Synchronizing smooth and topological 4-manifolds

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1 Overview of the Field

Much is still unknown about 4-dimensional manifolds. Indeed, while the 3-dimensional Poincaré conjecture was settled in the affirmative by Perelman at the turn of the 21st century, the smooth 4-dimensional Poincaré conjecture remains open. Furthermore, the question of classifying the set of smooth structures on a 4-manifold remains a mystery; there is no smooth 4-manifold for which the set of smooth structures is completely known. Knot and link concordance has proved to be a powerful tool for understanding 4-manifolds; for example, any topologically but not smoothly slice knot gives rise to an exotic $\mathbb{R}^4$.

A primary goal of this workshop was to bring together researchers in smooth and topological 4-manifolds in order to facilitate interactions between experts in both areas. The list of invited participants included both well-established senior experts in the field together with rising new talent, including a large number of postdocs and graduate students. We made a particular effort to ensure that all of the young mathematicians present were given the opportunity to present.

From the smooth perspective, Heegaard Floer homology has been particularly successful in yielding new insights into concordance. Several new invariants and results were presented at this workshop, particularly the talks of Feller, Hendricks, Levine, Sato, Stipsicz, and Wang. In particular, the Upsilon invariant of Ozsváth-Stipsicz-Szabó has yielded many new results (e.g. [1], [2], [3]) and the $\underline{V}_0$ and $\overline{V}_0$ invariants of Hendricks-Manolescu seem well-positioned to yield new results as well.

The global structure of the knot concordance group can be understood by considering geometric filtrations related to gropes and Whitney towers, and the closely related solvable filtration. After a survey on this theory by Jae Choon Cha, recent progress on our understanding of these filtrations was presented by Taehee Kim, relating to a filtration of concordance classes of knots that models the doubly slice question, and Chris Davis presented work on the difference between 0.5 and 1-solvability. Slava Krushkal presented recent progress on the surgery conjecture in 4-manifolds. This conjecture is key to understanding topological manifolds with arbitrary fundamental groups, and the techniques used to study it have strong connections to link concordance.

David Gay, Rob Kirby, Jeff Meier, Juanita Pinzón Caicedo, and Alex Zupan made up a strong contingent of experts in the recently blossoming theory of trisections of 4-manifolds. This was an opportunity, through a series of excellent talks, for many not directly involved with the development of the theory to learn about recent advances, and for the practitioners to get together. This is an exciting theory which enables one to understand 4-manifolds in terms of curves on surfaces, and it is hoped that it will enable us to attack some of the hard problems in 4-manifold topology that have hitherto proved intractable.

The conference helped fuel a great deal of intra- and inter- action involving those currently thinking about these different aspects of knot concordance and 4-manifolds.

2 Presentations

- **Jae Choon Cha** opened the workshop with an excellent overview of topological concordance of knots and links. He discussed disk embedding in dimension 4 and filtrations arising from gropes and Whitney towers. His talk also featured numerous open problems (see Section 3.1), which helped to kick off a week of fruitful interactions and discussions amongst the participants.

- **Christopher Davis** spoke on 1-solvability and genus one algebraically slice knots; this is joint with Taylor Martin, Carolyn Otto, and Jung Hwan Park. In the 1990's Cochran Orr and Teichner introduced a filtration of knot concordance indexed by half integers (the solvable filtration.) Since then this filtration has been a convenient setting for many advances in knot concordance. There are now many results in the literature demonstrating the difference between the nth and $(n.5)$th terms in this filtration, but none regarding the difference between the $(n.5)$th and $(n+1)$st. Davis, Martin, Otto, and Park prove that every genus one (0.5)-solvable knot is 1-solvable. They also provide a new sufficient condition for a high genus (0.5)-solvable knot to be 1-solvable and give some possible candidates for knots which are (0.5)-solvable but not 1-solvable.

- **Andrew Donald** spoke on a slicing obstruction from the 10/8 theorem. This is joint with Faramarz Vafaee. A smooth knot slicing obstruction can be derived from Furuta's 10/8 theorem using 0-surgery on knots. They show that this detects torsion elements in the concordance group and can be used to find topologically slice knots which are not smoothly slice.

- **Peter Feller** spoke on joint work with David Krcatovich in which they use the Upsilon invariant to provide bounds on cobordisms between knots that "contain full-twists". They recover and generalize a classical consequence of the Morton-Franks-Williams inequality for knots: positive braids that contain a full twist realize the braid index of their closure. They also provide inductive formulas for the Upsilon invariants of torus knots and compare the Upsilon function to the Levine-Tristram signature profile.

