Twenty-five years of representation theory of quantum groups

Pavel Etingof (MIT),
Victor Ginzburg (University of Chicago),
Nicolas Guay (University of Alberta),
David Hernandez (Université de Paris 7),
Alistair Savage (University of Ottawa)

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1 Overview of the Field

Quantum groups first appeared under different disguises in the first half of the 1980's, in particular in the work of physicists and mathematicians interested in the quantum inverse scattering method in statistical mechanics. For instance, one of the simplest examples, the quantized enveloping algebra of $\mathfrak{sl}_2$, first surfaced in a paper of P. Kulish and N. Reshetikhin in 1981 about integrable systems. Quantized enveloping algebras of Kac-Moody algebras, the most studied examples of quantum groups, are non-commutative and non-cocommutative Hopf algebras discovered around 1985 independently by V. Drinfeld and M. Jimbo. The address of V. Drinfeld at the International Congress of Mathematicians in 1986 brought quantum groups to the attention of mathematicians worldwide and laid the foundations of this theory. In one fell swoop, Drinfeld gave birth to a new branch of algebra which started growing at tremendous speed and has remained a very active area of research to this day.

Quantum groups have found applications in other branches of mathematics, topology being one notable example present at the workshop. Quantum groups have even made their appearance in subjects more distant from algebra, for instance in probability theory, harmonic analysis and number theory (the last one being the subject of the BIRS workshop "Whittaker functions, crystal bases and quantum groups" held in June 2010). Some ideas that have been fruitful in the study of quantum groups have gained in popularity and have started spreading across mathematics, for instance the philosophy of categorification.

Several different approaches to the study of representations of quantum groups have been successful. An algebro-combinatorial approach has led to the discovery of crystal bases and of more general crystal structures, which have had a profound influence in Lie theory. Geometric realizations of representations have been obtained via equivariant K-theory of appropriate varieties, e.g. Steinberg varieties and quiver varieties. Of course, Lie theorists have been very much interested in highest weight modules for quantum groups, their characters and also in some specific families of finite dimensional modules, e.g. Weyl modules and Kirillov-Reshetikhin modules. The representation theory of quantum groups is a lively area of research and it is reasonable to expect it to remain this way for the foreseeable future.

2 Recent developments and open problems

One of the most exciting recent developments surrounding quantum groups is the categorification of quantized Kac-Moody enveloping algebras by M. Khovanov and A. Lauda and, independently, by R. Rouquier. Their work has led to the introduction of a very interesting new family of algebras, the Khovanov-Lauda-Rouquier algebras or quiver Hecke algebras, which have a rich representation theory. These have had applications to the study of other algebras of interest in Lie theory: for instance, they have been used by J. Brundan and A. Kleschev to create new gradings on blocks of Hecke algebras. Of course, all this leads to the open problem of categorifying many other interesting algebraic structures in representation theory (e.g. highest weight modules, Heisenberg algebras, etc.). Moreover, these ideas have also lead to applications in topology (e.g. categorification of knot invariants) and open problems in this direction were mentioned in the talks of C. Stroppel and B. Webster.

One of the first celebrated theorems about quantum groups is the Khono-Drinfeld theorem which states that the monodromy representation of the braid group $B_{\mathfrak{g}}$ coming from the Knizhnik-Zamolodchikov connection is isomorphic to a representation constructed from the R-matrix of $U_q\mathfrak{g}$. Recent work of V. Toledano Laredo on the trigonometric Casimir connection is in the same vein and leads naturally to an analogous conjecture about the monodromy of this connection whose proof in the $\mathfrak{sl}_2$-case was announced at the workshop.

Cluster algebras have been one of the hottest topics in algebra in the last ten years, rapidly developing in connection with various subjects. Rather recent work of D. Hernandez, B. Leclerc, H. Nakajima and others has investigated connections between these and finite-dimensional representations of quantum affine algebras. Starting from the observation that the cluster exchange relations are of the same form as the T-system equations satisfied by Kirillov-Reshetikhin modules, D. Hernandez and B. Leclerc [17] have conjectured that the Grothendieck ring of a certain tensor category of finite dimensional modules is isomorphic to a cluster algebra (this is another example of categorification, called monoidal categorication). Their work and subsequent work of Nakajima [25], imply their conjecture at "level 1" (and imply as a byproduct the positivity conjecture for finite type cluster algebras). Furthermore, it has been shown recently that certain quantized coordinate rings have the structure of a quantum cluster algebra. Cluster algebras were the subject of the BIRS workshop "Cluster algebras, representation theory, and Poisson geometry" in September 2011. Several participants of our workshop participated in the September workshop and talked about the connection with quantum groups.

There has been a lot of progress made recently on other aspects of quantum group theory, for instance on generalization of Kirillov-Reshetikhin modules, on quantization of structures analogous to Lie bialgebras, on crystals, on quantum toroidal algebras and on ring theoretical properties of quantized coordinate rings. The meeting featured talks on all these subjects and many of these ended with important conjectures. Moreover, the presentation of IḞrenkel paved the way for a completely new direction of research for years to come. We have outlined in the previous paragraphs three exciting avenues for future advances in quantum group theory. More information about recent developments and open problems is contained in the following section.

3 Content of the talks

The speakers all gave talks about their recent research projects, sometimes on topics that have now become "classical" in the representation theory of quantum groups (e.g. crystals), and sometimes on topics that are somewhat more remote but inspired by work originally done on quantum groups (for instance, categorification of various mathematical structures). Four of the speakers were postdocs (Sachin Gautam, David Jordan, Peter Tingley, Charles Young). There was also one poster presented by another postdoc, Stewart Wilcox, based on his Ph.D. thesis about rational Cherednik algebras: these are related to the representation theory of quantum groups in various ways – for instance, the Knizhnik-Zamolodchikov connection plays a very important role for quantum groups and for rational Cherednik algebras and the Ph.D. thesis of Stewart Wilcox provides a detailed study of that connection in a very important special case.

