Geometric Tomography

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1 Overview of the Field

Geometric Tomography is the area of Mathematics where one investigates properties of solids based on the information about their sections and projections. It shares ideas and methods from many fields of Mathematics, such as Differential Geometry, Functional Analysis, Harmonic Analysis, Combinatorics and Probability. But the most significant overlap is with Convex Geometry and in particular with the classical Brunn-Minkowski theory. The workshop brought together a number of top researchers as well as students and postdocs with the aim of discussing most recent developments in the area.

The topics of the workshop included harmonic analysis on the sphere, spherical operators and special classes of bodies, geometric inequalities, discrete geometry, probability and random matrices.

2 Presentation Highlights

We start the description with the results of Mark Agranovsky. Let f be a continuous function on the unit sphere S^{n-1}, and let $F_a f$ be the *shifted* spherical Funk transform with respect to the center $a \in \mathbb{R}^n$, i.e.,

$$(F_a f)(E) = \int\limits_{S^{n-1}\cap E} f(x) dA_E.$$

Here E is a k-dimensional subspace passing through a and dA_E is the surface area measure on the $(k-1)$-dimensional sphere $S^{n-1} \cap E$. It is known that every operator F_a with $|a| \neq 1$ has a non-trivial kernel. On the other hand, the kernel is trivial, provided $|a| = 1$. Given $A = \{a_1, \ldots, a_s\} \subset \mathbb{R}^n$, define

$$F_A f = \{F_{a_1} f, \ldots, F_{a_s} f\}.$$

The problem is to describe all sets such that $\ker F_A = \{0\}$. Agranovsky's approach relies on the action of the group $Aut(B^n)$ of automorphisms of the unit ball and exploits group-invariance arguments. Denote by $G(A)$ the group generated by the symmetries $\tau_{a_j} : S^{n-1} \to S^{n-1}$,

$$\tau_{a_j}(x) = x + 2\frac{1 - \langle x, a_j\rangle}{|a - x|^2}(a - x), \quad j = 1, \ldots, s.$$

In the case of two points, $s = 2$, it is shown that the paired transform $f \to (F_{a_1} f, F_{a_2} f)$ fails to be injective iff the group $G(A)$ generated by τ_{a_1}, τ_{a_2}, is finite. In the general case it is also shown that if $\ker F_A \neq \{0\}$, then $G(A)$ is a Coxeter group (every subgroup with two generators is finite). It would be very interesting to see if the converse is true.

Jan Boman presented uniqueness results related to supports of distributions. Let $f \neq 0$ be a compactly supported distribution in $\mathbb{R}^n$, $n \geq 2$, and let Rf be its Radon transform. It is shown that if the Radon transform is supported on the set of tangent planes to the boundary ∂D of a bounded convex domain D, then ∂D must be an ellipsoid. As a corollary one gets a new proof of a recent theorem of Koldobsky, Merkurjev, and Yaskin, who settled a special case of a conjecture of Arnold that was motivated by a famous lemma of Newton. The following questions are left open and deserve an attention. Let D be a domain in $\mathbb{R}^n$ and let $\overline{D_0}$ be its sub-domain. Does there exist a non-trivial function f supported by $\overline{D}$ such that Rf vanishes for every line that meets D_0? What about polygons? Which subsets of the manifold of lines in the plane can be the support of Rf for some compactly supported function or distribution f in $\mathbb{R}^2$?

Mark Rudelson spoke about their joint results with Herman König. They considered the problem of maximal and minimal (in volume) non-central sections of the cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ by subspaces $E \subset \mathbb{R}^n$, $\dim(E) = n - d$, $d = 1, \ldots, n-1$. It was proved by Vaaler that the minimal central section of the cube is the one orthogonal to the coordinate direction, and after K. Ball we know that

$$\mathrm{vol}_{n-d}(Q_n \cap E) \leq (\sqrt{2})^d.$$

Several results for non-central sections that have distance $t > \frac{\sqrt{n-1}}{2}$ from the origin were also obtained by Moody, Stone, Zach and Zvavitch. Rudelson and König proved that

$$\mathrm{vol}_{n-d}(Q_n \cap (x + E)) > c(d)$$

for $|x| \leq \frac{1}{2}$, $x \in E^{\perp}$. They got a control on $C(d)$ in the case $d = 1$ by showing that

$$\mathrm{vol}_{n-1}(Q_n \cap (x + E)) > \frac{1}{17}$$

and also obtained a similar result for the complex cube with $\frac{1}{27}$ instead of $\frac{1}{17}$.One of the open problems is to consider the case of general polytopes instead of Q_n.