- **Stefan Friedl** spoke on a conjectural "if and only if criterion" for topological concordance to the unknot and the Hopf link; the conjecture was originally stated by Peter Teichner and Friedl in 2004. In his talk, Friedl provided evidence for the conjecture and reported on rather preliminary work with Patrick Orson on extending this conjecture to the concordance to the Hopf link.

- **Kristen Hendricks** gave a talk on involutive Heegaard Floer homology, in which she and Ciprian Manolescu use the conjugation symmetry on the Heegaard Floer complexes to define a three-manifold invariant. Within this package of invariants are two new invariants of homology cobordism, one of which (unlike other invariants arising from Heegaard Floer homology) detects non-sliceness of the figure-eight knot. These homology cobordism invariants give rise to knot concordance invariants by considering surgery along the knot; the computation of these knot invariants depends on the knot Floer complex CFK^∞, together with an endomorphism ι_K.

- **Francesco Lin** gave a talk on Pin(2) monopole Floer homology, the Morse-theoretic analogue of Manolescu's Pin(2)-equivariant Seiberg-Witten-Floer homolog. It can be used to provide an alternative disproof of the longstanding Triangulation Conjecture. He also discussed some computational tools for the theory.

- **Taehee Kim** spoke on joint work with Jae Choon Cha on unknotted gropes and Whitney towers in 4-space. Gropes and Whitney towers are primary tools for studying 4-dimensional topology. As an effort to understand gropes and Whitney towers via the structure of their complements, they introduce notions of unknotted gropes/Whitney towers in 4-space. This is motivated by Freeman's result that an embedded 2-sphere in 4-space is topologically unknotted if its complement has infinite cyclic fundamental group. As an application, they establish grope and Whitney tower bi-filtrations of knots in 3-space by taking a slice of unknotted gropes/Whitney towers. Using the amenable signature theorem by Cha, which is based on the work of Cha and Orr, they prove that these bi-filtrations have rich structures.

- **Slave Krushkal** gave two talks, the first on 1/2 -π_1-null surgery kernels, and the second on a homotopy$^+$ solution to the A-B slice problem, both joint work with Mike Freedman. It has been known for a long time that π_1-null surgery kernel imply surgery (pg 94 Freedman-Quinn book) and also that weaker grope based kernels are"universa" for surgery (if you can solve these problems, you can solve all unobstructed surgery problems.) Freedman and Krushkal have shown that a kind of kernel"half way betwee" the two is still universal.

Four-dimensional surgery is a fundamental technique underlying geometric classification results for topological 4-manifolds. It is known to work in the topological category for a class of "good" fundamental groups. This result was originally established in the simply-connected case by Freedman in 1981, and it is currently known to hold for groups of subexponential growth and a somewhat larger class generated by these. The A-B slice problem is a reformulation of the surgery conjecture for free groups, which is the most difficult case. In this talk Krushkal showed that the A-B slice problem admits a link-homotopy+ solution. The proof relies on geometric applications of the group-theoretic 2-Engel relation. He also discussed implications for the surgery conjecture.

- **Adam Levine** spoke on satellite operators and piecewise-linear concordance. He shows that there exists a knot in a homology sphere Y, which is the boundary of a contractible 4-manifolds, such that K does not bound a piecewise-linear disk in any homology 4-ball bounded by Y. His proof relies on a computation of the concordance invariants τ and ε using bordered Floer homology, which shows that a certain satellite operator does not induce a surjection on the knot concordance group.

- **Lukas Lewark** spoke on joint work with Peter Feller on upper bounds for the topological slice genus of knots. In 1981, Freedman proved that knots with trivial Alexander polynomial bound a locally flat disc in the four-ball. As a consequence, Feller showed that the degree of the Alexander polynomial constitutes an upper bound for the topological slice genus of a knot. Lewark and Feller proved a stronger bound, which is still determined solely by the knot's Seifert form. Their work leads to upper bounds for the slice genus of torus knots (Baader, Feller, Lewark, Liechti) and two-bridge knots (Feller, Mccoy), and for the stable slice genus of alternating knots (Baader, Lewark).