Monday, August 8

The workshop started with a presentation by Professor Evegeny Mukhin (Indiana University - Purdue University Indianapolis) entitled *Representations of Toroidal Quantum* $\mathfrak{gl}[1]$ about his joint work with B. Feigin, E. Feigin, M. Jimbo and T. Miwa [12, 13]. Toroidal quantum algebras (double affine quantum groups) have been investigated for about twenty years and connections with theoretical physics (e.g. Fock space constructions) have been established. They are very interesting mathematical objects, but also very mysterious since they are hard to investigate. Until this recent work, quantum toroidal algebras were defined only for finite dimension simple Lie algebras and for the Lie algebra $\mathfrak{gl}[n]$ with $n \geq 2$. It turns out that Fock spaces depending on a continuous complex parameter also appear in the representation theory of the toroidal quantum $\mathfrak{gl}[1]$ and can be shown to be tame, irreducible highest weight modules. Furthermore, E. Mukhin presented constructions of various tame, irreducible representations with natural bases parameterized by certain plane partitions (e.g. Macmahon modules) with, possibly, various boundary conditions. They can be identified with subspaces of certain tensor products of Fock modules. He also gave character formulas after various specializations of parameters. As a byproduct, he explained how to obtain Gelfand-Zetlin like bases in a family of lowest weight irreducible $\mathfrak{gl}_\infty$-modules. A connection with double affine Hecke algebras for the symmetric group was also mentioned: the spherical subalgebra of type A is a quotient of the quantum toroidal algebra of $\mathfrak{gl}[1]$. Double affine Hecke algebras are another fascinating topic in representation theory which has garnered a lot of attention in the last fifteen years and were the subject of another talk (by E. Vasserot). It was mentioned at the end of his talk that all the results he presented should be extendible to the quantum toroidal algebra for $\mathfrak{gl}[n]$ for any $n \geq 1$.

The second talk, *Quantum foldings* by Professor Jacob Greenstein (University of California at Riverside), was the first of a few on the subject of crystals. The discovery of crystal bases by M. Kashiwara in the 1990's is one of the most important in the representation theory of quantum groups and has found a lot of applications in representation theory and in combinatorics. He presented the results of his joint work [2] with A. Berenstein on the following natural question. A classical result in Lie theory stipulates that a simple finite dimensional Lie algebra that is not simply laced can be constructed as the subalgebra $\mathfrak{g}^\sigma$ of a Lie algebra $\mathfrak{g}$ of type A,D or E fixed by a diagram automorphism σ of the latter. This construction is called "folding" and extends to Kac-Moody Lie algebras when σ is admissible. It is well-known that foldings do not admit direct quantum analogues, so it is natural to ask if there is a less direct way to relate the representation theory of $U_q(\mathfrak{g})$ and of $U_q(\mathfrak{g}^\sigma)$. The answer is yes if $\mathfrak{g}^\sigma$ is replaced by $(\mathfrak{g}^\vee)^\sigma$ where $\mathfrak{g}^\vee$ is the Langlands dual Lie algebra: it can be shown that there exists an embedding of crystals (or of Lusztig's canonical bases) for $\mathfrak{g}$ and $\mathfrak{g}^\sigma$. The aim of their work is to introduce algebraic analogues and generalizations of foldings in the quantum setting which yield new flat quantum deformations of non-semisimple Lie algebras and of Poisson algebras. Perhaps the most spectacular example is an algebra that can be regarded as a new algebra of quantum n by n matrices. Their work has led them to introduce new interesting examples of so-called überalgebras (which appear to have interesting ring theoretical properties as mentioned in the talk of M. Yakimov): he explained a general setting where such algebras can be useful (namely as a substitute when a linear map between algebras does not extend to an algebra homomorphism) and he provided an explicit construction of an überalgebra in the case of the dual pair of type (D_{n+1}, C_n) via a flat deformation of the enveloping algebra of $\mathfrak{sl}_n \ltimes (\mathbb{C}^n \otimes \mathbb{C}^n)$. A link with Hall algebras and with cluster algebras (a very hot topic over the past twelve years) was briefly mentioned at the end of his talk.

Dr. Anthony Licata (Stanford University) was the first to speak on the subject of categorification. The title of his talk on joint work with Sabin Cautis was *Heisenberg Categorification and Hilbert Schemes*. Given a finite subgroup G of $SU(2)$, he defined a monoidal category whose Grothendieck group is isomorphic to the homogeneous Heisenberg algebra corresponding to G. A very interesting property of this category is that it acts on the direct sum of the derived categories of Hilbert schemes of points on the resolution of the corresponding simple singularity; from this, one can recover a known representation of the Heisenberg algebra on the direct sum of the Grothendieck groups of these resolutions. Their work sheds some light on the connection between two realizations of Fock space representations of the associated affine Kac-Moody algebra. The popularity of the subject of categorification in representation theory has grown tremendously since the seminal works of Khovanov-Lauda and Rouquier a few years ago on the categorification of quantum groups. The work of A. Licata and S. Cautis uses ideas derived from the discoveries of M. Khovanov and A. Laura and can be viewed as an indirect application to a problem in classical representation theory of ideas stemming from quantum groups.

