Gorenstein Homological Algebra

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1 Overview of the Field

Homological algebra is at the root of modern techniques in many areas of mathematics including commutative and non commutative algebra, algebraic geometry, algebraic topology and representation theory. Not only that all these areas make use of the homological methods but homological algebra serves as a common language and this makes interactions between these areas possible and fruitful. A relative version of homological algebra is the area called Gorenstein homological algebra. This newer area started in the late 60s when Auslander introduced a class of finitely generated modules that have a complete resolution. Auslander used these modules to define the notion of the G-dimension of a finite module over a commutative noetherian local ring. Then Auslander and Bridger extended the definition to two sided noetherian rings (1969). The area really took off in the mid 90s, with the introduction of the Gorenstein (projective and injective) modules by Enochs and Jenda ([1]). Avramov, Buchweitz, Martsinkovsky, and Reiten proved that if the ring R is both right and left noetherian and if G is a finitely generated Gorenstein projective module, then Enochs' and Jenda's definition agrees with that of Auslander's and Bridger's of module of G-dimension zero. The Gorenstein flat modules were introduced by Enochs, Jenda and Torrecillas as another extension of Auslander's Gorenstein dimension.

The Gorenstein homological methods have proved to be very useful in characterizing various classes of rings. Also, methods and results from Gorenstein homological algebra have successfully been used in algebraic geometry, as well as in representation theory. But the main problem in using the Gorenstein homological methods is that they can only be applied when the corresponding Gorenstein resolutions exist. So the main open problems in this area concern identifying the type of rings over which Gorenstein homological algebra works or, more generally, other categories where these methods can be applied. Of course one hopes that this is the case for any ring. The existence of the Gorenstein resolutions are still open problems. And they have been studied intensively in recent years. These are also the problems we considered.

2 Recent Developments and Open Problems

As already mentioned, we considered some of the main open problems in Gorenstein homological algebra - the existence of the Gorenstein resolutions.
We recall that the Gorenstein (projective, injective, flat) modules are defined in terms of totally acyclic complexes. We recall that an acyclic complex P of projective R-modules (R is an arbitrary ring) is called *totally*

acyclic if the complex $Hom(P, Q)$ is still exact for any projective module Q. A totally acyclic complexes of injective modules is defined dually. And an exact complex F of flat left R-modules is said to be F*-totally acyclic* if $I \otimes F$ is exact for any injective right R-module I. A module M is Gorenstein injective if and only if it is a cycle of a totally acyclic complex of injective modules. Dually, a module G is Gorenstein projective if it is a cycle of a totally acyclic complex of projective modules. And a Gorenstein flat module is a cycle of an F-totally acyclic complex of flat modules.
A Gorenstein ring (in the sense of Iwanaga, [5] and [6]) is a two sided noetherian ring R that has finite self injective dimension on both sides. Over such a ring the exact complexes of projective (injective, flat) modules have some very nice homological properties. More precisely, over a Gorenstein ring every acyclic complex of projective (injective) modules is totally acyclic. And every acyclic complexes of flat modules is F-totally acyclic over any Gorenstein ring. So over such a ring the class of Gorenstein projective (injective, flat) modules coincides with that of the cycles of acyclic complexes of projectives (injective, flat modules respectively).

It is a natural question to consider whether or not these conditions actually characterize Gorenstein rings, or more generally whether or not it is possible to characterize Gorenstein rings in terms of acyclic complexes of (Gorenstein) injectives, (Gorenstein) projectives and (Gorenstein) flats. We considered this question and we gave several characterizations of the rings with this property.

3 Presentation Highlights

Since this was a Research in Teams workshop for three people, there were no formal presentations.

4 Scientific Progress Made

The major contribution made at this RIT was the completion of a paper that has been in progress for a while. This paper is submitted now.
The participants also worked on a second paper that is still in progress.

Participants Estrada and Iacob are two of the three authors of the paper "Totally acyclic complexes" ([2]). As noted above the main problem considered here is : What is the most general type of ring R with the property that the Gorenstein projective (injective, flat respectively) are simply the cycles of exact complexes of projective (injective, flat respectively) modules?
It is worth mentioning that over such rings the class of Gorenstein injective modules is both covering and enveloping, and the class of Gorenstein flat modules is preenveloping. This guarantees the existence of the minimal Gorenstein injective left and right resolutions, and also, the existence of the right Gorenstein flat resolutions.