- **Andrew Lobb** gave a talk on Khovanov-Rozansky smooth sliceness obstructions; this is joint work with Lukas Lewark. Rasmussen's invariant from perturbed Khovanov cohomology is a concordance homomorphism to the integers which also gives a lower bound on the smooth slice genus. Khovanov-Rozansky $sl(n)$ cohomology generalizes Khovanov cohomology (which appears as the case $n = 2$) and perturbations of it give rise both to a slew of concordance homomorphisms which are also lower bounds as well as to lower bounds which are not equivalent to concordance homomorphisms. For the case $n = 2$ there is essentially only one perturbation, while already perturbations of the case $n = 3$ exhibit complicated behavior.

- **Jeff Meier**, **Juanita Pinzón Caicedo**, and **Alex Zupan** all gave talks on trisections of 4-manifolds. Pinzón Caicedo spoke on joint work with Nick Castro in which they develop a definition of relative trisections for 4-manifolds with boundary, and prove a uniqueness result in terms of stabilizations. Meier spoke on joint work with Zupan, where they adapt the theory of trisections to the relative setting of knotted surfaces in the four-sphere. Their theory serves as a four-dimensional analogue to bridge splittings of classical knots and links: every such surface admits a decomposition into three standard pieces called a bridge trisection. Zupan spoke on joint work with Meier and Trent Schirmer. They show that a given link has Stable Generalized Property R if and only if a certain infinite family of induced trisections is nonstandard.

- **Matthias Nagel** spoke on unlinking information from 4-manifolds. He explained how to obtain lower bounds on unlinking numbers through 4-manifold techniques using a generalization of a theorem of Cochran-Lickorish. He demonstrated the method using links from Kohn's table whose unlinking numbers have only recently been determined through these methods.

- **Daniel Ruberman** spoke on two results related to 4-manifolds with boundary. The first, joint with Dave Auckly, Hee Jung Kim, and Paul Melvin, is a construction of diffeomorphisms of finite order on the boundary of certain contractible manifolds that change their smooth structure relative to the boundary. Tange has recently announced a similar result. They show in fact that for any finite group G acting on the 3-sphere, there is a G-action on the boundary of a contractible manifold, such that every element changes the smooth structure relative to the boundary. Their construction initially produces reducible boundaries, and then they show how to make these hyperbolic. The second set of results, joint with Arunima Ray, is concerned with two analogues of Dehn's lemma for 4-manifolds. They give examples of a reducible 3-manifold Y bounding a 4-manifold W that does not split smoothly as a boundary-connected sum, even though the reducing sphere in Y is null-homotopic in W. By a different construction, they find a contractible 4-manifold W with boundary a 3-manifold Y containing an essential torus that doesn't bound (smoothly, in one version; topologically in another version) a solid torus in W.

- **Kouki Sato** gave a talk on Heegaard Floer correction terms of 1-surgeries along $(2,q)$-cablings. The Heegaard Floer correction term (d-invariant) is an invariant of rational homology 3-spheres equipped with a Spin^c structure. In particular, the correction term of 1-surgeries along knots in the 3-sphere is a ($2\mathbb{Z}$-valued) knot concordance invariant d_1. In this work, Sato estimates d_1 for the $(2,q)$-cable of any knot K. This estimate does not depend on the knot type of K. If K belongs to a certain class which contains all negative knots, then equality holds. By using this estimate, Sato obtain two corollaries. One of the corollaries shows that the relationship between d_1 and the Heegaard Floer tau invariant is very weak in general. The other one gives infinitely many knots which cannot be unknotted either by only positive crossing changes or by only negative crossing changes.

- **Minkyoung Song** gave a talk on invariants and structures of the homology cobordism group of homology cylinders. The homology cobordism group of homology cylinders is enlargement of both the mapping class group and the concordance group of string links in homology $D^2 \times I$. Song studies the structure of the group via a filtration of extended Milnor invariants combined with Johnson homomorphisms, and also obtains deeper information invisible to previously known invariants by employing Hirzebruch-type intersection form defect invariants

- **Laura Starkston** spoke on line arrangements in the topological, smooth, and symplectic categories. This is joint work with Danny Ruberman. A complex line arrangement is a collection of complex projective lines in $\mathbb{CP}^2$ which may intersect at points of multiplicity greater than two. The combinatorial arrangements which can be geometrically realized and their space of realizations have been studied classically. They define symplectic, smooth, and topological versions of complex line arrangements in $\mathbb{CP}^2$, and studied their realizability. While one might hope that these more flexible categories allow us to realize any combinatorics, they showed that there are obstructions to topological realizations of many combinatorial arrangements. Many open questions remain about realizability in different categories.

