Geometric Scattering Theory and Applications

Peter Hislop (University of Kentucky),
Rafe Mazzeo (Stanford University),
Peter Perry (University of Kentucky),
Antonio Sá Barreto (Purdue University)

March 14, 2010–March 19, 2010

1 Overview of the Field

Classical scattering theory, by which we mean the scattering of acoustic and electromagnetic waves and quantum particles, is a very old discipline with roots in mathematical physics. It has also become an important part of the modern theory of linear partial differential equations. *Spectral geometry* is a slightly more recent subject, the goal of which is to understand the connections between the behavior of eigenvalues of the Laplace-Beltrami operator Δ_g on a compact Riemannian manifold (M, g) and various features of the geometry and topology of this manifold. *Geometric scattering theory* has developed over the past few decades as a unification and extension of these two fields, though certain aspects of the field go back much further. On the one hand, scattering theory is the natural replacement for the study of eigenvalues on complete, noncompact manifolds since the spectrum of the Laplace-Beltrami operator may often contain only continuous spectrum, whereas there is still a rich theory for some of the other objects in scattering theory described below. On the other hand, the study of scattering theory in the setting of Riemannian manifolds adds many new subtleties and problems over those encountered for traditional Schrödinger operators on Euclidean space by allowing for spaces with various more intricate types of asymptotic geometries. This broader perspective has turned out to be surprisingly revealing and to shed light on many of the classical problems in scattering on Euclidean spaces. This 'unification' of scattering theory and spectral geometry was proposed as a systematic area of study in a series of lectures given by R. Melrose at Stanford in 1994 [11]. Since that time the field has grown substantially, partly along some of the lines that Melrose had foreseen, but in many exciting and unexpected directions as well.

Geometric scattering theory encompasses the study, in the broadest sense, of the spectrum of the Laplace-Beltrami operator Δ_g and other natural elliptic operators on complete, noncompact Riemannian manifolds (M, g) with geometries which are 'asymptotically regular' at infinity. While there is no precise definition of this condition, it encompasses many natural settings and includes such cases as manifolds which are asymptotically Euclidean or conic, asymptotically cylindrical or periodic, asymptotically (either real or complex) hyperbolic, or modeled on other locally or globally symmetric spaces of noncompact type. The objects of study include the resolvent of the Laplace-Beltrami operator Δ_g given by

$$R(\lambda) = (\Delta_g - \lambda)^{-1},$$

which is a priori defined as a holomorphic family of L^2 bounded operators when λ is away from the spectrum. In many cases, the resolvent can be extended to act between other function spaces, and in that sense can be continued to a meromorphic operator-valued function. The poles of this meromorphic extension are called *resonances*. Resonances are the natural generalizations of L^2 eigenvalues. Another central object in geometric scattering theory is the *scattering operator*, which can be defined as follows. For manifolds which are (asymptotically) Euclidean, and endowed with a system of polar coordinates (r, θ) at infinity, a solution of $(\Delta_g - \lambda)u = 0$ has an asymptotic expansion of the form

$$u(r,\theta) \sim r^{\frac{1-n}{2}} \left(f(\theta) e^{ir\sqrt{\lambda}} + g(\theta) e^{-ir\sqrt{\lambda}} \right) + \mathcal{O}(r^{-\frac{1+n}{2}}),$$

where $n = \dim M$. It is known that (at least when λ is not an eigenvalue or a resonance), if f is any function on the sphere at infinity, then there is a uniquely associated 'generalized eigenfunction' u with eigenvalue λ having the above form. Hence, for for a function f on the sphere at infinity, there is a uniquely associated matching coefficient g. The scattering operator is the map

$$(S(\sqrt{\lambda})f)(\theta) = g(\theta).$$

For manifolds with other types of asymptotic geometry, there is a corresponding definition based on the fact that generalized eigenfunctions have an asymptotic expansion at infinity similar to the one above. Like the resolvent, the scattering operator also has a meromorphic extension in many situations, and its poles are (usually) the same as the set of resonances. The resolvent and scattering operator carry the same information, in principle, but both are very interesting objects of study in their own right. The scattering operator is traditionally studied in the physics literature because it is presumably the one which is actually observable.

