statistical Mathematics)
Marinucci, Domenico (University of Rome)
Mazza, Christian (Universite de Fribourg)
Nardi, Yuval (Carnegie Mellon University)
Ninomiya, Yoshiyuki (Kyushu University)
Renaud, Olivier (Universit de Genve)
Rootzen, Holger (Chalmers Institute of Technology)
Samorodnitsky, Gennady (Cornell University)
Schneider, Rolf (University of Freiburg)
Schwartzman, Armin (Harvard School of Public Health)
Siegmund, David (Stanford University)
Spodarev, Evgeny (Ulm University)
Sun, Jiayang (Case Western Reserve University)
Takemura, Akimichi (Universit of Tokyo)
Taylor, Jonathan (Stanford University)
Tecuapetla Gomez, Inder Rafael (Cornell University)
Teguia, Alberto M (Duke University)
Vadlamani, Sreekar (Stanford University)
Vitale, Rick (University of Connecticut)
Weil, Wolfgang (Universitaet Karlsruhe)
Wigman, Igor (CRM, Universite de Montreal)
Wilson, Richard (University of Queensland)
Wschebor, Mario (Universidad de la Republica Uruguay)
Xiao, Yimin (Michigan State University)
Younes, Laurent (John Hopkins University)
Zelditch, Steve (Johns Hopkins University)

References

[1] R.J. Adler and J.E. Taylor. *Random Fields and Geometry*. Springer, 2007.

[2] J. Sun. Tail probabilities of the maxima of Gaussian random fields. *Ann. Probab.*, 21(1):34–71, 1993.

[3] J. Taylor, A. Evans, and K. Friston. A tribute to: Keith worsley – 1951-2009. *NeuroImage*, 46(4):891 – 894, 2009.

[4] J.E. Taylor. *Euler Characteristics for Gaussian Fields on Manifolds*. PhD thesis, McGill University, 2001.

[5] J.E. Taylor and R.J. Adler. Gaussian processes, kinematic formulae and Poincaré's limit. *Ann. Probab.*, 37(4):1459–1482, 2009.

[6] J.E. Taylor, K. Friston, and A. Evans. Obituary, Keith Worsley. *IMS Bulletin*, 38(5):10, 2009.

[7] J.E. Taylor, A. Takemura, and R.J. Adler. Validity of the expected Euler characteristic heuristic. *Ann. Probab.*, 33(4):1362–1396, 2005.

[8] K.J. Worsley. Local maxima and the expected Euler characteristic of excursion sets of χ^2, f and t fields. *Adv. in Appl. Probab.*, 26:13–42, 1994.

[9] K.J. Worsley. Boundary corrections for the expected Euler characteristic of excursion sets of random fields, with an application to astrophysics. *Adv. Appl. Probab.*, 27:943–959, 1995.

[10] K.J. Worsley. Estimating the number of peaks in a random field using the Hadwiger characteristic of excursion sets, with applications to medical images. *Ann. Statist.*, 23:640–669, 1995.