Herman König gave a talk about non-central sections of the simplex, cube and cross-polytope. Let $a \in S^{n-1}$ and $t \in \mathbb{R}$. Given a convex body $K \subset \mathbb{R}^n$ consider the parallel section and the perimeter functions,

$$A_K(a,t) = \mathrm{vol}_{n-1}(\{x \in K : \langle x, a \rangle = t\}),$$

$$P_{\partial K}(a,t) = \mathrm{vol}_{n-2}(\{x \in \partial K : \langle x, a \rangle = t\}).$$

König was motivated by the aforementioned results of Moody, Stone, Zach and Zvavitch, who proved that

$$A_{Q_n}(a,t) \leq A(a^{(n)}, t)$$

for $a^{(n)} = \frac{(1,\ldots,1)}{\sqrt{n}}$, provided $\frac{\sqrt{n-2}}{2} < t \leq \frac{\sqrt{n}}{2}$, and by the recent results of Liu and Tkocz, who proved that

$$A_{B_1^n}(a,t) \leq A_{B_1^n}(e_1, t),$$

for $\frac{1}{\sqrt{2}} < t \leq 1$.

Let $\Delta^n = \{x \in \mathbb{R}_+^{n+1} : \sum_{j=1}^{n+1} x_j = 1\}$ be a simplex and let $a \in S^n \subset \mathbb{R}^{n+1}$ be such that $\sum_{j=1}^{n+1} a_j = 0$. It was shown by Webb that

$$A_{\Delta^n}(a, 0) \leq A_{\Delta^n}(\tilde{a}, 0) = \frac{\sqrt{n+1}}{\sqrt{2}(n-1)!}$$

for $\tilde{a} = \frac{(1,-1,0,\ldots,0)}{\sqrt{2}}$. König's results include the estimate

$$A_{\Delta^n}(\tilde{a}, t) \leq A(a^{[n]}, t) = \frac{\sqrt{n+1}}{(n-1)!}\Big(\frac{n}{n+1}\Big)^{\frac{n}{2}}\Big(\sqrt{\frac{n}{n+1}} - t\Big)^{n-1},$$

where

$$a^{[n]} = \left(\sqrt{\frac{n}{n+1}}, -\frac{1}{\sqrt{n(n+1)}}, \ldots, -\frac{1}{\sqrt{n(n+1)}}\right) \in S^n, \qquad n \geq 3.$$

A similar result is obtained for $P_{\partial\Delta^n}(a,t)$ as well as several new results for $A_{B_1^n}(a,t)$, $P_{\partial B_1^n}(a,t)$ and for $A_{Q_n}(a,t)$, $P_{\partial Q_n}(a,t)$. Several questions about local minima and maxima of the parallel section and the perimeter functions are left open.

Tomasz Tkocz together with Han Huang, Boaz Slomka and Beatrice-Helen Vritsiou attacked the famous illumination problem, posed independently by Levi (1955), Hadwiger (1957) and Gohberg-Marcus (1960). Let $K \subset \mathbb{R}^n$ be a convex body. How many translates $N(K)$ of the interior of K are needed to cover K, or, equivalently, how many external sources $l(K)$ of light are needed to illuminate ∂K? The conjecture is that

$$N(K) = l(K) \leq 2^n$$

with equality iff K is a cube (up to an affine map). Many partial results are known, in particular,

$$N(K) \leq 4^n \frac{1+o(1)}{\sqrt{\pi}} \sqrt{n} \log n.$$

Following the ideas of independent approaches of Artstein-Avidan-Slomka and Naszodi, it was proved by Huang, Slomka, Tkocz and Vritsiou that

$$N(K) \leq C 4^n e^{-c\sqrt{n}}.$$

They do it by showing that

$$\frac{|K|}{|K \cap -K|} \leq 2^n e^{-c\sqrt{n}},$$

provided the barycenter of K is at the origin.

Galyna Livshyts brought a discussion of the Log-Brunn-Minkowski conjecture and related questions. Let $0 \leq \lambda \leq 1$ and let

$$\lambda K +_0 (1-\lambda) L = \bigcap_{u \in S^{n-1}} \{x \in S^{n-1} : |\langle u, x\rangle| \leq h_K(u)^{\lambda} h_L(u)^{1-\lambda}\}$$

be the logarithmic sum of two convex bodies K and L, where $h_K(u) = \sup\limits_{x \in K} \langle x, u\rangle$ is the support function of K.

Böröczky, Lutwak, Yang and Zhang asked if

$$|\lambda K +_0 (1-\lambda) L| \geq |K|^{\lambda} |L|^{1-\lambda}.$$

Galyna presented several related results, in particular, she asked if

$$8|K|\text{span}(e_1, e_2)^{\perp}| + \int\limits_{S^{n-1}} \frac{(|u_1| + |u_2|)^2}{h_K(u)} dS_K(u) \leq 4 \frac{|K|e_1^{\perp}| + |K|e_2^{\perp}|}{|K|}.$$

Here $K|e_1^{\perp}$ stands for the orthogonal projection on the subspace $e_1^{\perp}$.

Eli Putterman continued the discussion about log-Brunn-Minkowski inequality. He showed how one can obtain the global log-BM from the local log-BM.