- **András Stipsicz** gave a talk on the Upsilon invariant, which provides a homomorphism from the smooth knot concordance group to the group of piecewise-linear functions from $[0,2]$ to $\mathbb{R}$. The invariant comes from applying a 1-parameter family of linear transformations to the knot Floer bifiltered chain complex CFK^∞.

- **Shida Wang** spoke on semigroups of iterated torus knots and the Upsilon invariant. He discussed the usage of semigroups and some subtleties in the computation of the Upsilon invariant for torus knots. He also gave some nontriviality results on the kernel of the Upsilon invariant.

3 Recent Developments and Open Problems

The following open problems were discussed at the workshop.

3.1 Jae Choon Cha's talk

1. The π_1-null disc lemma for free groups, or perhaps for amenable groups.

2. Are good boundary links freely slice?

3. Whitehead double conjecture: A Whitehead double of a link is freely slice if and only if the link is homotopically trivial.

4. Is the map from $\mathcal{C}$ to the concordance of knots in $\mathbb{Z}$-homology 3-spheres an injection?
5. Develop a homology surgery machinery in dimension 4.
6. Construct higher order invariants of boundary links beyond those determined by the map to the free group.
7. Is there finite order in $\mathcal{C}$ other than the order 2 elements given by negative amphichiral knots?
8. Do all algebraic order 4 knots have infinite order in the knot concordance group?
9. Can the amenable signature theorem be used for non-solvable groups to obstruct slicing, or perhaps bounding certain types of Whitney towers?
10. Is the intersection of the grope, Whitney tower or n-solvable filtration trivial?
11. Does homology cobordism of zero surgeries preserving the homotopy classes of the meridians imply concordance?

3.2 Jae Choon Cha problem session

1. Is it true that K slice if and only if $BD_n(K)$ is slice for all n? It is known that $BD_n(K)$ is $\mathbb{Q}$-slice (bounds a disc in a rational homology 4-ball). Then $BD_{n-1}(K)$ is rationally slice. Thus $K\#K^r$ is $\mathbb{Q}$-slice.
2. He drew a knot a bit like a twist knot with an a-fold clasp and $-a$ twists. With $a = 1$ it is the figure eight. Now let K_a be the $(2,1)$ cable of that knot. It is known that K_a is $\mathbb{Q}$-slice. The answer is probably no because Miyazaki, using Casson and Gordon, showed that K_1 is not homotopy ribbon.

   Also, M_{K_a} is 0-surgery cobordant to $S^1 \times S^2$ via W such that $H_1(W) = \mathbb{Z} \oplus \mathbb{Z}_2$ and $H_2(W) = \mathbb{Z}$. The meridians are not homotopic. When you find a new concordance invariant please compute it on a couple of these knots and email Jae Choon.

3.3 Jeff Meier

1. Is there a unique surface with bridge trisection number 3?
2. Is a knot of a connect sum of n copies of $\mathbb{RP}^2$s in S^4 with $\pi_1 = \mathbb{Z}/2$ unknotted i.e. (topologically) equivalent to the standard embedding of such a surface?

3.4 Mark Powell

1. Are the iterated graded quotients of the bipolar filtration of topologically slice knots non trivial, or even better of infinite rank?

3.5 Adam Levine

1. If K_1 is concordant to K_2, then for all n, $S^3_n(K_1)$ is homology cobordant to $S^3_n(K_2)$. Is the converse true?

   Note that for all n, there exists K_1 and K_2 such that $S^3_n(K_1)$ is homology cobordant to $S^3_n(K_2)$ but K_1 and K_2 are not concordant. (Cochran-Franklin-Hedden-Horn for $n = 0$). But you have to change the knots for different n. Challenge: find a single pair that works for multiple n.

   Let P be a winding number 1 satellite operator. Let $P(K,t)$ be the t-twisted satellite of K. Suppose that $P(U,-n)$ is slice. Then for any knot K, $S^3_n(K)$ is homology cobordant to $S^3_n(P(K))$.
2. Can we find P such that $P(U,-n)$ is slice for all n, but $P \cup \eta$ (where η is the meridian of the solid torus in which P sits) not concordant to the Hopf link?

3.6 Matt Hedden

1. Same question as Adam for branched covers of knots. If all the branched covers are homology cobordant then are the knots concordant?

2. Follow up from Kent (see below, Matt stood up once after Adam and once after Kent). Does there exist a P in a solid torus such that $P\colon \mathcal{C} \to \mathcal{C}$ is a non-trivial homomorphism, not equal to the identity nor to zero? He would conjecture no.