The following talk by Professor Weiqiang Wang (University of Virginia), *Spin fake and generic degrees for the symmetric group*, was not only about the symmetric group itself, but also about associated Hecke algebras, in particular the Hecke-Clifford algebras and the spin Hecke algebras. Quantum groups have been known to be related to Hecke algebras since the paper [18] of M. Jimbo, one of the first on the subject of quantized enveloping algebras. The fake degrees are graded multiplicities of an irreducible module of a Weyl group in its coinvariant algebra. The generic degrees arise from Hecke algebras, and their evaluation at a prime power are degrees of irreducible characters of finite Chevalley groups. In his talk, based on joint work with Jinkui Wan, W. Wang formulated and computed the spin analogues of fake and generic degrees for the symmetric group and related Hecke algebras. Of importance in this work is the Morita equivalence between the spin Hecke algebras and the Hecke-Clifford algebras introduced by S. Sergeev and G. Olshanski over twenty years ago in their work on (quantum) supergroups of type Q. He gave explicit formulas for the fake degrees (in terms of a Schur function) and stated that it was equal to the generic degree in this super context. He also presented results about trace forms on spin Hecke and Hecke-Clifford algebras.

Professor Milen Yakimov (Louisiana State University) gave the last talk of the first day about *Ring theory of quantum solvable algebras*. It was the only one on this topic, although a large body of literature has been produced about the subject. The area of quantum groups has supplied a very large number of examples where general methods for studying noncommutative rings can be tested and developed, using representation theoretic methods. A. Joseph, T. Hodges, T. Levasseur, and Y. Soibelman obtained a great deal of information about the spectra of quantized coordinate rings of simple Lie groups, and K. Goodearl and G. Letzter developed a general stratification theory putting the area in the framework of quantum affine toric varieties. The De Concini-Kac-Procesi quantum nilpotent algebras were another large class of algebras which were heavily investigated.

This talk was a great overview of known results on spectra of quantum solvable algebras, normal separation, the extension of Gabber's catenarity theorem to these classes, and the classification of their automorphism groups along the Andruskiewitsch-Dumas conjecture. It was partly an historical review about important theorems from the 1990's, and partly one about the recent work of the speaker. For instance, he stated his very recent theorem to the effect that the center of A. Joseph's localization of a quantized coordinate ring is a Laurent polynomial algebra; another of his theorems provides a homeomorphism between the space of maximal ideal of a quantized function algebra and a Laurent polynomial ring, which answers a question of Goodearl and Zhang to the effect that maximal ideals in such a ring all have finite codimension. He finished with an overview of open problems in the ring theoretic side of quantum groups: for example, he stated a conjecture to the effect that a Cauchon-Goodearl-Letzter extension (which a type of iterated skew polynomial algebra) is catenary.

Tuesday, August 9

The second day started with a talk of Dr. Charles Young from the University of York about his joint work with Evgeny Mukhin on *Extended T-systems*. Kirillov-Reshetikhin modules form a family of representations of quantum affine algebras with a lot of interesting properties and which have been studied rather extensively. One of their most important properties is that they satisfy certain systems of recurrence relations called T-systems (this is the Kirillov-Reshetikhin conjecture, proved by H. Nakajima and D. Hernandez). The goal of their work is to replace Kirillov-Reshetikhin modules by a wider class of modules from which it is possible to build generalized T-systems. Such a wider class of modules includes all minimal affinizations (Kirillov-Reshetikhin modules are special types of minimal affinizations) and so-called snake modules. Their work is restricted to categories of finite-dimensional representations of quantum affine algebras of types A and B, but he commented on what can be expected in other types (based on some "experimental" preliminary results). He outlined the proofs of their results using the theory of q-characters and also briefly explained how to compute the q-character of a snake module by computing a certain sum over non-overlapping tuples of paths, so that snake modules are in fact thin and special. Their main theorem is the existence of a short exact sequence for snake modules, modulo the direction of the arrows. He concluded with a couple of open problems, one of which was about hypothetical relations between the generalized T-systems and the cluster algebra conjecture of D. Hernandez and B. Leclerc.

C. Young was followed by Professor Anne Schilling (University of California at Davis) with a talk entitled *Crystal energies via the charge in types A and C*. Hers was the first of a few talks about the combinatorial approach to representation theory. The discovery of crystal bases by M. Kashiwara in the early 1990's [19] is one of the most important in the history of quantum groups and has had applications to more classical (i.e. non-quantum) problems in representation theory. The axiomatic notion of crystal can be extolled from his result on crystal bases and the rich combinatorial structure of crystals has been studied quite a lot for more than fifteen years.

The energy function of affine crystals is an important grading used in one-dimensional configuration sums of statistical mechanical models and generalized Kostka polynomials. It is defined by the action of the affine Kashiwara crystal operators through a local combinatorial rule and the R-matrix. Nakayashiki and Yamada have related the energy function in type A to the charge statistic of Lascoux and Schuetzenberger. Computationally, it is much more efficient to compute charge than energy since its definition involves a recursive definition of local energy and the combinatorial R-matrix, for which not in all cases efficient algorithms exist. In her talk, she related energy to a new charge statistic in type C which comes from the Ram-Yip formula for Macdonald polynomials. This involves in particular the generalization of parts of the Kyoto path model for perfect crystals to the nonperfect setting, which yields an isomorphism between affine highest weight crystals and tensor products of Kirillov-Reshetikhin crystals. This is joint work with Cristian Lenart [23].