Apostolos Giannopoulos gave several results related to the conjecture posed by V. Milman. Let K be a symmetric convex body. For an n-tuple $C = (C_1, \ldots, C_s)$ of convex symmetric bodies C_j, $j = 1, \ldots, s$, consider the norm of the vector $T = (t_1, \ldots, t_s) \in \mathbb{R}^n$ defined by

$$\|T\|_{C,K} = \frac{1}{\prod_{j=1}^{s} |C_j|} \int\limits_{C_1} \cdots \int\limits_{C_s} \|\sum_{j=1}^{s} t_j x_j\|_K dx_1 \ldots dx_s.$$

Is it true that if $C_1 = \ldots = C_s = C$, $|C_j| = 1$, then $\|T\|_{C,K}$ is equivalent to the Euclidean norm up to a logarithm in dimension? Giannopoulos, Chasapis and Skarmogiannis gave an alternative proof of the lower estimate of Gluskin and Milman,

$$\|T\|_{C,K} \geq \frac{n}{e(n+1)}\|T\|_2.$$

The upper bound is reduced to obtaining a constant of the order $L_C\sqrt{n}M(K_{\text{iso}})$, $M(K) = \int\limits_{S^{n-1}} \|\xi\|_K d\sigma(\xi)$, provided K is in isotropic position, and L_K is the isotropic constant. They hope that

$$L_C\sqrt{n}M(K_{\text{iso}}) \leq c(\log n)^b$$

for some absolute constant $b > 0$, however the best currently know estimate of Giannopoulos and E. Milman is

$$M(K_{\text{iso}}) \leq c\frac{(\log n)^{\frac{2}{5}}}{\sqrt[10]{n}L_K}.$$

The presented upper bound is

$$\|T\|_{C,K} \leq c\max\{\sqrt[4]{n}, \sqrt{\log(1+s)}\}L_C\sqrt{n}M(K)\|T\|_2,$$

provided $C_1 = \ldots = C_s = C$ and C is in isotropic position. Other non-trivial upper bounds are obtained in the unconditional, ψ_2 and co-type-2 cases.

Carsten Shütt presented their joint results with Matthias Reitzner and Elisabeth Werner about convex hull of random points on the boundary of a simple polytope. Given a convex body $K \subset \mathbb{R}^n$ and N random points $x_1, \ldots, x_N$ in K, let $K_N = [x_1, \ldots, x_N]$ be the convex hull of these points. What is the expected number of vertices $\mathbb{E}f_0(K_N)$, facets $\mathbb{E}f_{n-1}(K_N)$ and the volume difference $\text{vol}_n(K) - \mathbb{E}\text{vol}_n(K_N)$? They show that choosing N points on the boundary of a simple convex polytope $P \subset \mathbb{R}^n$,

$$\mathbb{E}(f_{n-1}(P_N)) = c_{n,n-1}f_0(P)(\ln N)^{n-2}(1 + O(\ln N)^{-1}), \quad c_{n,n-1} > 0,$$

and

$$\text{vol}_n(P) - \mathbb{E}\text{vol}_n(P_N) = \frac{c_{n,P}}{N^{\frac{n}{n-1}}}(1 + O(N^{-\frac{1}{(n-1)(n-2)}}).$$

They expect that for arbitrary polytopes, one has

$$\text{vol}_n(P) - \mathbb{E}\text{vol}_n(P_N) = c_n\frac{\text{flag}(P)\text{vol}_n(P)}{N^{\frac{n}{n-1}}}(1 + O(N^{-\frac{1}{(n-1)(n-2)}})).$$

Here the flag is an n-tuple $(f_0(P), f_1(P), \ldots, f_{n-1}(P))$ of k-dimensional faces of P, and $\text{flag}(P)$ is the number of flags of P. It would be interesting to evaluate the constant c_n.

Grigoris Paouris talked about his joint results with Boris Hanin on non-asymptotic behavior of the spectrum of products of many random matrices (square or rectangular). Let A be a $n \times n$ random matrix with mean zero Gaussian entries $a_{ij} \sim \frac{1}{\sqrt{n}}N(0,1)$, and let $s_1(A) \geq \ldots \geq s_n(A)$ be its singular values. The Lyapunov exponents are defined as

$$\lambda_k = \frac{1}{n}\log s_k(X_N),$$

where X_N is the product of N independent random matrices, $X_N = A_N \cdot \ldots \cdot A_1$. They proved that for small n and large N if

$$H_{n,N}(t) = \frac{1}{n}\{j \leq n : \lambda_j \leq \log t\},$$

and $N \geq T^n \log n$, $\delta \sim \frac{1}{T}$, then

$$\|H_{n,N} - H\|_{L^\infty[\delta,1-\delta]} \leq \frac{1}{T}$$

with probability $\geq 1 - e^{-cn^4\frac{N}{T^4}}$. Here

$$H(t) = \int\limits_{-\infty}^{t} h(x)dx, \qquad h(x) = 2x1_{[0,1]}(x).$$

They conjecture that this statement should be true for not Gaussian variables only. It is also shown that for fixed n and N, one has

$$d(\Lambda_k, N(1,\varepsilon)) \leq C\frac{k^{\frac{3}{4}}\log N\sqrt{n}}{\sqrt{N}},$$

where $\Lambda_k = (\lambda_1,\ldots,\lambda_k)$ is a Gaussian Random vector with some mean and variance.

