Geometry of Rotation Sets

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During Research in Pairs week we focused on the problem of finding new (previously unknown) shapes of rotation sets of 2-torus homeomorphisms. By a result of Misiurewicz and Ziemian [8], such sets are always convex subsets of the plane, but the exact admissible shapes remain a major open problem in rotation theory, with the question whether a round disk could appear as such a set, being the most famous one.

Our approach was through a study of a parametric family of 1-dimensional maps on a wedge of two circles. Such an approach has been previously shown to be quite fruitful by a relatively recent paper of Boyland, de Carvalho and Hall [4], where it produced new rotation sets. These sets included non-polygonal rotation sets with an extreme point which is totally irrational and accumulated on both sides by other extreme points. This improved on a result of Kwapisz [7], who constructed a non-polygonal rotation set with two (partially) irrational extreme points, accumulated by other extreme points on one side. The paper of Boyland, de Carvalho and Hall also contained the first example for a parametric family of torus homeomorphisms with a Tal-Zanata property, i.e. the occurrence of discontinuity of in the sets of extreme points. The paper also contained a question whether an extreme point is irrational if and only if it is a limit extreme point. The validity and usefulness of the approach, through the study of parametric families of maps on the wedge of two circles, comes from the fact that these maps give rise to torus homeomorphisms, trough the consideration of their natural extensions (in the sense of Rohlin [**?**]), and a dynamical embedding theorem due to Brown, Barge and Martin [1],[6] (the so-called BBM attractors, recently studied extensively by Boyland, de Carvalho and Hall [3],[5]). The crucial result about this approach is that the rotation sets of the 1-dimensional maps coincide with those of their natural extensions.

During our stay in Banff we considered a rational version of Kwapisz map from [7] on a wedge of two circles $X = \mathbb{S}^1_h \vee \mathbb{S}^1_v$, where $\mathbb{S}^1_h \cap \mathbb{S}^1_v = O$. Let $\epsilon > 0$ be small and $a, b \in [0.1, 0.2] \cap \mathbb{Q}$. Given a $\delta > 0$ we let $[-\delta, \delta]_h = B(O, \delta) \cap \mathbb{S}^1_h$ and $[-\delta, \delta]_v = B(O, \delta) \cap \mathbb{S}^1_v$. Let $d = \min\{a, b\}$.

We define a map $F_{a,b} : X \to X$ as a composition $F_{a,b} = f_b \circ f_a$, where $f_a, f_b : X \to X$ are given as follows:

- $f_a(x) = x + a \bmod 1$ for all $x \in \mathbb{S}^1_h$.
- $f_a(x) = x$ for all $x \in \mathbb{S}^1_v \setminus [-0.01, 0.01]_v$.
- $f_a|([-0.01, -0.001]_v \cup [0.001, 0.01]_v)$ is linear onto $[-0.01, 0.01]_v$.
- $f_a([-0.001, 0]_v) = f_a([0, 0.001]_v) = [0, a]_h$ (linearly).
- $f_b(x) = x + b \bmod 1$ for all $x \in \mathbb{S}^1_v$.
- $f_b(x) = x$ for all $x \in \mathbb{S}^1_h \setminus [-0.01, 0.01]_h$.
- $f_b|([-0.01, -0.001]_h \cup [0.001, 0.01]_h)$ is linear onto $[-0.01, 0.01]_h$.

- $f_a([-0.001, 0]_h) = f_a([0, 0.001]_h) = [0, b]_v$ (linearly).