The first speaker of the afternoon was Professor Maxim Nazarov from the University of York who spoke about his joint work with S.Khoroshkin and E.Vinberg on a *Generalized Harish-Chandra isomorphism* [22]. The classical Harish-Chandra isomorphism is one of the most important results in the representation theory of semisimple Lie algebras. The motivation for seeking to generalize it was to obtain explicit realizations of all simple finite-dimensional modules of Yangians and of their twisted analogues. Yangians are one important family of quantum groups of affine types and the classification of their irreducible representations has been known for almost twenty-five years; it has also been known for a long time that, for the Yangians of $\mathfrak{sl}_2(\mathbb{C})$, those are isomorphic to tensor products of evaluation representations. However, finding explicit, concrete realization of irreducible representations of the other Yangians (twisted and non-twisted) has been a very challenging problem which has been solved in impressive recent work of M. Nazarov and S. Khoroshkin [21].

For any complex reductive Lie algebra $\mathfrak{g}$ and any locally finite $\mathfrak{g}$-module V, M. Nazarov and his collaborators extended to the tensor product A of $U(\mathfrak{g})$ with V the Harish-Chandra description of $\mathfrak{g}$-invariants in the universal enveloping algebra $U(\mathfrak{g})$. Their description of the algebra of invariants $A^{\mathfrak{g}}$ is in terms of the invariants of the action of Zhelobenko operators on the zero weight space of a certain quotient of A and their proof uses the Mickelsson algebras. The case relevant for Yangians is when V is a the Grassmann algebra of $\mathbb{C}^{mn}$ since, in this case, the (twisted) Yangian of $\mathfrak{g}$ admits a non-trivial homomorphism to $A^{\mathfrak{g}}$. He explained the role played by Howe duality in their work, and also how to construct a functor $\mathcal{F}$ from the category of finite dimensional modules over a certain finite dimensional reductive Lie algebra $\mathfrak{k}$ to the category of finite dimensional modules for a (twisted) Yangian $Y(\mathfrak{f})$ when $\mathfrak{k}$ and $\mathfrak{f}$ form an Howe dual pair. He finished by stating his main theorem which says that any irreducible representation of $Y(\mathfrak{f})$ can be obtained as the image of an intertwining operator between two modules over $Y(\mathfrak{f})$ obtained via the functor $\mathcal{F}$ applied to Verma modules over $\mathfrak{k}$.

The last two talks were more of a combinatorial nature. Professor Masato Okado of Osaka University delivered a presentation about *Open problems related to Kirillov-Reshetikhin crystals*. It is widely known that Kirillov-Reshetikhin modules of quantum affine algebras admit extremely rich structures, such as T-systems, fermionic character formulas, existence of crystal bases (Kirillov-Reshetikhin crystals), Kyoto path realization of affine highest weight crystals, existence of the corresponding geometric crystals, and positive birational Yang-Baxter maps (also called tropical R maps). There are many conjectures related to Kirillov-Reshetikhin crystals. Although some of these have been settled recently, many are still open. M. Okado reviewed recent progress and surveyed important open problems on this subject. The first conjecture he presented was divided into two parts: the first one about the existence of crystal bases for Kirillov-Reshetikhin modules (he recalled all the cases which have been proved so far), and a second part which says that, if a finite dimensional module over a quantum affine algebra has a crystal basis, then it must be a tensor product of Kirillov-Reshetikhin modules. His second conjecture gave a criterion for the perfectness of Kirillov-Reshetikhin crystals. The third one was about a connection, already established in type A and many other non-exceptional types, between perfect crystals and ground state paths; he illustrated it using an example in type $G_2^{(1)}$. Finally, he presented a conjectural fermionic character formula which has been resolved in some cases through the work of H. Nakajima, D. Hernandez, P. Di Francesco, R. Kedem, A. Schilling, M. Shimozono and himself. He explained some of the steps and ideas (Q-systems, rigged configurations) in the proof of those cases which have been established so far.

The day ended with a talk of a combinatorial and geometric nature by Professor Joel Kamnitzer (University of Toronto) about *Components of quiver varieties and affine Mirkovic-Vilonen polytopes*. G. Lusztig introduced quiver varieties whose components index the semicanonical basis for symmetric Kac-Moody Lie algebras. The speaker explained a method for understanding these components in finite and affine types using the combinatorics of Mirkovic-Vilonen polytopes. In the affine type, this gives a new combinatorics to describe crystals of affine Lie algebras, generalizing ideas of Beck-Chari-Pressley, Dunlap, and others.

He started by recalling Lusztig's bijection defined using the crystal structure on Lusztig's canonical basis for $U^+\mathfrak{g}$ (with $\mathfrak{g}$ finite) and stated a theorem of Lusztig and Berenstein-Zelevinsky which provides the connection between canonical bases and certain GGMS polytopes in $\mathfrak{h}^*_{\mathbb{R}}$ called Mirkovic-Vilonen (MV) polytopes. These are characterized by the fact that a GGMS polytope is an MV polytope if and only if all of its 2-faces are, and MV polytopes of rank 2 are known by an explicit rule. One of the goal of his research was to generalize them to the case when $\mathfrak{g}$ is an affine Kac-Moody algebra $\mathfrak{g}$. An important role is played by certain polytopes $Pol(M)$ associated to modules M over a preprojective algebra of the Dynkin quiver Q of $\mathfrak{g}$. His main theorem in the affine case consisted of two results, the first one being the existence of a bijection (constructed using these $Pol(M)$) between the canonical basis and affine MV polytopes and the second one stating that a decorated GGMS polytope is an affine MV polytope if and only if all of its 2-faces are. This is joint work with Pierre Baumann and Peter Tingley, who had more to say about this on Friday.

