CLASSIFICATION OF AMENABLE C*-ALGEBRAS

Final Report

Introduction

The conference brought together leading researchers and young mathematiciansworking on the classification theory of amenable C*-algebras. The talks surveyed some of the very recentbreakthroughs and offered a chart for future expected developments. This led to a consolidation of our understanding of the open problems and some of the promising ideas in the classification theory.From the point of view of the organizers, what was particularly satisfactory for this conference was the timeliness with which it was held, allowing us to draw on the insight gained through many recently announced deep results, and to stimulate the future work in the area by bringing together influential (established as well as beginning) researchers. Since it is well established that operator algebra theory thrives in frequent and diverse interaction with other mathematical subjects the program contained not only talks relating to the core of classification theory, but also presenting interesting applications of results or methods from other areas to ours, or from ours to other areas. All of these efforts are naturally interwoven with ties in many directions represented mainly by scientific collaborations among their proponents, but to organize this report we will attempt a taxonomy as follows:

1. Dimension theory, regularity properties and classification of amenable C*-algebras
2. Computations of the Cuntz semigroup
3. Invariants of non-simple C*-algebrasand applications to graph C*-algebras
4. Classification of dynamical systems and group actions on amenable C *-algebras
5. General theory

For each of these areas we shall attempt below a brief overview of what the conference talks and the general discussion seemed to indicate about the status quo and the direction of future work.

1 Dimension theory, regularity properties and classification of amenable C*-algebras

1.1 Talks

1.1.1 George Elliott

Inductive limits of matrix algebras over the circle

1.1.2 Guihua Gong

ASH-inductive limits: Approximation by Elliott-Thomsen building blocks

1.1.3 Huaxin Lin

Unitaries in simple C*-algebras of tracial rank one and homomorphisms into simple $\mathcal{Z}$-stable C*-algebras

1.1.4 Zhuang Niu

A remark on AH algebras with diagonal maps

1.1.5 Leonel Robert

Classification of inductive limits of 1-dimensional NCCW-complexes

1.1.6 Karen Strung

A technique to show certain C*-algebras are TAI after tensoring with a UHF algebra.

1.1.7 Wilhelm Winter

Dimension, $\mathcal{Z}$-stability, and classification, ofnuclear C*-algebras

1.2 Discussion

In the past five years the state of knowledge around Elliott's conjecture for simple C*-algebras has advanced rapidly, particularly in the case that the projections of the algebra separate its tracial functionals. For instance, we now know by ground-breaking work of Toms and Winter that the C*-algebras associated to minimal uniquely ergodic dynamics on finite-dimensional spaces are determined up to isomorphism by their graded ordered K-theory, as outlined in the talk given by Winter. At the centre of these developments are the Jiang-Su algebra $\mathcal{Z}$ and the attendant property of $\mathcal{Z}$-stability (a C*-algebra A is $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$). This sort of tensorial absorption property is ubiquitous in operator algebra classification: Connes's proof that an amenable II_1 factor $\mathcal{M}$ with separable predual is the hyperfinite factor $\mathcal{R}$ proceeded by showing first that $\mathcal{M}\bar{\otimes}\mathcal{R} \cong \mathcal{M}$; the Kirchberg-Phillips classification of simple purely infinite C*-algebras relies heavily on the fact that any such algebra A satisfies $A \cong A \otimes \mathcal{O}_\infty$ for the Cuntz algebra $\mathcal{O}_\infty$. But not all simple separable nuclear C*-algebras are $\mathcal{Z}$-stable, in contrast with the tensorial absorption properties of factors and purely infinite algebras. Why so? Very roughly, the latter two classes of algebras are non-commutative generalizations of low-dimensional spaces, while general C*-algebras may exhibit characteristics of higher-, even infinite-dimensional topological spaces. Here as in the classical case, one expects many strong theorems to hold only for C*-algebras which are finite-dimensional in a suitable sense. One is hence drawn to the conjecture that for A a unital simple separable nuclear C*-algebra, the following properties are equivalent:

(i) A has finite nuclear dimension;

(ii) $A \otimes \mathcal{Z} \cong A$;

(iii) A has strict comparison.

A detailed exposition of properties (i) and (iii) is beyond the scope of this report; let us mention only that nuclear dimension generalizes the classical covering dimension of a space to the realm of C*-algebras, and that strict comparison means, roughly, that the pre-order on Hilbert modules over A given by inclusion up to isomorphism is determined by the rank of the modules as measured by traces. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are known, and (iii) $\Rightarrow$ (ii) holds under some additional conditions. The implication (ii) $\Rightarrow$ (i) is only known for classes where Elliott's classification conjecture holds, so that Elliott's program is a central facet of the conjecture. The equivalence of (ii) and $mathrm(iii)$ would represent a broad generalization of Kirchberg's celebrated $\mathcal{O}_\infty$ stability theorem for nuclear simple separable purely infinite C*-algebras. A lot of focus in the field, and at the workshop in particular, is aimed at resolving this conjecture, relating $\mathcal{Z}$-stability to topological and homological notions of finite-dimensionality for C*-algebras, and understanding how these notions may be used to delimit the universe of classifiable simple C*-algebras in a useful way. The conjecture was addressed directly in the talks by Lin and Winter. It has proven fruitful to revisit some of the classes of simple C*-algebras given as inductive limit of building blocks in the light of recently acquired insight, and indeed five of the talks (by Elliott, Gong, Niu, Robert and Strung) were drawing on knowledge from the initial stages of the Elliott program to shed light on these issues, and address key aspects of the central conjecture mentioned above.semigroup and comparison of open projections