3.7 Kent Orr

1. Related to work of Davis and Ray. Rough question: can one put a structure on the "family of satellite operators" so that $\mathcal{C}^{top}$ is a "module" over this structure. One has to also consider string links, by the work of John Burke and Diego Vela.

2. Cochran-Harvey-Leidy. Fractal structure conjecture. The primary decomposition of the iterated quotients of the n-solvable filtration gives evidence. Can we define interesting metrics that enable us to make the fractal structure precise? Does the family of satellite operators have a fractal structure?

3.8 Danny Ruberman

1. Let Y be an integral homology 3-sphere that bounds a contractible 4-manifold W, i.e., $\partial W = Y$. Suppose that K in Y is a knot. Does K bound an embedded disc in W? (No locally flat requirement.)

   Recall the proof that any homology sphere bounds a contractible 4-manifold. Take a product $Y \times I$ then do surgery to make it a homology cobordism with $\pi_1 = 0$. Stack infinitely many and then use the proper h-cobordism theorem to recognise the end as $S^3 \times \mathbb{R}$ and add in a point.

2. Given (Y, K) as above, does there exist a 1-connected h-cobordism X from Y to Y containing a concordance from K to K?

3. Real version. Suppose Y has a $\mathbb{Z}_n$ action with fixed set a knot K. Does the $\mathbb{Z}_n$ action extend over $W \simeq *$? (Yes if $\mathbb{Z}_n$ acts freely.)

4. (Also from Adam Levine's talk.) Is every knot in an integral homology sphere topologically concordant in a homology cobordism to a knot in S^3?

3.9 Shida Wang

1. In his talk, Shida Wang conjectured that L-space iterated torus knots are the only L-space knots whose set of exponents of the Alexander function are closed under addition. Based on conversations with David Krcatovich, Wang tried more potential counterexamples to his conjecture.

3.10 Slava Krushkal

1. Are Whitehead doubles of all homotopically trivial links are topologically slice?

4 Scientific Progress Made

- Dave Gay and Rob Kirby finished the last piece of a puzzle and proved that a trisection of a finitely presented group determines a unique smooth, closed, oriented 4-manifold, in analogy with Heegaard splittings which can be regarded as a bisection of a 3-manifold group. This was a question that Dave Gay asked at an open problems session, and Gay writes that "the initial burst of discussion that followed from me asking the question really helped push us to figure this out. One nice nugget of a corollary is that the smooth 4-dimensional Poincare conjecture can now be stated as a purely group theoretic question."

- Alexander Zupan and Jeff Meier clarified a picture of a potentially non- standard trisection of S^4 while working late into the night at Banff. Meier also began a correspondence with Kouki Sato regarding the calculation of correction terms for certain 3-manifolds obtained as surgery on links of two components.
- Paul Melvin and Danny Ruberman found time to discuss their work on equivariant corks, and to make progress on sharpening the results that Ruberman spoke about in his lecture. Laura Starkston and Ruberman spent considerable time finishing the details of a ‘complexification’ result for real pseudoline configurations; this was part of Starkston’s lecture at the conference. They were able to improve their result to show that a topological configuration of circles in the real projective plane can be isotoped so that it is the intersection of the real projective plane with a collection of symplectic 2-spheres in the complex projective plane.
- Lukas Lewark and Andrew Lobb benefitted in particular from the discussions of the Upsilon invariant, which have influenced their current research.
- The work in Shida Wang’s talk was posted on the arXiv [5] after he collected people’s feedback in the workshop. After Shida Wang’s talk, David Krcatovich asked how to determine whether a numerical semigroup arises from that of an iteratered torus knot. In their further discussion, David Krcatovich wondered whether the Upsilon function of any knot is a linear combination of that of L-space iterated torus knots. These problems are still under investigation and may begin a new collaborative project.
- Peter Feller and Lukas Lewark continued on our project about upper bounds for the topological slice genus.
- Kristen Hendricks and Jen Hom conjectured a formula for the behavior of ι_K under connected sum. Further discussions after the workshop between Hendricks and Ian Zemke refined this conjecture, and Zemke is currently writing up a proof of the conjecture.
- Francesco Lin benefitted from discussions with Matt Hedden and Mark Powell, resulting in the [4], where he studies the behavior of Manolescu’s correction terms under certain Dehn surgeries, with applications to homology cobordism, Seifert fibered surgeries and concordance invariants.

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