Wednesday, August 10

The first presentation of the day was delivered by Professor Benjamin Enriquez (Université de Strasbourg) and pertained to *Solutions of some problems in the quantization of Lie bialgebras*. A famous conjecture of V. Drinfeld going back twenty years ago [4] states that any Lie bialgebra could be quantized. It was proved in the 1990's in a series of papers by P. Etingof and D. Kazhdan using the theory of associators, see for instance [9, 10]. B. Enriquez started by giving an overview of the quantization problem which consists of constructing a functor from a classical object (e.g. a Lie bialgebra) to a quantum object (e.g. a bialgebra) such that the classical object can be recovered by taking the semi-classical limit of the quantum one. The other two families of classical objects that he considered are the Lie quasibialgebras and the coboundary Lie bialgebras, the corresponding quantum objects being quasibialgebras and coboundary bialgebras.

The existence of quantization for coboundary Lie bialgebras and quasi Lie bialgebras was solved by himself and G. Halbout in the impressive papers [6, 7]. After his introduction to the progress made on those quantization problems, he presented the formalism of PROPS which are symmetric tensor categories equipped with a functor from the Schur category which is the identity on objects and introduces a certain PROP which is relevant for the quantization problem of Lie bialgebra. (PROPS playing an analogous role exist also for the other classes of objects mentioned above.) The rest of this talks was devoted to explaining the main ideas in the proofs of their very important results using the formalism of PROPS and translating the quantization problem into this language.

Afterwards, Professor Valerio Toledano Laredo of Northeastern University gave a talk entitled *Yangians, quantum loop algebras and trigonometric connections*. Let G be a semisimple complex algebraic group (or $GL_n(\mathbb{C})$) with maximal torus H. He started with a general result about flat connections on H_{reg} and explained how the trigonometric Casimir connection, which is flat and W-equivariant, is constructed using Yangians [27]. In the case of $GL_n(\mathbb{C})$ and when the fiber of the vector bundle affording this connection is a tensor product of evaluation modules, the connections thus obtained is essentially the dynamical differential equation of V. Tarasov and A. Varchenko. He stated a conjecture to the effect that the action of the affine braid group on a representation V of the Yangian $Y_\hbar(\mathfrak{g})$ which comes from the Casimir connection is equivalent to its action via quantum Weyl group operators on a representation $\mathcal{V}$ of the quantum loop algebra $U_\hbar(L\mathfrak{g})$. This is reminiscent of a famous fundamental theorem of V. Drinfeld and Kohno [5] about the Knizhnik-Zamolodchikov connections and can be viewed as an affine extension of it. Matching those two classes of representations involves in particular the construction of a functor relating finite-dimensional modules of those two quantum groups. He devoted a good amount of time explaining a theorem of his which provides an isomorphism between a completion of the Yangian of $\mathfrak{g}$ and a completion of the corresponding quantum loop algebra (see [16]). This is a Lie algebra analog of Lusztig's isomorphism between an affine Hecke algebra and a completion of its associated degenerate version. That isomorphism induces an equivalence of categories between graded representations of the Yangian and filtered representations of the quantum loop algebra and it also induces Drinfeld's degeneration map between the Yangian and the associated graded ring of the quantum loop algebra with respect to the evaluation ideal. All these results are of the utmost importance in the representation theory of quantum groups of affine type: it had been believed for a long time that such an isomorphism existed after completion and that the categories of finite dimensional representations of Yangians and of quantum loop algebras were almost "the same" since the simple objects in both cases are parametrized by the so-called Drinfeld polynomials, but this had never been giving a rigorous treatment. He finished by presenting explicitly that isomorphism in the $\mathfrak{sl}_2$-case.

This is based on joint work with Sachin Gautam of Columbia University who was the next speaker and talked about *Monodromy of the trigonometric Casimir connection for* $\mathfrak{sl}(2)$, which is another joint project. He explained the proof of their theorem to the effect that the monodromy of the trigonometric Casimir connection for a tensor product of evaluation modules of the Yangian of $\mathfrak{sl}(2)$ is described by the quantum Weyl group operators of the quantum loop algebra. One of the main ideas of the proof is to use commuting action of $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$ in a polynomial ring with kn variables to relate the Casimir connection for $\mathfrak{sl}_n$ with the KZ-connection on n points and the Drinfeld-Khono theorem. He also explained how to extend the proof to the case of $\mathfrak{gl}_2$. In the course of the proof, he obtained an explicit expression for the lattice part of the affine braid group action.

Thursday, August 11

Professor Catharina Stroppel (Universität Bonn) was the first speaker of the day and she talked about *Fractional Euler characteristics and categorified colored Jones polynomials*. Some of the most fascinating applications of quantum groups can be found in topology, especially in the construction of manifold invariants. In Khovanov's categorification of the Jones polynomial, a polynomial invariant of links is upgraded to an invariant with values in complexes of graded vector spaces such that taking the graded Euler characteristic gives back the original polynomial. One would like to extend this construction to other invariants like colored Jones or Turaev-Viro 3-manifold invariants. The problem hereby is that the polynomial invariant (or at least its construction) is not defined integrally anymore, but it is defined instead over the rational numbers, hence one would like to interpret rational numbers as Euler characteristics and linear maps with not necessarily integral matrix entries as maps induced by functors on the Grothendieck group. These questions and their relevance in existing categorifications were addressed in her talk.

She started by recalling basic ideas about categorification and sketched a categorification of the $U_q\mathfrak{sl}_2$-module $\mathbb{Q}(q)^{\otimes 2}$ using derived categories built from various blocks of the category $\mathcal{O}$ for $\mathfrak{gl}_n$. She then raised the question of categorifying the whole of $\mathrm{Rep}(U_q\mathfrak{sl}_2)$ with a view towards 3-manifolds invariants. She explained some ideas about how to make sense of fractional Euler characteristics by working with completed Grothendieck groups [1] and certain intermediate subcategories of bounded derived categories. This is certainly going to be very useful for further work on categorification. Afterwards, she stated two major theorems of hers, obtained alongside I. Frenkel and J. Sussan [15]: the first one states how to categorify the Jones-Wenzl projector using certain Serre quotients of blocks of the category $\mathcal{O}$ for $\mathfrak{gl}_n$, and the second one gives a construction of a categorification of the colored Jones polynomial using tensor products of $U_q\mathfrak{sl}_2$-modules.