By S_a we mean the set of points that under iterations of both f_a and f_b never leave $\mathbb{S}^1_h$. We define S_b analogously. These are points which never enter the interval $(-0.001, 0.001)$, on either $\mathbb{S}^1_h$ or $\mathbb{S}^1_v$, under any iterations of $F_{a,b}$. So, we define $S_a = \mathbb{S}^1_h \setminus \bigcup_{n\in\mathbb{N}} F^{-n}_{a,b}(-0.001, 0.001)$. It is not difficult to verify that this set is not empty. We let

$$\phi_a = \left(f_a|_{\mathbb{S}^1_h}\right) \circ \left(f_b|_{\mathbb{S}^1_h}\right) \text{ and } \phi_b = \left(f_b|_{\mathbb{S}^1_v}\right) \circ \left(f_a|_{\mathbb{S}^1_v}\right).$$

Note that ϕ_a and ϕ_b are nondecreasing degree 1 maps on $\mathbb{S}^1_h$ and $\mathbb{S}^1_v$ respectively. The graph of ϕ_a is shown below.

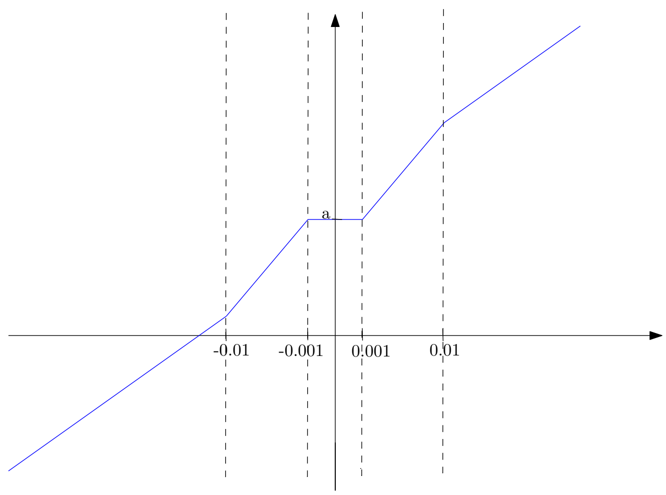

Figure 1: The graph of ϕ_a

Recall that for a continuous map F on the m-dimensional torus $\mathbb{T}^m$ its rotation set $\rho(F)$ is defined as the set of limits of statistical averages

$$\left(\frac{\tilde{F}^{n_l}(x_l) - x_l}{n_l}\right)_{l=1}^{\infty}, \quad x_l \in \mathbb{R}^m,\ n_l \to \infty,$$

where $\tilde{F}$ is a lift of F to the covering space $\mathbb{R}^m$. For a point $x \in \mathbb{T}$ the rotation vector of x is $\rho(x) = \lim_{n\to\infty} \frac{\tilde{F}^n(x)-x}{n}$, provided that the limit exists. It was shown by Misiurewicz and Ziemian in [8] that when $m = 2$ the rotation set is a compact and convex subset of $\mathbb{R}^2$ (it is not true for $m \geq 3$).

In our case, map F is a natural extension of the map $F_{a,b}$ to $\mathbb{T}^2$ as discussed above. Hence, our goal is to determine the shape of $\rho(F_{a,b})$. Although $\rho(\phi_a)$ and $\rho(\phi_b)$ could, in general, be line segments, the fact that ϕ_a and ϕ_b are nondecreasing implies, by Lemma 1.5 in [2], that their rotation sets are reduced to a point. Moreover, since for any x we have $x + a - 0.001 < F_{a,b}(x) < x + a + 0.001$ by Lemma 1.6 in [2] we obtain that $\rho(\phi_a) = r$ for some $r \in [a - 0.001, a + 0.001]$. The reason we are interested in the properties of maps ϕ_a and ϕ_b is that the intersections of $\rho(F_{a,b})$ with vertical and horizontal coordinate axes are the intervals $[0, \rho(\phi_a)]$ and $[0, \rho(\phi_b)]$ respectively. We prove the following.

Lemma 1. *If $a < a_1$ then $\rho(\phi_a) \leq \rho(\phi_{a_1})$. Suppose $n \in \mathbb{N}$ and $a \in [0.1, 0.2]$ are such that*

$$\frac{1}{n} \in [0.1, 0.2] \quad \textit{and} \quad \frac{1-0.001}{n} < a < \frac{1+0.001}{n}. \tag{1}$$

Then $\rho(\phi_a) = \frac{1}{n}$. In particular, $a \mapsto \rho(\phi_a)$ is not strictly increasing.