We proved first ([2], Proposition 3) that, over any ring R, an acyclic complex of projective modules is totally acyclic if and only if the cycles of every acyclic complex of Gorenstein projective modules are Gorenstein projective. The dual result for injective and Gorenstein injective modules also holds over any ring R ([2], Proposition 4). And, when R is a GF-closed ring, the analogue result for flat/Gorenstein flat modules is also true (Proposition 5). Then we showed (Theorem 2) that over a left noetherian ring R, a third equivalent condition can be added to those in Proposition 3, more precisely, we proved that the following are equivalent: 1. Every acyclic complex of injective modules is totally acyclic. 2. The cycles of every acyclic complex of Gorenstein injective modules are Gorenstein injective. 3. Every complex of Gorenstein injective modules is dg-Gorenstein injective.

Theorem 3 shows that the analogue result for complexes of flat and Gorenstein flat modules holds over any left coherent ring R. We prove (Corollary 1) that, over a commutative noetherian ring R, the equivalent statements in Theorem 3 hold if and only if the ring is Gorenstein. We also prove (Theorem 4) that when moreover R is left coherent and right n-perfect (that is, every flat right R-module has finite projective dimension $\leq n$) then statements 1, 2, 3 in Theorem 2 are also equivalent to the following: 4. Every acyclic complex of projective right R-modules is totally acyclic. 5. Every acyclic complex of Gorenstein projective right R-modules is in $\widetilde{\mathcal{GP}}$. 6. Every complex of Gorenstein projective right R-modules is dg-Gorenstein

projective.
Corollary 2 shows that when R is commutative noetherian of finite Krull dimension, the equivalent conditions (1)-(6) from Theorem 4 are also equivalent to those in Theorem 3 and hold if and only if R is an Iwanaga-Gorenstein ring. Our Corollary 2 improves on results by Iyengar and Krause ([4]) and by Murfet and Salarian ([7]). Iyengar and Krause proved that for a commutative noetherian ring R with a dualizing complex, the class of acyclic complexes of injectives coincides with that of totally acyclic complexes of injectives if and only if R is Gorenstein. Then Murfet and Salarian removed the dualizing complex hypothesis and characterized Gorenstein rings in terms of totally acyclic complexes of projectives. We are adding more equivalent characterizations, still under the assumption that R is commutative noetherain of finite Krull dimension.

In the second part of the paper we focus on two sided noetherian rings that satisfy the Auslander condition. We prove (Theorem 7) that for such a ring R that also has finite finitistic flat dimension, every complex of injective (left and respectively right) R-modules is totally acyclic if and only if R is an Iwanaga-Gorenstein ring.
The paper [2] was submitted for publication shortly after the workshop. A preprint is available on arXiv.

Further progress in studying the existence of the Gorenstein flat precovers was made by the three participants in the workshop at BIRS (Bravo, Estrada, Iacob), and a paper on these new results is being prepared ([3]). We consider a Grothendieck closed symmetrical monoidal category $(\mathcal{C}, -\otimes -, [-,-])$. This important class of categories includes, among others, categories of modules over a commutative ring, sheaves of $\mathcal{O}_X$-modules, quasi-coherent sheaves of $\mathcal{O}_X$-modules, comodules over a flat Hopf algebroid, and categories of representations by modules of arbitrary quivers. An object X of $\mathcal{C}$ is flat if the functor $X \otimes -$ is exact. Given a class of objects $\mathcal{A}$, let $\widetilde{\mathcal{F}}$ denote the class of all acyclic complexes of flat objects F such that $A \otimes F$ is still acyclic, for any $A \in \mathcal{A}$. We show that the class $\widetilde{\mathcal{F}}$ is covering. Moreover, if $\mathcal{C}$ has a flat generator then $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}^{\perp})$ is a hereditary perfect cotorsion pair and there is an induced cofibrantly generated abelian model category structure with $\widetilde{\mathcal{F}}$ as the class of cofibrant objects. These results are in [3] which is still in progress.

The three participants also considered duality pairs and used the connections between duality pairs and cotorsion pairs in order to prove the existence of precovers/preenvelopes with respect to some classes of modules: FP_n-injective and flat modules, Gorenstein AC-injective and flat modules, Ding-injective modules. This is still work in progress.

5 Outcome of the Meeting

We expect that the paper [2] submitted shortly after the workshop will be accepted in 2016.
The three participants in the workshop plan to finalize our joint work on the existence of the Gorenstein flat precovers, and submit it as soon as possible.

The fact that we could meet and discuss our ideas was very valuable. We thank BIRS for hospitality and for giving us the opportunity to advance our work in such a pleasant setting.

References

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