Ergodic Optimization

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1 Overview of the Field

The field is a relatively recently established subfield of ergodic theory, and has significant input from the two well-established areas of symbolic dynamics and Lagrangian dynamics. The large-scale picture of the field is that one is interested in optimizing potential functions over the (typically highly complex) class of invariant measures for a dynamical system. Tools that have been employed in this area come from convex analysis, statistical physics, probability theory and dynamic programming. The field also has both general aspects (in which the optimization is considered in the large on whole Banach spaces) and local aspects (in which the optimization is studied on individual functions). In the latter category, there has been input from physicists with numerical simulations suggesting that the optimizing measures are typically supported on periodic orbits. This should be contrasted with the situation typically found in the 'thermodynamic formalism' of ergodic theory, in which the measures picked out by variational principles tend to have wide support. Ergodic optimization may be viewed as the low temperature limit of thermodynamic formalism.

2 Recent Developments and Open Problems

Recent developments in the field have been multi-faceted: there has been a general goal of establishing results showing that for a typical potential function, the optimizing measures are supported on periodic orbits. Until now, all results of this type have been established on separable Banach spaces of functions, whereas the principal goals have been to establish them for certain non-separable Banach spaces. In recent progress, a first non-separable Banach space with this property was identified. Another aspect that has been of considerable interest has been studying the maximizing measures obtained as limits of Gibbs measures as the temperature is reduced to 0. Recent work has identified natural examples in which this limiting process fails. Work is ongoing to relate this to physical processes. Other groups of researchers have used convex analysis and measures of spread coming from economics to identify whole classes of functions sharing common optimizing measures. This is particularly interesting as the measures that appear in this study are widely known in a variety of other contexts.

Each speaker included open problems as part of their lectures. Additionally we scheduled a problem session for the presentation of problems which may not have found their way into lectures. Here are some of those problems.

2.1 Entropy of Nearest Neighbor SFT's

Definition: Let X be a $\mathbb{Z}^d$ nearest neighbor SFT with alphabet A, and multiple measures of maximal entropy.

For such X, where $d \geq 1$, define

$$\alpha_d := \inf_X\{(\log|A|) - h(X)\}$$

Ronnie Pavlov, in his talk, mentioned the following facts: $\alpha_d > 0$ for all d, $\alpha_1 = \log 2$, $\alpha_2 < \log 2$, and $\alpha_{d+1} \leq \alpha_d$ for all d. He asks if it is true, then, that $\lim_{d\to\infty} \alpha_d = 0$?

Equivalently, one asks if there exist $\mathbb{Z}^d$ nearest neighbor SFTs with multiple measures of maximal entropy and entropy arbitrarily close to the log of the alphabet size?

2.2 Some Questions Related to $\times 2$-invariant Measures

The following were contributed by Oliver Jenkinson.

2.2.1 Continuous f with Lebesgue measure as the unique $\times 2$-invariant f-maximizing measure

The following is Problem 3.9 in [1]

Problem:

Let $T(x) = 2x \pmod 1$. Explicitly exhibit a continuous function $f : [0,1] \to \mathbb{R}$ such that $\int f(x)\,dx > \int f\,d\mu$ for all T-invariant probability measures μ other than Lebesgue measure.

Some remarks:

(a) The strict inequality is key; if the inequality were weak then a constant function would suffice.

(b) It is known that such functions f exist (see [2, Cor. 1]).

(c) By an "explicit" representation of f we have in mind some sort of series expansion, for example a Fourier expansion.

(d) It is known that any such f cannot be too "regular"; for example f cannot be Hölder (see e.g. the discussion in [1, 2]). There are heuristic reasons (see [2]) for expecting such an f to be highly oscillatory.

Since periodic orbit measures are weak-$*$ dense in the set of T-invariant measures, the following weaker version of the above problem is perhaps no easier to solve.

Problem:

Let $T(x) = 2x \pmod 1$. Explicitly exhibit a continuous function $f : [0,1] \to \mathbb{R}$ such that $\int f(x)\,dx > \frac{1}{n}\sum_{i=0}^{n-1} f\big(T^i(\frac{j}{2^n-1})\big)$ for all $n \geq 1$ and $0 \leq j \leq 2^n - 1$.