Rafal Latala described his joint results with Petr Nayar and Marta Strzelecka. The general problem is to compare the strong and weak moments of random n-dimensional vectors X in $(\mathbb{R}^n, \|\cdot\|)$, say,

$$(\mathbb{E}\sup_{\|t\|_*\leq 1}|\langle t, X\rangle|^p)^{\frac{1}{p}} \leq C_{n,p}\sup_{\|t\|_*\leq 1}(\mathbb{E}|\langle t, X\rangle|^p)^{\frac{1}{p}},$$

where $\|\cdot\|_*$ is the dual norm. They show, in particular, that for any non-empty set $T\subset\mathbb{R}^n$ and $p\geq 2$, one has

$$(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^p)^{\frac{1}{p}} \leq 2\sqrt{e}\sqrt{\frac{n+p}{p}}\sup_{t\in T}(\mathbb{E}|\langle t, X\rangle|^p)^{\frac{1}{p}}.$$

Paouris proved that for a log-concave vector X,

$$(\mathbb{E}|X|^p)^{\frac{1}{p}} \leq c_1\mathbb{E}|X| + c_2\sup_{t\in T}(\mathbb{E}|\langle t, x\rangle|^p)^{\frac{1}{p}}.$$

Is it possible to take $c_1 = 1$? What other norms can you take? Let X be log-concave, $r<\infty$ and let $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in l_r. Is it true that

$$(\mathbb{E}\|X\|^p)^{\frac{1}{p}} \leq Cr(\mathbb{E}\|X\| + \sup_{\|t\|_*\leq 1}(\mathbb{E}|\langle t, x\rangle|^p)^{\frac{1}{p}})$$

holds with C instead of Cr?

Alexander Litvak reported about his results with Konstantin Tikhomirov. The general problem is as follows. Let B be a random matrix $n\times n$ with i.i.d. ± 1 entries. What is $P_n := \mathbb{P}(B \text{ is singular})$? Equivalently, let $X_1,\ldots,X_n$ be independent random vectors uniformly distributed on the vertices of the n-dimensional cube $[-1,1]^n$. What is the probability that the vectors are linearly independent? It is known that

$$P_n \geq (1-o(1))2n^2 2^{-n},$$

and the conjectures are that

$$P_n \leq (\frac{1}{2}+o(1))^n = 2^{-(1+o(1))n}$$

(solved recently by Tikhomirov) and

$$P_n \leq (1+o(1))2n^2 2^{-n}.$$

They also ask the same question about Bernoulli $0/1$ random matrices and the conjecture is that

$$P_n = (1+o(1))\mathbb{P}\{\exists \text{a zero row or a zero column}\} = (1+o(1))2n(1-p)^n.$$

Basak and Rudelson proved that

$$P_n \leq e^{-cnp},$$

provided $p\geq \frac{C\ln n}{n}$ and that

$$P_n \leq (1+o(1))2n(1-p)^n,$$

provided $p\leq \frac{\ln n}{n} + o(\ln\ln n)$. Tikhomirov proved that

$$P_n \leq (1-p+o(1))^n$$

for $p\in(0,\frac{1}{2}]$. Recently, Litvak and Tikhomirov showed that for $C\frac{\ln n}{n}\leq p\leq c$,

$$P_n \leq (1+o(1))2n(1-p)^n.$$

Arnaud Marsiglietti gave a talk that was devoted to his joint results with James Melbourne about localization technique for discrete log-concave random variables. The main idea is to translate the "continuous" results from convex geometry to the "discrete" ones. In this lecture the results related to the 1993 Theorem of Lovasz and Simonovits were presented. A function $f : \mathbb{N} \to (0, \infty)$ is called discrete log-concave if

$$f(n)^2 \geq f(n-1)f(n+1)$$

$\forall n \in \mathbb{N}$ and f has a contiguous support, i.e., $\forall a \leq b$, $a, b \in \{f > 0\}$ iff $\forall k \in \{a, \ldots, b\}$, k must belong to $\{f > 0\}$. A random variable is discrete log-concave if its probability mass function is discrete log-concave. Let $N \in \mathbb{N}$ and let $[N] = \{0, \ldots, n\}$. For a measure γ with contiguous support and a function h define the set

$$P_h^\gamma([N]) = \{P_X \in P([N]) : X \quad \text{log-concave}/\gamma \quad \text{and} \quad \mathbb{E}(h(X)) \geq 0\},$$

where $P([N])$ stands for the set of all probability measures with the support on $[N]$, and $X \sim p$ is log-concave with respect to an integer-valued measure γ with mass function q means that $\frac{p}{q}$ is log-concave. It is proved that if $\varphi : P_h^\gamma([N]) \to \mathbb{R}$ is convex, then