The above lemma completely characterizes the behavior of the rotation set of $F_{a,b}$ along the coordinate axes for parameters sufficiently close to points of the form $\frac{1}{n}$ for a positive integer n. We are able to show that when $a = b = \frac{1}{n}$ the rotation set of $F_{a,b}$ is just a quadrilateral with vertices $(0,0)$, $(\frac{1}{n},0)$, $(0,\frac{1}{n})$ and $(\frac{1}{2n-1},\frac{1}{2n-1})$. First, we try to understand how a small perturbation of the parameters affects the shape of the rotation st. We focus on the symmetric case $a = b$ with a being in a small neighborhood of $\frac{1}{n}$. We prove that the rotation set of $F_{a,a}$ is no longer a quadrilateral when $a \neq \frac{1}{n}$.

Lemma 2. *Suppose $a = b \in [0.1, 0.2]$ and $a \in (\frac{1-0.001}{n}, \frac{1}{n})$ for some natural n. Then $\rho(\phi_a) = \frac{1}{n}$ and $\rho(\phi_b) = \frac{1}{n}$, but $(\frac{1}{2n-1},\frac{1}{2n-1})$ is not a vertex or an interior point of $\rho(F_{a,b})$.*

Next, we attempt to track the boundary point of $\rho(F_{a,a})$ which lies on the diagonal axis of symmetry as the parameter a approaches $\frac{1}{n}$. Although we are fairly successful in this endeavor, our methods are very technical and it is not clear to us at the moment how to apply them to describe the change of other points in the rotation set when the parameter changes. On the other hand, we do gain some much needed intuition into the evolution of the rotation set when $a \to \frac{1}{n}$, which makes the proof of our next result a valuable step forward.

Lemma 3. *Suppose $a = b \in [0.1, 0.2]$ and $a \in (\frac{1-0.001}{n}, \frac{1}{n})$ for some natural n. Then there exists a strictly decreasing sequence $(\varepsilon_k)_{k\geq 2} \subset (0, 0.001)$ such that for any $a_k \in \left(\frac{1-\varepsilon_k}{n}, \frac{1}{n}\right)$ we have*

- $\left(\frac{k}{2kn-k+1}, \frac{k}{2kn-k+1}\right) \in \rho(F_{a_k,a_k})$;
- $\left(\frac{k}{2kn-k-1}, \frac{k}{2kn-k-1}\right) \notin \rho(F_{a_k,a_k})$

The proof of the above lemma is based on an inductive argument. Suppose $a = b \in [0.1, 0.2]$ and $a \in (\frac{1-0.001}{n}, \frac{1}{n})$ for some natural n. Denote by $\varepsilon = 1 - na$, so that $0 < \varepsilon < 0.001$ and $a = \frac{1-\varepsilon}{n}$. For each $k \geq 2$ we will find ε_k such that whenever $\varepsilon \leq \varepsilon_k$ there is periodic point with rotation vector $\left(\frac{k}{2kn-k+1}, \frac{k}{2kn-k+1}\right)$, but for $\varepsilon > \varepsilon_k$ no such points exist. The periodic point will travel between the circles in the following manner:
- when k is even, i.e. $k = 2m$ for some $m \in \mathbb{N}$

$$S_v \to \underbrace{S_h \xrightarrow[\text{rotations}]{\text{two}} S_v \to \cdots \to S_h \xrightarrow[\text{rotations}]{\text{two}} S_v}_{m-1 \text{ times}} \xrightarrow[\text{rotations}]{\text{two}} S_v;$$

- when k is odd, i.e. $k = 2m+1$ for some $m \in \mathbb{N}$

$$S_v \to S_h \xrightarrow[\text{rotation}]{\text{one}} \underbrace{S_v \xrightarrow[\text{rotations}]{\text{two}} S_h \to \cdots \to S_h \xrightarrow[\text{rotations}]{\text{two}} S_v}_{m \text{ times}} \xrightarrow[\text{rotation}]{\text{one}} S_v.$$

Conclusion When $m > 1$ and $k = 2m$ (or $k = 2m+1$) we obtain an iterative formula for the periodic point $x_{2kn-k+1}$. Since ε has to satisfy additional inequalities when m increases, the "cut off" for acceptable values of ε will decrease, most likely to zero. This can be computed and made precise. However, it is a lot of work, and a more feasible approach to determine the exact rotation sets (not just diagonal points) would be very welcome here.