2.2.2 Positive entropy uniquely maximizing measures for analytic functions?

Again, let $T(x) = 2x \pmod 1$. It is known that there exist real analytic functions whose unique maximizing measure is strictly ergodic but non-periodic. Examples of such functions can be found within the one-parameter family $f_\theta(x) = \cos 2\pi(x-\theta)$: for certain values of θ the f_θ-maximizing measure is a Sturmian measure supported on a Cantor set. Non-periodic Sturmian measures are, in a sense, the closest to periodic among all non-periodic measures; for example the symbolic complexity of a non-periodic Sturmian orbit (which is a generic orbit for the corresponding Sturmian measure) is as small as it can be among non-periodic orbits. Non-periodic measures with higher complexity can also arise as maximizing measures for (higher degree) trigonometric polynomials; for example measures which are combinatorially equivalent to an interval exchange can occur. All these measures seem to have rather low symbolic complexity, however, so that the following question is open.

Problem:

If $T(x) = 2x \pmod 1$, can a positive entropy T-invariant measure uniquely maximize a real analytic function f?

2.2.3 Sturmian measure with largest variance

Among $\times 2$-invariant probability measures, it is known (see [3]) that Sturmian measures have the smallest variance (where variance denotes the quantity $\int (x - \int x\, d\mu(x))^2\, d\mu(x)$). More precisely, the variance of a Sturmian measure (around its mean) is strictly smaller than for all other invariant measures with the same mean value (i.e. barycentre). The variance of a Sturmian measure depends continuously on its parameter (or 'rotation number'), and the following is open (see [4]):
Problem:
Which Sturmian measure has largest variance?

We remark that symmetry considerations mean that if S_ϱ has largest variance then so does $S_{1-\varrho}$. Numerical experiment suggests that the relevant value of ϱ is approximately $676/1761 \approx 0 \cdot 383873$.

2.3 Zero Temperature Limits

The following was submitted by Jean-René Chazottes (of the Centre de Physique Thorique, École Polytechnique) and Michael Hochman (Institute for Advanced Study, Princeton), who were not actually in conference but who recognized the importance of our meeting and the relevance to their work.

In a recent paper (Commun. Math. Phys. 297 (2010)), they exhibited Lipschitz potentials on the full shift $\{0,1\}^{\mathbb{N}}$ such that the associated Gibbs measures fail to converge as the temperature goes to zero. In higher dimension, namely on the configuration space $\{0,1\}^{\mathbb{Z}^d}$, $d \geq 3$, they showed that this non-convergence behavior can occur for the equilibrium states of finite-range interactions, that is, for locally constant potentials. Here are several open questions posed in that paper.

2.3.1 The one-dimensional case

In the previously mentioned paper they obtained the following result.
Theorem A:
There exist subshifts $X \subseteq \{0,1\}^{\mathbb{N}}$ so that, for the Lipschitz potential $\varphi_X(y) = -d(y, X)$, the family $\{\mu_{\beta\varphi}, \beta > 0\}$ does not converge (weak-*) as $\beta \to +\infty$.

In this statement, $d(\cdot,\cdot)$ is the usual distance on $\{0,1\}^{\mathbb{N}}$. This theorem holds more generally for one-sided or two-sided mixing shifts of finite type.

2.3.2 Topological dynamics of X

Let μ be an ergodic probability measure for some measurable transformation of a Borel space, and $h(\mu) < \infty$. By the Jewett-Krieger theorem there is a subshift X on at most $h(\mu)+1$ symbols whose unique shift-invariant measure ν is isomorphic to μ in the ergodic theory sense. For the potential φ_X, all accumulation points of $\mu_{\beta\varphi}$ are invariant measures on X, so they all equal ν; thus $\mu_{\beta\varphi} \to \nu$ as $\beta \to +\infty$. This shows that the zero-temperature limit of Gibbs measures can have arbitrary isomorphism type, subject to the finite entropy constraint, and raises the analogous question for divergent potentials:
Problem 1:
Given arbitrary ergodic measures μ', μ'' of the same finite entropy, can one construct a Hölder potential φ whose Gibbs measures $\mu_{\beta\varphi}$ have two ergodic accumulation points as $\beta \to +\infty$, isomorphic respectively to μ', μ''?