The approach to scattering theory outlined so far is called stationary scattering theory since dynamics has played no explicit role. There are two different ways that dynamics can enter the picture. The first is that the long-term behavior of the geodesic flow on (M, g) has a profound affect on the scattering operator and resonances. One of the important and broad areas of investigation in this field is to determine the relationships between these various objects in scattering theory and the dynamical properties of geodesic flow. Questions here stretch from the field now called quantum chaos, which has deep connections with number theory, to the large area around the Selberg and Arthur trace formulæ and the many more general trace formulæ coming from the physics literature. The second way that scattering theory relates to dynamics is that all of stationary scattering theory can be recast in terms of behavior of solutions to the associated wave equation

$$\Box u := (\partial_t^2 - \Delta_g)u = 0.$$

The scattering operator, for example, can be understood as a means of comparing the long-time evolution of the Cauchy data, i.e. the map

$$\left(u|_{t=0}\,, \partial_t u|_{t=0}\right) \longmapsto \left(u|_{t=T}\,, \partial_t u|_{t=T}\right),$$

to the corresponding Cauchy data evolution for a 'free' operator, e.g. the wave operator on a space which contains only information about the asymptotic geometry of (M, g), but discards all the interior geometric and topological 'complexities' which cause the scattering. There are many relationships between these two notions of dynamics, of course, most centered around the fundamental principle that singularities of the solutions $u(t, z)$ of this wave equation are invariant under the Hamiltonian flow associated to the principle symbol of $\Box$ on $T^*(\mathbb{R} \times M)$, which is closely related to the geodesic flow on M.

Taking this broader perspective–i.e., including the wave operator along with the resolvent of the Laplace-Beltrami operator (and also the heat operator, time-dependent Schrödinger operator, etc.)–leads to many new questions as well as important connections to other fields. Amongst the most important of these is the connection with the study of solutions of the wave equation associated to a Lorentzian metric, in particular on spaces which do not naturally split as $\mathbb{R} \times M$. The most natural and important examples of such Lorentzian spaces are the vacuum solutions of the Einstein equations, sometimes called cosmological spacetimes. There is extensive ongoing work aimed at understanding the stability under the nonlinear Einstein evolution of even the most simple of these spacetimes. For example, the proof of the stability of Minkowski space, by Christodoulou and Klainerman [3], over twenty years ago, is still being digested by the community, and simplifications and extensions of that work are still of immediate interest. The current main focus here is to prove the stability of the Kerr family of spacetimes, which are rotating black holes. The current state of knowledge now rests on a detailed understanding of the linear wave evolution on these spaces, but many of the nonlinear aspects of the problem remain out of reach. Geometric scattering has a lot to say about all of this. In particular, something still poorly understood is how one should define resonances for these operators when the time variable does not split off as a factor. This is quite important because the location of resonances affects the rates of linear wave decay, which in turn is important to understand precisely as input to the nonlinear theory.

Let us discuss only one further aspect of geometric scattering theory, which is the connection between scattering theory on asymptotically hyperbolic manifolds and conformal geometry. That there should be some connection between asymptotically hyperbolic and conformal geometries was presciently foreseen by Fefferman and Graham in the early 1980's [4, 5], which now falls under the rubric of Fefferman's ambient metric program. This correspondence was discovered independently by the string theorists and is known in that community as the holographic principle, also called the AdS/CFT or Maldacena correspondence. The mathematical aspects of this were established in the seminal paper of Graham and Zworski [7], which explained, amongst other things, how the higher order conformally covariant operators (the so-called GJMS operators) of conformal geometry can be realized as residues of poles of the scattering operator for the Laplace-Beltrami operator on an associated asymptotically hyperbolic Einstein space. Other major discoveries here include the study of renormalized volumes, as defined by Graham and Witten [6] and Juhl's theory of generalized intertwining operators [10].

We have necessarily omitted many important aspects, questions and tools of the subject, but the description above gives some indication of the vitality of the subject and its deep connections with many other parts of mathematics.

2 Description of the meeting

The BIRS meeting itself was intended as a gathering of researchers from disparate parts of the field, to help disseminate advances in one part of the subject to specialists in other parts, and also to help introduce the many younger researchers in the field to the broader areas of investigation and the many senior researchers. Particular effort was focussed on inviting young researchers, spotlighting their work, and giving them an opportunity to interact with senior researchers in their fields. In all of this, the meeting was a great success, as we describe below.

A meeting with very similar themes was held at BIRS in the Spring of 2003; this was the second meeting in the history of BIRS, and we hope that BIRS will continue to be the venue for future meetings which follow the continuing developments and successes of this field.

Focus of the meeting

Set into the backdrop of this general field of geometric scattering, and based on the very enthusiastic response by researchers, in particular the junior ones, the organizers decided to focus on just a few of these themes, with emphasis on the contributions of younger participants complemented by talks from senior researchers chosen for their expository abilities who provided overview of some of the most important new directions.