1.2.1 Francesc Perera

Semigroup valued lower semicontinuous functions (with applications to theCuntz semigroup)

1.3 Discussion

Cuntz introduced his semigroup in 1978 to study traces and their generalizations, but only in recent years has it been realized that this object is a fruitful vessel also for classification theory. The semigroup is related to the K_0-group, but instead of recording the structure of finitely generated projective modules over a C^*-algebra, it records the structure of its countably generated modules. Whereas it was recognized early that this object carried a lot of the information of the underlying C^*-algebra, for many years it was thought that it would be too difficult to compute to be of theoretical or practical use in classification theory.The examples given by Villadsen, Rørdam and Toms have established that the classical invariants based on K-theory and traces, shown to be complete by Elliott in a multitude of important cases, do not suffice in general, and since the Cuntz semigroup was used to establish the existence of such examples in Toms's work there has been intense interest in understanding how to compute or describe it, and how to derive and improve known classification results using this object. A key result was obtained by Perera and Toms to the effect that when a C^*-algebra is "nice" in the sense of being $\mathcal{Z}$-stable, then the Cuntz semigroup is determined by the non-stable K-theory and the trace space of the C^*-algebra in question.Recently, a lot of work has gone into the analysis of the Cuntz semigroup of a C^*-algebra of the form $\mathrm{C}(X, A)$ and talks both by Perera and Tikuisis reported on progress in this direction, in the first case obtained in joint work with Antoine and Santiago. Also, Ortega in his talk explored the possibilities for understanding or computing the Cuntz semigroup by means of the concept of open projections studied by Akemann and Pedersen.reas the classification theory for non-simple **stably finite** C^*-algebras was developed in parallel with the simple case, and by use of more or less the same type of invariants, completely new ideas go into the case of non-simple **purely infinite** C^*-algebras. The early breakthroughs of Kirchberg, who defined and employed ideal-related KK-theory to attack these types of problems, gave access to profound isomorphism results and complete classification for $\mathcal{O}_2$-stable nuclear C*-algebras by means of their primitive ideal space, but is has been a great challenge to complement Kirchberg's theory by finding the K-theoretical invariants which lead to ideal-related KK-isomorphism in Kirchberg's sense. Asking that the ideal lattice be finite is natural in this context, but many fundamental questions remain open even for very small such lattices.Work of Meyer and Nest (outlined in Meyer's talk) represented the next big breakthrough in this area, and was the main suject of discussion in this particular section of the conference. Fixing a finite ideal lattice, Meyer and Nest provide a machinery for analyzing whether or not the family of K-groups associated to subquotients (along with the natural maps between them) allow the establishing of a universal coefficient theorem which may then in combination with Kirchberg's result lead to classification. Meyer and Nest proved that in the linear case, this is always the case, but also gave examples to show that for other ideal lattices, these invariants are not enough. As Meyer reported, Bentmann and Köhler have characterized precisely which spaces share this property with the linear case, and further studied Meyer and Nest's analysis which leads to positive results when one adds more groups to the invariant. Also, work by Arklint, Restorff, and Ruiz (reported in Arklint's talk) has demonstrated than in some cases where classification by the natural invariant is known to fail in general, one still gets a complete invariant in the real rank zero case.In a parallel effort, much recent work has gone into the classification theory for non-simple graph C*-algebras. This well-studied class of C*-algebras cuts across the traditional boundaries of classification theory in the non-simple case, since some simple subquotients may be AF and others purely infinite, but nevertheless work by Eilers and Tomforde showed that classification was possible also here, employing the Corona Factorization Property and ideas by Rørdam, refined in a paper by Eilers, Restorff, and Ruiz. The talks of Ruiz and Tomforde outlined recent progress in this area of research which apart from its applications to graph algebra theory seems to carry a lot of insight into the boundaries of non-simple classification theory away from the stably finite case. Also, the question of which invariants may be used in this special case were discussed in Arklint's talk in the context of the results outlined in the previous paragraph.The classification theory for non-simple C*-algebras does not as yet come equipped with as precise range results as those which have been known for decades in the simple case. Most notably, the class of (non-Hausdorff) spaces which may occur as the primitive ideal space of a C*-algebra remains unknown, but recent work reported at the workshop by Kirchberg has remedied the situation under the natural restriction of amenability. Also, Tomforde reported on range results in the context of the classification of graph C*-algebras, obtained in joint work with Eilers, Katsura and West.On the one hand, operator algebras associated to dynamical systems have proved to be extremely important and challenging examples which have inspired much deep work and lead to beautiful results, and on the other hand, results and invariants obtained in the context of operator algebras have proved to be useful at the core of the theory of ergodic theory and dynamical systems.The backdrop for our workshop in this particular context was two very satisfactory, and in some sense final, results pointing mainly in the aforementioned direction. Firstly, the results by Toms and Winter (also mentioned above) had established (by an inventive reinterpretation of an idea of Putnam) that crossed products given by minimal $\mathbb{Z}$-actions on spaces with finite dimension were, in fact, classifiable by the Elliott invariant, by invoking many of the most important recent additions to classification theory: the nuclear dimension of Kirchberg, Winter, and Zacharias the theory of recursively subhomogeneous algebras of Phillips, the tracial rank zero classification by Lin, and the notion of $\mathcal{Z}$-stability. And secondly, the results by Giordano, Matui, Putnam, and Skau had finally established that any minimal $\mathbb{Z}^n$-action on a Cantor set was in fact strongly orbit equivalent to an action of $\mathbb{Z}$ (just as in the measurable case), thus reducing the classification of $\mathbb{Z}^n$-actions up to this equivalence relation, and the classification of the associated crossed products, to the fundamental case resolved by Giordano, Putnam, and Skau.With these long-standing open problems resolved, it is natural to break new ground and study to what extent the methods developed may carry over to higher generality. The talks of Matui and Phillips presented quite complete results on classifying actions on simple purely infinite C*-algebras by $\mathbb{Z}^n$ and by finite groups, respectively, refining the notion of Rohlin property and leading to algebraic challenges related to those described in the previous section. Hirshberg presented joint work with Winter and Zacharias showing the preservation of the – for classification – key property of finite nuclear dimension under passage to certain crossed products. And Sierakowski described joint work with Rørdam explaining when thecrossed product of a C*-algebra by an exact discrete group is purely infinite (simple or non-simple). There is some hope that these results may combine with classification theory for non-simple C*-algebras as mentioned above.