$$\sup_{P_X \in P_h^\gamma([N])} \varphi(P_X) \leq \sup_{P_{X^\sharp} \in A_h^\gamma([N])} \varphi(P_{X^\sharp}),$$

where

$$A_h^\gamma([N]) = P_h^\gamma([N]) \cap \{X \text{ log-affine}/\gamma\}$$

(i.e., inequalities become equalities). As one of the applications in the case $\varphi(P_X) = P_X(A)$, $A \subset \mathbb{R}$, one can prove bounds or every log-concave discrete X,

$$\mathbb{P}(X > t) \leq c_1 e^{-\frac{c_2 t}{\mathbb{E}(X)}}, \quad (\mathbb{E}[X^s])^{\frac{1}{s}} \leq C(r,s)(\mathbb{E}[X^r])^{\frac{1}{r}},$$

with some explicit constants. It would be interesting to verify if they are sharp.

Peter Pivovarov described his joint results with Jesus Rebollo Bueno about stochastic Prékopa-Leindler inequality for log-concave functions. For $x, y \in \mathbb{R}^n$, and $\lambda \in [0,1]$, let

$$(f \star_\lambda g)(v) = \sup\{f(x)^\lambda g(y)^{1-\lambda} : v = \lambda x + (1-\lambda)y\}.$$

Given a log-concave integrable function $f : \mathbb{R}^n \to [0, \infty)$, define

$$G_f = \{(x, z) \in \mathbb{R}^n \times [0, \infty) : z \leq f(x)\},$$

and let the vectors $\{(X_i, Z_i)\}_{i=1}^N \subset \mathbb{R}^n \times [0, \infty)$ be i. i. d. uniform in G_f. For two integrable log-concave functions $f, g : \mathbb{R}^n \to [0, \infty)$, $\lambda \in (0,1)$ and $N, M > n+1$, it is proved that for all $\alpha > 0$,

$$\mathbb{P}\Big(\int_{\mathbb{R}^n} ([f]_N \star_\lambda [g]_M)(v) dv > \alpha\Big) \geq \mathbb{P}\Big(\int_{\mathbb{R}^n} ([f^*]_N \star_\lambda [g^*]_M)(v) dv > \alpha\Big),$$

where

$$[f]_N(x) = e^{\sup\{z : (x,z) \in H_f\}}, \quad H_f = \text{conv}\{(X_1, \log Z_1), \ldots, (X_N, \log Z_N)\},$$

and $*$ stands for a non-decreasing rearrangement. In particular, for one function they get a stochastic functional Groemer-type inequality.

Petros Valettas gave a talk about lower deviation estimates in normed spaces. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and let G be a Gaussian vector. The main goal is to provide upper bounds for $\mathbb{P}(\|G\| \leq \delta \mathbb{E}(\|G\|))$. Jointly with Grigoris Paouris they obtain the estimate

$$\mathbb{P}(f(G) \leq \mathbb{E}f(G) - t\sqrt{\text{Var}[f(G)]}) \leq e^{-ct^2}, \quad t > 0.$$

Giorgos Chasapis presented several results about random polytopes related to Lutwak's conjecture. Let K be a convex body and let $1 \leq l \leq n-1$. Define

$$\Phi_{[l]}(K) = |K|^{-\frac{1}{n}} \Big(\int_{G_{n,l}} |P_F K|^{-n} d\nu_{n,l}(F) \Big)^{-\frac{1}{ln}},$$

the l-th normalized affine quermassintegral. Prove that

$$\Phi_{[l]}(K) \geq \Phi_{[l]}(B_2^n).$$

Another conjecture of Dafnis and Paouris is that there exist two constants c_1 and c_2 such that for all l,

$$c_1\sqrt{\frac{n}{e}} \leq \Phi_{[l]}(K) \leq c_2\sqrt{\frac{n}{e}}.$$

Piotr Nayar's talk was devoted to Khinchin's inequality which is stated as follows. Let $\{\varepsilon_j\}_{j=1}^N$ be i.i.d random variables with $\mathbb{P}(\varepsilon_j = \pm 1) = \frac{1}{2}$ for $j = 1, \dots, N$, i.e., a sequence with Rademacher distribution. Let $0 < p < \infty$ and let $x_1, \dots, x_N \in \mathbb{C}$. Then

$$A_p\Big(\sum_{j=1}^n |x_j|^2\Big)^{\frac{1}{2}} \leq \Big(\mathbb{E}\Big|\sum_{j=1}^n \varepsilon_j x_j\Big|^p\Big)^{\frac{1}{p}} \leq B_p\Big(\sum_{j=1}^n |x_j|^2\Big)^{\frac{1}{2}},$$

where A_p and B_p are some absolute constants. Together with Tomasz Tkocz Petr Nayar gave a beautiful elementary proof of this inequality for even p. It would be very interesting to obtain an elementary proof for odd p.