2.3.3 Maximization of marginal entropy

Let φ be a Hölder potential and $\mathcal{M}$ the set of invariant probability measures μ for which $\int \varphi d\mu$ is maximal. It is known that if μ is an accumulation point of $(\mu_{\beta\varphi})_{\beta>0}$ then $\mu \in \mathcal{M}$ and furthermore μ maximizes $h(\mu)$ subject to this condition.

In the example constructed in the proof of Theorem A, the potential φ had two φ-maximizing ergodic measures μ', μ'', and the key property that we utilized was that their marginals at certain scales had sufficiently different entropies. In fact, the measure maximizing the marginal entropy on $\{0,1\}^n$ for certain n was alternately very close to μ' and to μ''.

It is an interesting question if such a connection between zero-temperature convergence and marginal entropy exists in general. Let φ be a Hölder potential, and for each n let $\mathcal{M}_n^*$ denote the set of marginal distributions produced by restricting $\mu \in \mathcal{M}$ to $\{0,1\}^n$. The entropy function $H(\cdot)$ is strictly concave on $\mathcal{M}_n^*$, and therefore there is a unique $\mu_n^* \in \mathcal{M}_n^*$ maximizing the entropy function. Let

$$\mathcal{M}_n = \{\mu \in \mathcal{M} \,:\, \mu|_{\{0,1\}^n} = \mu_n^*\}.$$

This is the set of φ-maximizing measures which maximize entropy on n-blocks. Note that the diameter of $\mathcal{M}_n$ tends to 0 as $n \to \infty$ in any weak-* compatible metric. Hence we can interpret $\mathcal{M}_n \to \mu$ in the obvious way.

Problem 2:

Is the existence of a zero-temperature limit for φ equivalent to existence of $\lim \mathcal{M}_n$? More generally, do $(\mu_{\beta\varphi})_{\beta\geq 0}$ and $(\mathcal{M}_n)_{n\geq 0}$ have the same accumulation points?

2.3.4 The multi-dimensional case

The case $d \geq 3$:

Recall that a shift of finite type $X \subseteq \{0,1\}^{\mathbb{Z}^3}$ is a subshift defined by a finite set L of patterns and the condition that $x \in X$ if and only if no pattern from L appears in x. Given $L \subseteq \{0,1\}^E$ one can define the finite-range interaction $(\Phi_B)_{B\subseteq\mathbb{Z}^d,|B|<\infty}$ by

$$\Phi_E(x) = \begin{cases} -1/|E| & x|_E \in L \\ 0 & \text{otherwise} \end{cases}$$

and $\Phi_B = 0$ for $B \neq E$; the associated potential on $\{0,1\}^{\mathbb{Z}^d}$ is

$$\varphi_L(x) := \sum_{B\ni 0} \frac{1}{|B|}\,\Phi_B(x) = \begin{cases} -1 & x|_E \in L \\ 0 & \text{otherwise.} \end{cases}$$

Clearly an invariant measure μ on $\{0,1\}^{\mathbb{Z}^d}$ satisfies $\int \varphi_L d\mu = 0$ if and only if μ is supported on X; thus the shift-invariant ground states are precisely the shift invariant measures on X.

In the paper mentioned in the introduction we obtained the following result.

Theorem B:

For $d \geq 3$ there exist locally constant (i.e. finite-range) potentials φ on $\{0,1\}^{\mathbb{Z}^d}$ such that for any family $(\mu_{\beta\varphi})_{\beta>0}$ in which $\mu_{\beta\varphi}$ is an equilibrium state (i.e. a shift-invariant Gibbs state), the limit $\lim_{\beta\to+\infty}\mu_{\beta\varphi}$ does not exist.

Comments.