The last part of her talk dealt with applications of these results, in particular to categorification of $3j$-symbols, Θ-networks (interpreted as the Euler characteristic of $\mathrm{Ext}^*(L, L)$ for some simple Harish-Chandra bimodule L) and tetrahedron networks. Her work in this direction has applications in representation theory: for instance, $3j$ symbols can be viewed as generalizations of the Kazhdan-Lusztig polynomials.

She was followed by Professor Wolfgang Soergel from the University of Freiburg who spoke on *Koszul duality in positive characteristic*, mainly in the context of the category $\mathcal{O}$ of a reductive algebraic group. The main new point was a formality result for the derived category of sheaves on the complex analytic flag variety with coefficients in a finite field, constructible along the Bruhat stratification: the extension algebra of parity sheaves as a dg-ring with trivial differential already describes this triangulated category. The method to prove this is splitting by the action of the Frobenius, which can be done under very mild and explicit restrictions on the characteristic.

In the afternoon, Professor Igor Frenkel from Yale University gave an intriguing presentation on *Quantum groups associated to the split real Lie groups, their representations and future perspectives*. He outlined the beginning of a new ambitious program to study the representation theory of certain quantum groups when the norm of q is 1, extending to the case $|q| = 1$ many results already known when $|q| < 1$. Slightly more precisely, one of the main goals of his program is to construct q-deformations of minimal (spherical) principal series representations for split real Lie groups with properties similar to those in the compact case. This involves notions from functional analysis, namely the theory of positive self-adjoint operators. His program is expected to be connected to canonical bases, topological invariants and some notion of continuous categorification. He gave a very brief overview of work from the past twenty-five years which should be relevant for his program (e.g. Drinfeld double, equivalence of categories between quantum groups and affine Lie algebra).

Most of his talk focused on the case of $\mathfrak{sl}_2$ (both on $U_q\mathfrak{sl}_2$ and on $F_q^+(GL_2(\mathbb{R}))$), starting with an observation of Faddeev to the effect that a pair of quantum torus algebras (or modular double of quantum plane algebra) can be derived from the simplest Heisenberg algebra. This suggests to consider a similar modular double for $U_q\mathfrak{sl}_2$ and a family of representations realized via operators on $L^2(\mathbb{R})$ belonging to the quantum plane algebra and satisfying a certain transcendental relation. He presented concrete, explicit formulas involving properties of the quantum dilogarithm (a quotient of two q-deformations of gamma functions) which are relevant for the $\mathfrak{sl}_2$-picture. One surprising aspect is that only continuous series representations appear in the decomposition of the tensor product of two continuous series representations.

The last part of his talk was about the modular double of $F_q^+(GL_2(\mathbb{R}))$. He finished by stating a Peter-Weyl type theorem which provides a decomposition of $L^2(F_q(GL_2^+(\mathbb{R})))$ as a representation of the quantum group $U_{q\tilde{q}}\mathfrak{sl}_2(\mathbb{R})$. (This is partly based on work of Ivan Ip.)

Professor Alexander Molev (University of Sydney) followed with a talk about the *Feigin-Frenkel center for classical types*. For each simple Lie algebra $\mathfrak{g}$, consider the corresponding affine vertex algebra $V(\mathfrak{g})$ at the critical level. The center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of this vertex algebra is a commutative associative algebra whose structure was described about two decades ago by a remarkable theorem of B. Feigin and E. Frenkel [11] which is fundamental in the study of affine Lie algebras and has applications in the celebrated Langlands program [14]. That theorem states that the center is generated by rank($\mathfrak{g}$) Segal-Sugawara vectors and the translation operator given by the derivative. However, only recently simple formulas for the generators of the center were found for the Lie algebras of type A following Talalaev's discovery of explicit higher Gaudin Hamiltonians. AṀolev was able to obtain explicit formulas for generators (Sugawara operators) of the centers of the affine vertex algebras $V(\mathfrak{g})$ associated with the simple Lie algebras $\mathfrak{g}$ of types B, C and D and he presented those formulas at the workshop. (For $\mathfrak{gl}_N$, such formulas were obtained earlier by T. Talalaev using the Bethe subalgebra of the Yangian of $\mathfrak{gl}_N$.) The construction relies on the Schur-Weyl duality involving the Brauer algebra, and the generators are expressed as weighted traces over tensor spaces and, equivalently, as traces over the spaces of singular vectors for the action of the Lie algebra $\mathfrak{sl}(2)$ in the context of the Howe duality. He presented concrete examples to show what his formulas looked like. His explicit formulas could also be used to give a simpler proof of the theorem of B. Feigin and E. Frenkel. Applying the state-field correspondence map to a complete set of Segal-Sugawara operators in $\mathfrak{z}(\widehat{\mathfrak{g}})$ yield generators for the center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$. It also leads to an explicit construction of a commutative subalgebra of the universal enveloping algebra $U(\mathfrak{g}[t])$ and to higher order Hamiltonians in the Gaudin model associated with each Lie algebra $\mathfrak{g}$. It would be possible to introduce analogues of the Bethe subalgebras of the Yangians $Y(\mathfrak{g})$ and show that their graded images coincide with the respective commutative subalgebras of $U(\mathfrak{g}[t])$.