We describe now the sets of topics discussed in the lectures of this meeting, divided into slightly arbitrary groups:

- Classical geometric scattering: Guillarmou, Borthwick, Christiansen, Datchev, Nonnenmacher, Aldana, Marazzi;
- Connections with conformal geometry and AdS/CFT: Graham, Juhl, Gover, Hirachi;
- Connections with mathematical relativity: Tohaneanu, Alexakis, Wunsch, Baskin, Wang, Vasy, Häfner;
- Other: Zelditch (Quantum ergodic restriction theorems), Albin (Signature theorems on stratified spaces).

Presentation Highlights

Rather than discuss all the talks in detail, let us focus on a representative sample of these presentations.

- Andreas Juhl lectured on his surprising new discoveries about the the universal recursive structure of Branson's Q-curvature, which shows that the Q-curvature in a given dimension and order can be written in terms Q-curvatures and GJMS operators of lower degrees and orders. The Q-curvature has emerged as a basic curvature invariant in conformal geometry, but its geometric meaning is still mostly mysterious. Juhl's results are likely to play a major role in the continuing efforts to understand its geometric content and uses.
- Colin Guillarmou presented his joint work with Rafe Mazzeo [8] which extends the theory of the meromorphic continuation of the resolvent of the Laplacian to arbitrary geometrically finite hyperbolic manifolds. This completes an old program in the subject, historically one of the first in geometric scattering theory, by finally incorporating intermediate rank cusps with irrational holonomy.
- Spyros Alexakis presented his recent work with A. D. Ionescu and Sergiu Klainerman which gives a solution to the old conjecture that any solution of the vacuum Einstein equation which is equal to the Kerr solution outside the event horizon is in fact globally equal to the Kerr solution. Their work requires an assumption (smallness of the Mars tensor) which they hope eventually to remove. Nonetheless, the basic analysis here contains an important new unique continuation theorem.
- Stéphane Nonnenmacher lectured on his work with Johannes Sjöstrand and Maciej Zworski in which they show that the study of the resolvent, and hence of scattering operator and resonances, can be reduced to the study of a family of 'open quantum maps', which are finite-dimensional operators constructed by quantizing the Poincaré map associated with the geodesic flow near the set of trapped trajectories.

Scientific Progress Resulting from the Meeting

The following items are distilled from an email survey of participants following the conference.

- Pierre Albin, Hans Christianson, and Colin Guillarmou were able to make progress on their ongoing project concerning the existence of quasimodes for the Schrödinger operator near a hyperbolic geodesic.
- Dean Baskin was able to use ideas from conversations with Colin Guillarmou and from Guillarmou's talk to obtain some new results about resonances for the wave operator on the AdS-Schwarzschild spacetime.
- Rod Gover and Jean-Philippe Nicolas initiated a collaboration to apply ideas from conformal geometry to understand decay of curvature and other fields in relativity.
- Hans Christianson continued his joint work with Steven Zelditch on quantum unique ergodic restriction theorems; he was also able to work with Pierre Albin, Colin Guillarmou and Jeremy Marzuola on the construction of soliton-like solutions to the nonlinear Schrödinger equation on compact manifolds. Finally, he made progress with Jared Wunsch on proving local smoothing for manifolds with degenerate trapping, and also with Kiril Datchev and Colin Guillarmou on the random walk operator on manifolds with cusp ends.
- Stéphane Nonnenmacher initiated work on a number of open problems with Tanya Christiansen (on "generic" fractal Weyl laws), with Frééric Naud (on spectral gap results of Bourgain-Gamburd-Sarnak and their possible application to resonance-free regions in congruence quotients of $\mathbb{H}^2$), and with Andras Vasy on high-energy resolvent estimates and their application to similar problems
- Jean-Phillipe Nicolas and Dietrich Häfner were able to use time during the meeting to finish a joint paper; they also began a new research project with Rod Gover.
- David Borthwick, Tanya Christiansen, Peter Hislop, and Peter Perry made progress on a joint work concerning generic properties of the distribution of resonances for manifolds hyperbolic at infinity

- Clara Aldana and Pierre Albin continued their joint work on isoresonant surfaces. This is a generalization of older work of Borthwick and Perry.
- Robin Graham engaged in extended conversations with Yoshihiko Matsumoto, a current graduate student of Kengo Hirachi, and provided significant assistance on his current thesis work.

Responses from participants

One of the most positive outcomes of the meeting was the interaction between senior and junior researchers. The conference was structured to highlight the work of younger researchers and provide opportunities for new collaborations. Here are some representative comments of younger researchers about the conference.

> "The conference was very useful to me because it gave me the opportunity to give a talk about my work on determinants of Laplacian and Ricci flow ... to a very suitable audience. ... On the other hand, I got to talk to people on my area who I had not met before."
>
> "In addition to a number of stimulating talks, it was a great opportunity to continue or begin new research."
>
> "My participation in this workshop allowed me to have some fresh insight in to a field to which I am a relative newcomer. I saw connections between one of my current research projects and the work of several speakers... The workshop gave me the opportunity to meet new people and establish more significant relationships with people I had met previously."