The previous statement is rather subtle. If there were a unique Gibbs state for each β then there would be a unique choice for $\mu_{\beta\varphi}$, and the previous result could be formulated more transparently: there exist locally constant potentials such that $\lim_{\beta\to+\infty}\mu_{\beta\varphi}$ does not exist. But we believe that in our example uniqueness does not hold at low temperatures.

Our result is about continuous families and does not contradict the fact that for each given family $(\mu_{\beta\varphi})_{\beta>0}$ of equilibrium states, there always exists a subsequence $(\beta_i)_{i\in\mathbb{N}}$ such that the limit $\lim_{i\to\infty}\mu_{\beta_i\varphi}$ exists. This is due to compactness of the space of probability measures.

There is nothing new in the fact that one can choose *some* divergent family $\beta \mapsto \mu_{\beta\varphi}$ of equilibrium states. Think *e.g.* of the Ising model below the critical temperature (β large enough): One can choose a family which alternates between the $+$ and $-$ phases. However it is also possible to choose families which converge to one of the ground states. Let us insist that in contrast to this kind of situation we prove the existence of examples where it is *not possible* to choose *any* family which converges to a ground state.

The case $d = 2$:

Let us say a few words about the limitations of the previous result. First, it seems likely that our examples support non-shift-invariant Gibbs states, i.e. Gibbs states which are not equilibrium states, and, furthermore, we do not know if the statement extends to them. Hence the requirement of shift-invariance. As for the restriction $d \geq 3$, the method used in our construction, which produces a potential of the form φ_L above, does not work at present in $d = 2$.

Problem 3:

For $d \geq 2$, do there exist finite-range potentials on the d-lattice such that every family of Gibbs states $\{\mu_{\beta\Phi}, \beta > 0\}$ fails to converge as $\beta \to +\infty$?

3 Meeting Highlights

The meeting featured some 13 lectures of a very high standard covering a wide range of topics in the area. The schedule allowed plenty of time for informal discussion between the participants and there was a good deal of lively discussion between talks. The participants included researchers at all ranks: from student to senior researcher.

The following lectures were given:

- M. Allahbakhshi, 'Measures of relative maximal entropy'
- V. Anagnostopoulou, 'First order stochastic dominance'
- G. Contreras, 'Maximizing measures and Lagrangian subactions'
- C. Gonzalez-Tokman, 'Approximating invariant densities of metastable systems'
- R. Iturriaga, 'Selection of a Hamilton Jacobi solution via a discount factor'
- O. Jenkinson, 'Ergodic Dominance'
- I. Morris, 'Joint spectral radius and its connection with ergodic optimisation'
- R. Pavlov, 'Shifts of finite type with nearly full entropy'
- M. Pollicott, 'Dynamical Zeta Functions'
- A. Quas, 'Rates of approximation of optimizing measures by periodic orbit measures'
- J. Siefken, 'Ergodic optimization and super-continuous functions'
- P. Thieullen, 'Rotation vector for minimizing configurations of the multi-dimensional Frenkel-Kontorova model'
- F. Vivaldi, 'Minimal modules of periodic orbits'

4 Outcome of the Meeting

The meeting was very successful at presenting a broad spectrum of work related to the central topic. Many of the participants were meeting for the first time and there was a substantial impetus towards collaboration and future meetings. A number of the participants commenced initial discussions that show some promise of development into full collaborative work.

As usual, BIRS was a superb place to organize a meeting. Many participants were there for the first time. New and returning participants alike were eager to come to BIRS. From an organizational point of view, it was extremely smooth. Brenda's assistance on the ground was invaluable and friendly as ever.

References

[1] O. Jenkinson, Ergodic optimization, *Discrete & Continuous Dynamical Systems*, **15** (2006), 197–224.

[2] O. Jenkinson, Every ergodic measure is uniquely maximizing, *Discrete & Continuous Dynamical Systems*, **16** (2006), 383–392.

[3] O. Jenkinson, A partial order on $\times 2$-invariant measures, *Math. Research Lett.*, **15** (2008), 893–900.

[4] O. Jenkinson, Balanced words and majorization, *Discr. Math. Alg. Appl.*, **1** (2009), 485–498.