The day ended with a presentation by Professor Eric Vasserot (Université de Paris 7) on *Cyclotomic rational double affine Hecke algebras and categorification*. It was proved by P. Shan that, from the category $\mathcal{O}$ of a cyclotomic rational double affine Hecke algebra H, one can obtain a categorification of the quantum Fock space. It is conjectured in [28] that this category is equivalent to a certain subcategory of a parabolic category $\mathcal{O}$ at negative level of an affine Kac-Moody algebra. This subcategory can be seen as a higher analogue of the q-Schur algebra because the category of modules over the latter is known to be equivalent to a highest weight subcategory of the affine category $\mathcal{O}$ of GL_N at a negative level. This is a consequence of the famous work of D. Kazhdan and G. Lusztig [20] which provides an equivalence between the category of finite dimensional representation of $U_q\mathfrak{gl}_N$ and a certain affine category $\mathcal{O}$ for $\mathfrak{gl}_N$.

He started by recalling general results about rational Cherednik algebras, in particular a theorem of his and P. Shan [26] which describes the support of modules obtained via the induction and the restriction functors. He then gave a present statement of a conjecture of P. Etingof [8] to the effect that the number of irreducible modules in the category $\mathcal{O}$ for H with a given support is equal to the dimension of a certain vector space obtained from weight spaces of a representation of an affine Lie algebra. The rest of his talk was devoted to his proof [26] of this conjecture for cyclotomic H using a categorification of the Heisenberg algebra action on the Fock space and a reinterpretation of the support of modules in $\mathcal{O}$ in terms of actions of affine Lie algebras on the Fock space. This is a major result in the representation theory of Cherednik algebras since, from the conjecture of P. Etingof, one obtains a formula for the number of irreducible finite dimensional modules over H.

Friday, August 12

The first two talks on Friday morning were delivered by postdoctoral researchers. Dr. David Jordan of the University of Texas at Austin spoke on *Quantized multiplicative quiver varieties*. He started by recalling a "diamond" of degenerations relating the quantized enveloping algebra $U_q\mathfrak{g}$ with $\mathfrak{U}\mathfrak{g}$, $\mathbb{C}[G]$ and $\mathrm{Sym}(\mathfrak{g})$ for $\mathfrak{g}$ a semisimple Lie algebra. His goal was to obtain a similar diamond for quiver varieties, $\mathfrak{U}\mathfrak{g}$, $\mathbb{C}[G]$ and $\mathrm{Sym}(\mathfrak{g})$ being replaced by, respectively, the quantized quiver varieties of Gan-Ginzburg, the multiplicative quiver varieties of Crawley-Boevey-Shaw and Lusztig's quiver varieties. The role of $U_q\mathfrak{g}$ is played by new algebras $D_q(Mat_d(Q))$ associated to a quiver Q and dimension vector d which can be defined explicitly in terms of generators and relations. An important theorem about these is that they yield a flat (PBW) q-deformation of the algebra of differential operators on the space of matrices associated to Q and that, furthermore, a certain localization of $D_q(Mat_d(Q))$ quantizes a quasi-Poisson structure on an open subset of the cotangent bundle of that space of matrices. $D_q(Mat_d(Q))$ admits a q-deformed moment map from the quantum group $U_q(\mathfrak{gl}_d)$, acting by base change at each vertex. The quantum Hamiltonian reduction, $A_d^\xi(Q)$, of D_q by μ_q at the character ξ is simultaneously a quantization of the multiplicative quiver variety, and a q-deformation of the quantized quiver variety associated to Q.

Specific examples of the data (Q, d, ξ) yield q-deformations of important algebras in representation theory: for example, the spherical double affine Hecke algebra of type A_n may be obtained in this way when Q is the Calogero-Moser quiver. (This is a deformation of a construction of the spherical rational Cherednik algebra by Gan and Ginzburg.) Given the ubiquity of quiver varieties in geometric representation theory, it is natural to anticipate further connections. He ended his talk by presenting briefly the construction of a functor from the category of modules over $D_q(Mat_d(Q))$ to the category of representations of the elliptic Weyl group.

Afterwards, Dr. Peter Tingley of the Massachusetts Institute of Technology delivered a presentation on *Combinatorics of affine* $\mathfrak{sl}(2)$ *MV polytopes* based on his joint work with Pierre Baumann, Thomas Dunlap and Joel Kamnitzer. MV polytopes give a useful realization of finite type crystals, which are combinatorial objects related to representations of complex simple Lie algebras and their quantized enveloping algebras. Recent work of Baumann and Kamnitzer constructs MV polytopes from Lusztig's quiver varieties, which are well defined outside of finite type. This work has now been extended to give a definition of MV polytopes in all symmetric affine cases, and to show that understanding the resulting combinatorics reduces to understanding the $\mathfrak{sl}(3)$ and affine $\mathfrak{sl}(2)$ cases. (This was explained on Tuesday by J. Kamnitzer.) In his talk, P. Tingley gave a simple characterization of the polytopes and a description of the crystal operators in the affine $\mathfrak{sl}(2)$ case, thereby completing the picture in all symmetric affine cases. He also explained what the combinatorics meant in terms of quiver varieties.