It is well known that the points which determine the rotation set are the rotation vectors of periodic points, specifically the ones which correspond to the boundary. Assuming that such points would change in a manner similar to the diagonal points investigated above, we make the following conjecture regarding the shape of the rotation set in the symmetric case $a = b$.

Conjecture. *Suppose $a = b \in [0.1, 0.2]$ and $a \in (\frac{1-0.001}{n}, \frac{1}{n})$ for some natural n. Then the rotation set $\rho(F_{a,b})$ is a pentagon. Consequently, from continuity of the rotation sets with nonempty interior, there exist rotation sets with irrational extreme points that are not limit extreme points.*

One can give a formula for rotation vectors of periodic points. Suppose $a \in [0.1, 0.2]$ and such that $|na - 1| < 0.001$ for some $n \in \mathbb{N}$. Let x be a periodic point with period p. Denote by $i = i(x)$ and $j = j(x)$ the number of full rotations of x on the circles $\mathbb{S}^1_v$ and $\mathbb{S}^1_h$ respectively. We say that a point has a *transition* when it leaves one circle, makes at least one rotation on the other circle and then comes back to the first circle and stays on it for at least one more iteration of F. We denote by $t = t(x)$ the number of transitions of x. Clearly for the rotation vector of x to not lie on the coordinate axes we must have at least one transition, i.e. $t \geq 1$. Since we require the point to complete at least one rotation on the other circle, we see that $t \leq \min\{i, j\}$.

Lemma 4. *Let x be a periodic point and i, j, t and n be as above. Than the rotation vector of x is*

$$\rho(x) = \left(\frac{i}{(i+j)n - t}, \frac{j}{(i+j)n - t} \right).$$

It follows from the previous lemma and the restriction $0 \leq t \leq \min\{i, j\}$ that the rotation vectors of all periodic points are in the pentagon with vertices $(0, 0)$, $\left(\frac{1}{n}, 0\right)$, $\left(0, \frac{1}{n}\right)$, and $\left(\frac{1}{2n-1}, \frac{1}{2n-1}\right)$. It also follows from Lemma 2 that the point on the axis of symmetry of the pentagon closest to the vertex $\left(\frac{1}{2n-1}, \frac{1}{2n-1}\right)$ is of the form

$$\left(\frac{m}{2mn - (m-1)}, \frac{m}{2mn - (m-1)} \right).$$

It is easy to see that rotation vectors of all periodic points on the line connecting the vertices $\left(\frac{1}{2n-1}, \frac{1}{2n-1}\right)$ and $\left(\frac{1}{n}, 0\right)$ must have exactly j transitions. For fixed i the point moves closer to the first vertex when j increases. Therefore, whenever $a \neq \frac{1}{n}$, the closest rotation vector of a periodic point on the right face of the pentagon has the form

$$\left(\frac{k+1}{(2k+1)n - k}, \frac{k}{(2k+1)n - k} \right).$$

The main question is how the numbers m and k are related, for the fixed angle of rotation a.

Our work to determine the answer to this question is ongoing, and progress is currently being made. The week we spent in Banff was invaluable for this project. The in-person interaction, even for a short period, was essential to move forward. We had to be together in a room with a large blackboard for several days to work out the proofs of statements presented in this report. This enabled us, at the end, to formulate our main conjecture. The level of detail discussed as well as the needed duration and pace of interaction could not be accommodated by remote screen sessions. We extend our sincere gratitude for this opportunity to the director, scientific committee, and staff of the BIRS research center.

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