2 General theory

2.1 Talks

2.1.1 Bruce Blackadar

On the work of Simon Wassermann

2.1.2 Ilijas Farah

Classification of C*-algebras and descriptive set theory

2.1.3 Thierry Giordano

A generalization of the Voiculescu-Weyl-von Neumann theorem

2.1.4 Ian Putnam

Relative K-theory of some groupoid C*-algebras

2.1.5 Iain Raeburn

C*-algebras related to dilation matrices

2.1.6 Hannes Thiel

A characterization of semiprojectivity for commutative C*-algebras

2.1.7 Simon Wassermann

Simple non-amenable C*-algebras with no proper tensor factorisations

2.1.8 Stuart White

Near inclusions of C*-algebras

2.2 Discussion

A number of talks were presented w hich in a multitude of ways stressed the interrelation between classification theory and other parts of operator algebras or indeed other areas of mathematics. An entire afternoon was committed to the celebration of Simon Wassermann's sixtieth birthday and his work of which the emphasis on tensor products and exactness has played an important role in the development of classification, as manifestly present in the idea of $\mathcal{Z}$-stability. Wassermann himself presented new results on the class of C*-algebras which are prime in the sense that they can not be written as a tensor product of other C*-algebras, and Blackadar gave an overview of Wassermann's work and its impact.In recent years, ties between operator algebras and descriptive set theory have been found and developed in the work of Akemann, Farah, and Weaver, and as an exciting development this venture has been taken into the realm of classification by work of Farah, Toms, and Törnquist. Farah reported on this in his talk.The notion of semiprojectivity, coined by Blackadar in 1985, has played an important role in classification theory and is known to hold for a large class of household C*-algebras. However, much has been unclear regarding the exact boundaries of the class of semiprojective C*-algebras until recently, when a number of questions have been resolved in the aftermath of a conference in Copenhagen. Thiel reported on his solution with Sørensen of the old problem of deciding precisely which spaces X give a semiprojective C*-algebra $C(X)$.When two C*-algebras A and B on the same Hilbert space are contained in each other up to a fixed error ϵ, must they then be the same or at least share properties as ϵ tends to zero? This is a classical question in C*-algebra theory which has seen a renaissance recently by joint work of Christensen, Sinclair, Smith, White, and Winter. Although the final resolution of a main problem in this area reported on in White's talk does not use classification results, the use of classification theory (and in particular the Elliott intertwining argument) was instrumental in reaching these results.The talks of Putnam and Raeburn in different ways addressed the problem of applying methods from classification to understanding the structure of KMS states on certain C*-algebras, and in Giordano's talk the question of finding "localized" versions of the classical Voiculescu-Weyl-von Neumann theorem by, e.g., specializing the targets was discussed.