The meeting concluded with a talk by Professor Ben Webster of Northeastern University about *Categorification, Lie algebras and topology*. It is a long established principle that an interesting way to think about numbers is as the sizes of sets or dimensions of vector spaces, or better yet, as the Euler characteristic of complexes. You cannot have a map between numbers, but you can have one between sets or vector spaces. For example, Euler characteristic of topological spaces is not functorial, but homology is functorial. One can try to extend this idea by taking a vector space and trying to make a category by defining morphisms between its vectors. This approach (interpreted suitably) has been a remarkable success within the representation theory of semi-simple Lie algebras and their associated quantum groups. This speaker gave an introduction to this area, with a view toward applications in topology, in particular to replacing polynomial invariants of knots that come from representation theory with vector space valued invariants that reduce to knot polynomials under Euler characteristic.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\dot{U}_q\mathfrak{g}$ be Lusztig's quantized enveloping algebra obtained by adding extra idempotents. The speaker started with an overview of the Chang-Rouquier-Khovanov-Lauda diagrammatic categorification of the universal enveloping algebra $\mathfrak{U}\mathfrak{g}$ and of $\dot{U}_q\mathfrak{g}$, insisting on the wonderful observation that the grading on their category $\mathcal{U}$ (whose Grothendieck group is $\mathfrak{U}\mathfrak{g}$) easily yields a graded version

$\widetilde{\mathcal{U}}$ whose Grothendieck group is $\dot{U}_q\mathfrak{g}$. He then explained briefly how to categorify, using again diagrams, the irreducible finite dimensional representations of $\mathfrak{g}$ with a given highest weight. He recalled how some of these categorifications where already known from previous work on the symmetric groups and Hecke algebras.

By analogy with how one obtains polynomial knot invariants from ribbon tensor categories of representations of $U_q\mathfrak{g}$, he proposed to construct quantum knot homologies from a categorification of tensor product of simples modules of $\dot{U}_q\mathfrak{g}$. This is envisioned as one step in a bigger schema to construct quantum knot homologies via conjectural ribbon 2-categories of representations of $\mathcal{U}$. For $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$, some of these categorifications can be obtained from familiar categories in classical representation theory (e.g. blocks of a parabolic category $\mathcal{O}$).

He finished by stating an amazing theorem, related to the conjecture of P. Etingof whose proof was explained on the previous day by E. Vasserot, which says that the category of finite dimensional representations of a symplectic reflection algebra for the wreath product of the symmetric group S_n with a finite subgroup Γ of $SL_2(\mathbb{C})$ is derived equivalent to a certain weight space of a categorification of a simple $\mathfrak{g}$-module.

4 Outcome of the meeting

The workshop was an occasion for researchers to get an overview of some important recent progress in the representation theory of quantum groups. Many positive comments have been received by the organizers to that effect. For instance, Professor Evegeny Mukhin (Indiana University - Purdue University Indianapolis) commented that “The conference was great. Everything was right – the organization, the participants, the timing, etc. It was my fourth time at BIRS and I would come again anytime. I have surely learned many things and even started a couple of new projects. It is too early to say how they will go, but it is certainly a very interesting twist.” Some participants also took the opportunity to exchange ideas relevant to their research, work on projects with their collaborators or start new ones, as exemplified by the following testimonials.

“I had some fruitful discussions with my collaborator Catharina Stroppel, in which we made some progress towards the categorification of matrices using Harish-Chandra bimodules. I also had some useful conversations with Weiqiang Wang, through which I learnt of some exciting new progress that he has made in the representation theory of Lie superalgebras.” Professor Jonathan Brundan, University of Oregon.

“It was a great workshop and very productive for me. Peter Tingley and I worked on our papers on affine MV polytopes, which should be completed soon. Ben Webster and I worked on our project involving quantization of slices in the affine Grassmannian using Yangians. During a hike, I talked to Evgeny Mukhin about monodromy of Bethe vectors. This conversation continued with Valerio Toledano Laredo the next day, and I hope that it will lead to a future project.

On the bus to the airport, I had a very good discussion with Jon Brundan, which gave me a much better understanding of how the category $\mathcal{O}$ work on categorification (see papers of Brundan, Kleshchev, Stroppel, Frenkel, Sussan, etc.) fits in with the affine Grassmannian and I hope this will lead to some future projects.” Professor Joel Kamnitzer, University of Toronto.

“The workshop has been a wonderful experience for me. It brought together leaders in several areas of representation theory; their talks provided a comprehensive and inspiring picture of the cutting edge research in the field. I received a valuable feedback for my own talk from several participants, including David Jordan, Valerio Toledano Laredo, Maxim Nazarov, Weiqiang Wang and Milen Yakimov. Through conversations with these people I discovered new approaches to develop a promising direction of research which will involve quantum versions of centers of vertex algebras.” Professor Alexander Molev, University of Sydney.

“I was able to meet with my collaborator Anne Schilling and we could discuss our ongoing research about the conjecture on the equality of one-dimensional sums and generating functions of rigged configurations. The presentations delivered by Toledano Laredo, Gautam and Frenkel were completely new to me and they will inspire my future research. The presentation delivered by Kamnitzer will become useful for my research about the combinatorial structure of Kirillov-Reshetikhin crystals.” Professor Masato Okado, Osaka University.

“Igor Frenkel made some useful remarks regarding my talk afterwards. Also, Peter Tingley had interesting comments regarding my talk on how to possibly generalize work to the nonperfect setting. We are still in discussion about this via e-mail. It was very useful for me to discuss some aspects of the crystal commutator with Joel Kamnitzer.” Professor Anne Schilling, University of California at Davis.

"The conference had many interesting talks, and was productive for me in several ways. Most particularly, I was able to meet with Joel Kamnitzer, and we made significant progress on our joint project on affine MV polytopes. Also, Anne Schilling's presentation gave me some new ideas which may be useful in understanding a relationship between Demazure characters and Macdonald polynomials in type C." Dr. Peter Tingley, Massachusetts Institute of Technology.

"From Eric Vasserot's talk I learned about results which will be very relevant to classifying the possible support sets of modules over the cyclotomic rational Cherednik algebra. I was also inspired by the talks in general to try to use more abstract machinery." Dr. Stewart Wilcox, University of Alberta.

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