3 Outcome of the Meeting

While it is hard to quantify specific outcomes of any given meeting, there are some indicators including papers written as a direct outgrowth of conversations at the conference, or at the very least, the formation of new collaborations which will eventually lead to publications. (We are aware of several papers resulting from such interactions which have not yet been completed, which is reasonable given the relatively brief time between the conference and when this report is being written.) We have already indicated several new and ongoing collaborations which were certainly expedited by this conference. Less tangible outcomes include the possibility of cross-disciplinary interaction, which was in any case one of the stated goals of the meeting. For example, we feel that the heavy emphasis we placed to spotlight the techniques of geometric scattering to problems in mathematical relativity not only helped illuminate the great progress that has been made at this interface between fields in the past few years, but has stimulated many of the young researchers to work on problems in these directions.

Publications and preprints

Dmitry Jakobson and Frédéric Naud were able to prove a "spectral gap" estimate for resonances of convex co-compact subgroups of arithmetic groups [9]. More specifically they prove a lower bound on the distance between the first non-trivial scattering resonance and remaining resonances. The spectral gap for resonances is analogous to the spectral gap between the lowest and next eigenvalue of the Laplacian, but the problem is much subtler since the resonances are determined by a non-self-adjoint eigenvalue problem. Jakobson and Naud exploit known Selberg trace formula techniques

Hans Christianson, Colin Guillarmou, and Laurent Michel completed a paper [2] about the spectral gap for the operator associated to random walks on finite-volume, non-compact surfaces with hyperbolic cusps. The study of this type of operator is relatively new in microlocal theory, and was brought to the attention of Gilles Lebeau by the probabilist Persi Diaconis a few years ago. They had given a thorough analysis of it in Euclidean space, but through that work it was realized that there are some important corresponding analytic questions that should be studied for other classes of manifolds. This paper is an important first step in that direction.

David Borthwick, Tanya Christiansen, Peter Hislop and Peter Perry [1] prove that the counting function for resonances on manifolds of constant negative curvature "near infinity" generically saturates known upper bounds for resonances. Deterministic lower bounds on counting for resonances are typically very difficult to obtain (and reasonable conjectures for the lower bounds are known to be false in many cases of interest). The approach of proving "generic" lower bounds provides a powerful tool for the study of resonances which has now been extended to an important geometric setting. This approach may lead to similar generic lower bounds on the counting function of resonances in strips in terms of the fractal dimension of trapped sets of geodesics.

References

[1] David Borthwick, Tanya Christiansen, Peter D. Hislop, Peter Perry, Resonances for manifolds hyperbolic at infnity: optimal lower bounds on order of growth. `arXiv:1005.5500v2 [math.DG]`. To appear in *International Mathematics Research Notices* **2011**.

[2] Hans Christianson, Colin Guillarmou, Laurent Michel, Random walks on surfaces with hyperbolic cusps. `arXiv:1005.2754v1 [math.SP]`.

[3] Demetrios Christodoulou, Sergiu Klainerman, The global nonlinear stability of the Minkowski space. Princeton Mathematical Series, **41**. Princeton University Press, Princeton, NJ, 1993.

[4] Charles Fefferman, C. Robin Graham, Conformal invariants. The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque 1985, Numero Hors Serie, 95–116.

[5] Charles Fefferman, C. Robin Graham, Q-curvature and Poincaré metrics. *Math. Res. Lett.* **9** (2002), no. 2-3, 139–151.

[6] C. Robin Graham, E. Witten, Conformal anomaly of submanifold observables in AdS/CFT correspondence. Nuclear Phys. B **546** (1999), no. 1-2, 52–64.

[7] C. Robin Graham, Maciej Zworski, Scattering matrix in conformal geometry. *Invent. Math.* **152** (2003), no. 1, 89–118.

[8] Colin Guillarmou, Rafe Mazzeo, Specral analysis of the Laplacian on geometrically finite hyperbolic manifolds, `arXiv:1002.2165v1 [math.SP]`

[9] Dmitry Jakobson, Frédéric Naud. On the resonances of convex co-compact subgroups of arithmetic groups. `arXiv:1011.6264v1 [math.SP]`.

[10] Andreas Juhl, On the recursive structure of Branson's Q-curvature, `arXiv:1004.1784v1 [math.DG]`.

[11] Richard Melrose, *Geometric Scattering Theory*, Stanford Lectures, Cambridge University Press, Cambridge, 1995