Elisabeth Werner gave a talk about her joint work with O. Giladi, H. Huang and C. Schütt on constraint convex bodies with maximal affine surface area. Given a convex body $K \subset \mathbb{R}^n$, the L_p-affine surface area is defined as

$$\mathrm{as}_p(K) = \int_{\partial K} \frac{k(x)^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x), \quad p \neq -n,$$

where the integration is with respect to the usual surface area measure μ_K over the boundary ∂K, N is the normal vector, and k is the Gauss curvature. The quantity is affine (linear) invariant, but not continuous in K, so the question is if one can get continuous affine invariants. They study the inner and outer maximal and minimal surface areas

$$\mathrm{IS}_p(K) = \sup_{C \subset K}(\mathrm{as}_p(C)), \qquad \mathrm{OS}_p(K) = \sup_{C \supset K}(\mathrm{as}_p(C)),$$

$$\mathrm{is}_p(K) = \inf_{C \subset K}(\mathrm{as}_p(C)), \qquad \mathrm{os}_p(K) = \inf_{C \supset K}(\mathrm{as}_p(C)),$$

for relevant ranges of p. In particular, they showed that for $p \in [0, n]$, $p \in [n, \infty]$, $[-n, 0]$, the maps $K \to \mathrm{IS}_p(K)$, $K \to \mathrm{OS}_p(K)$ and $K \to \mathrm{os}_p(K)$ are continuous in the Hausdorff metric correspondingly. They also study the bodies at which the corresponding $\sup$ or $\inf$ is reached. Following the results of Barany in the case $n = 2$, $p = 1$, they attack the questions about estimating the "size" of $\mathrm{IS}_p(K)$, $\mathrm{OS}_p(K)$ and $\mathrm{os}_p(K)$ in all dimensions for all relevant p.

Maria de los Angeles Alfonseca-Cubero spoke about her joint results with F. Nazarov, D. Ryabogin and V. Yaskin on a local solution to the eighth Busemann-Petty problem. In 1956, Busemann and Petty posed ten problems about symmetric convex bodies, of which only the first one has been solved. Their fifth and the eighth problems are as follows. If for an origin-symmetric convex body $K \subset \mathbb{R}^n$, $n \geq 3$, one has

$$\forall \theta \in S^{n-1} \quad h_K(\theta) vol_{n-1}(K \cap \theta^{\perp}) = C,$$

where the constant C is independent of θ, must K be an ellipsoid? If for an origin-symmetric convex body $K \subset \mathbb{R}^n$, $n \geq 3$, one has

$$f_K(\theta) = C(vol_{n-1}(K \cap \theta^{\perp}))^{n+1} \qquad \forall \theta \in S^{n-1},$$

where the constant C is independent of θ, must K be an ellipsoid? Here f_K is the curvature function, which is the reciprocal of the Gaussian curvature viewed as a function of the unit normal vector. They prove that if an origin-symmetric convex body $K \subset \mathbb{R}^n$, $n \geq 3$, satisfies one of the above conditions and is sufficiently close to the Euclidean ball in the Banach-Mazur metric, then K must be an ellipsoid.

Sudan Xing talked about dual curvature measures and the Orlicz-Minkowski problem which is about finding necessary and sufficient conditions on a finite Borel measure μ and a function $\varphi : (0,\infty) \to (0,\infty)$ so that there exists a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior and $\mu = \tau\varphi(h_K)S(K,\cdot)$ for some constant $\tau > 0$. Here h_K is the support function of the body K and $S(K,\cdot)$ is the surface area measure of K.

Shiri Artstein's talk was devoted to polarity, transportation and potentials. She started describing the "parallels" between the relations of the Legendre transform

$$\mathcal{A}\varphi(x) = \sup \frac{(\langle x, y\rangle - 1)_-}{\varphi(y)}$$

(both $\mathcal{L}$ and $\mathcal{A}$ are order reversing involutions) and the Prekopa-Leindler inequality and the polarity transform $\mathcal{A}\varphi$ and the inequality recently discovered with D. Florentin and A. Segal. Next, she talked about the source of other order reversing involutions, coming from a cost function $c(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty,\infty]$. Given a function $\varphi : \mathbb{R}^n \to [-\infty,\infty]$ she defined the c-transform of φ as

$$\varphi^c(y) = \inf_x (c(x,y) - \varphi(x))$$

and explained that different choices of the cost function lead to different transforms. In particular, the choice $c(x,y) = -\langle x, y\rangle$ yields $-\varphi^c = \mathcal{L}(-\varphi)$ and the choice $c(x,y) = -\ln(\langle x,y\rangle - 1)$ yields $e^{-\varphi^c} = \mathcal{A}(e^{-\varphi})$. Finally, she presented a result on the $\mathcal{A}$-transport of measures that is analogous to the famous Brenier, Mc-Cann and Cafarelli theorem (which measures can be mapped to one another using the "dual" gradient , i.e., the $\mathcal{A}$-gradient?)

Luis Carlos Garcia Lirola talked about volume product and metric spaces. Given a finite metric space (M,d), $M = \{a_0,\dots,a_n\}$ one can associate a polytope $P = P(M) \subset \mathbb{R}^n$ as

$$B_{\mathcal{F}(M)} = \operatorname{conv}\left\{\frac{e_i - e_j}{d(a_j,a_j)}\right\}, \qquad i \neq j.$$

The volume product of a metric space is

$$\mathcal{P}(M) = |B_{\mathcal{F}(M)}| \cdot |B_{\mathrm{Lip}_0(M)}|,$$

where

$$B_{\mathrm{Lip}_0(M)} = \Big\{f : \frac{f(a_i) - f(a_j)}{d(a_j,a_j)} \le 1 \ \forall i \neq j\Big\},$$

and a function f is identified with a vector $(f(a_1),\dots,f(a_n)) \in \mathbb{R}^n$, $B^*_{\mathrm{Lip}_0(M)} = B_{\mathcal{F}(M)}$. Together with M. Alexander, M. Fradelizi and A. Zvavitch they showed, in particular, if M is a finite metric space with minimal volume product such that $B_{\mathcal{F}(M)}$ is a simplicial polytope, then M is a tree (and so $\mathcal{P}(M) = \frac{4^n}{n!}$). In addition, they obtained several results related to a metric-graph characterization of $B_{\mathcal{F}(M)}$ being a Hanner polytope.

Michael Roysdon presented a slicing inequality of the Rogers-Shephard type. Michael generalized the original result of Mark Rudelson who proved that, for any m-dimensional subspace H of $\mathbb{R}^n$ and any convex body $K \subset \mathbb{R}^n$, one has

$$\mathrm{vol}((K + (-K) \cap H) \le \left[c \cdot \min\{n/m, \sqrt{m}\}\right]^m \sup_{y\in\mathbb{R}^n} \mathrm{vol}(K \cap (y + H)),$$

where the volume here is interpreted as the restriction of the Lebesgue measure of the m-dimensional subspace H and $c > 0$ is some absolute constant.

Michael worked on the case, when $\min\{n/m, \sqrt{m}\} = n/m$ for general measures μ on $\mathbb{R}^n$ having radially decreasing densities. The main result presented in the talk asserts that for any convex body $K \subset \mathbb{R}^n$, any measure μ on $\mathbb{R}^n$ having a radially decreasing density, and any m-dimensional subspace H of $\mathbb{R}^n$, and any measures η defined on $\mathbb{R}^n$ whose density satisfies certain concavity conditions and such that $\eta(K) > 0$ one has

$$\mu((K + (-K)) \cap H) \le \frac{\binom{n+m}{m}}{\eta(K)} \int_K \mu((-y + K) \cap H)\, d\eta(y).$$

The inequality was further extended to the setting of $(1/s)$-concave functions, with $s \in (0, \infty)$. In particular, it was shown that, for any such function and any measure μ, one has

$$\int_H \sup_{x=x_1-x_2} (f(x_1)^{1/s} + f(x_2)^{1/s})^s d\mu(x) \leq C(n,m,s) \cdot \sup_{y\in\mathbb{R}^n} \left\{ \int_{(\mathrm{supp}(f)-y)\cap H} f(x) d\mu(x) \right\},$$

where $C(n,m,s) > 0$ is a constant depending only on n, m and s.

Martin Henk delivered a lecture about slicing properties of the lattice point enumerator based on ongoing joint work with Ansgar Freyer. He presented the results about the discrete Meyer inequality for $n \geq 3$. Let K be an origin-symmetric convex body, and let

$$c(n) = \inf \left\{ \frac{G(K)^{\frac{n-1}{n}}}{\prod_{j=1}^{n} G(K \cap e_j^{\perp})^{\frac{1}{n}}} \right\},$$

where $G(K) = \sharp(K \cap \mathbb{Z}^n)$ stands for the lattice point enumerator. Gardner, Gronchi and Zong proved that

$$c(2) = 3^{-\frac{1}{2}} \quad \text{and} \quad c(n) \leq \frac{(n!)^{\frac{1}{n}}}{n}.$$

In general, $c(n) \leq 3^{\frac{1-n}{n}}$. Henk and Freyer showed that $c(n) \geq 4^{-(n+o(n))}$ and for the class of unconditional bodies $c(n) \geq 3^{-n}$.

Krzysztof Oleszkiewicz gave a lecture about some recent developments of harmonic analysis on the discrete cube $\{-1,1\}^n$. He discussed an improvement of a result of Friedgut, Kalai and Naor who have shown that if the variance of the absolute value of a sum of weighted Rademacher variables is much smaller than the variance of the sum, then one of the summands dominates the sum. He also gave new proofs of some results of K. Tanguy. Let

$$\mathrm{Inf}_{ij}(f) = \sum_{i,j\in S} (\widehat{f}(S))^2,$$

where $\widehat{f}(S) = \mathbb{E}(f \cdot w_S)$ and w_S is the Walsh function associated with a subset S of $[n]$. If $\forall i, j$,

$$Inf_{ij}(f) \leq \frac{1}{1000}(\frac{lnn}{n})^2,$$

then f is close to ± 1 or $\pm r_k$ (Rademacher functions of order k). He asked if given two finite 1-separated sets A, B in a Banach space $(F, \|\cdot\|)$, the Minkowski sum set $A + B$ contains a 1-separated subset of cardinality $|A| + |B| - 1$. Right after the conference Fedor Nazarov gave a negative answer to this question for A , B consisting of 3 points.

Yair Shenfeld talked about polytope extremals of the Alexandrov-Fenchel inequality. He presented results about 1985 Conjecture of R. Schneider. Let

$$V(K, L, C_1, \ldots, C_{n-2}) = \int_{S^{n-1}} h_K dS_{L,C_1,\ldots,C_{n-2}}.$$

If C_j are full-dimensional and we have an equality in

$$V(K, L, C_1, \ldots, C_{n-2})^2 \geq V(K, K, C_1, \ldots, C_{n-2}) V(L, L, C_1, \ldots, C_{n-2}),$$

then $h_K = h_{cL+t}$. Together with Ramon Van Handel they prove that the conjecture is true if all C_j are equal to each other, or if C_j are polytopes.

Oscar Adrian Ortega Moreno discussed the results related to the classical Tarski plank problem, asking if an n-dimensional convex body is covered by a collection of planks, then the sum of the widths of the planks should be at least the minimal width of the convex body they cover. Following Jiang, Polyanski he reproves the conjecture of Toth about zones on the unit sphere S^{n-1}. He asks if, given vectors $v_1, \ldots, v_n \in S^{n-1}$, there exists $v \in S^{n-1}$ such that

$$\prod_{j=1}^{n} |\langle v_j, v\rangle| \geq n^{-\frac{n}{2}}.$$

Gideon Schehtman's talk was devoted to the dimension reduction in the trace class norm. Let (M, d) be a metric space, and let $(X, \|\cdot\|)$ be a normed space. One says that M embeds into X with distortion C if there is a function $f : M \to X$ such that

$$d(x, y) \leq \|x - y\| \leq Cd(x, y) \qquad \forall x, y \in M.$$

The best C is denoted by $C_X(M)$. The interest is in $k_n^C(X)$-the smallest k such that for all $S \subset X$ with $|S| = n$ there is a subspace $Y \subset X$ of dimension k such that $C_Y(S) \leq C$ (one thinks that, say, $C = 2$). Together with A. Naor and G. Pisier they proved the strengthening of a Brinkman-Charikar result that

$$k_n^C(S_1) \geq n^{\frac{\alpha}{C^2}}$$

for a universal $\alpha > 0$. Here S_1 is the trace class (Schatten-Von-Neumann 1, Nuclear norm). The meaning of this result is that for all n there are n points in S_1 such that if Y is a subspace of S_1 of dimension k into which these n points embed with distortion C, then $k \geq n^{\frac{\alpha}{C^2}}$. Given k what is the order of the smallest m such that $\forall k$-dimensional subspace of S_1 2-embeds into S_1^m? He conjectured that there is no polynomial bound on m in terms of k.

Semyon Alesker spoke about a complex analogue of the algebra of even valuations on convex sets. Valuations on convex sets are a classical object in convexity with traditionally strong relations to integral geometry. A valuation is a finitely additive measure on the class of all convex compact sets in $\mathbb{R}^n$. Translation invariant valuations continuous in the Hausdorff metric are studied particularly well. During the last 25 year there was a considerable progress in their study and in their integral geometric applications. It was realized that continuous valuations are particularly rich in structure. Some years ago the speaker has introduced a canonical product on them. Several non-trivial properties of it has been found, as well as applications to integral geometry. The first part of the talk contained a review of some of the relevant background on valuations and the product on them. The focus was on versions of the Poincaré duality and hard Lefschetz theorem. They served as a motivation for the new results. The main new result was the introduction of a complex non-Archimedean analogues of the algebra of even translation invariant valuations. While at the moment these algebras lack a geometric interpretation, they have non-trivial algebraic properties. In particular they satisfy versions of the Poincaré duality and hard Lefschetz theorem. Behind these properties stay results on the Radon and cosine transform on Grassmannians over local fields.

3 Outcome of the Meeting

The meeting was very successful. We brought together mathematicians from many countries and many research areas, such as convex geometry, discrete geometry, probability, functional and harmonic analysis. Besides the leading scientists, we also had 1 undergraduate student, 4 graduate students and 4 postdocs or recent PhDs participating in the workshop. Female participation was about 22%. The friendly atmosphere created during the workshop helped many participants not only to identify the promising ways to attack old problems but also to get acquainted with many open new ones.