Almost Periodicity in Aperiodic Order

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1 Overview of the Field

Objects with long-range order but no translational symmetry have been of longstanding interest, but the field of Aperiodic Order was catalyzed by the discovery of quasicrystals in the early 1980s. An important goal of this field is to gain a better understanding of mathematical diffraction theory, especially for systems with a significant Bragg (or point) diffraction spectrum but crystallographically forbidden symmetry. Aperiodic order connects various different areas of mathematics, such as harmonic and Fourier analysis, spectral theory of dynamical systems, discrete geometry, number theory, and topology, to name just a few. It was the primary goal of this meeting to bring people from various subareas of Aperiodic Order together.

2 Recent Developments, Open Problems and Presentations

2.1 Pure point spectrum

In recent years, the connection between almost periodicity and pure point diffraction has become clearer. This connection has already appeared implicitly in the work of Hof and Solomyak, and explicitly in the work of Lagarias and Baake–Moody. Building and expanding on extensive previous work in this direction, the connection was recently fully characterized by Lenz, Spindeler and Strungaru.

Indeed, a measure ω has pure point diffraction exactly when it is mean almost periodic, while pure point diffraction and the so-called consistent phase property essentially is equivalent to Besicovitch mean almost periodicity. For dynamical systems, pure point spectrum is equivalent to the Besicovitch mean almost periodicity of almost all elements, while pure point spectrum, unique ergodicity and continuity of eigenfunctions can be characterized in terms of Weyl mean almost periodicity.

Some of the most interesting examples of highly ordered aperiodic point sets are of number-theoretic origin. Systems like square-free integers (the support of the Möbius function) and the visible points of a lattice fit into the larger class of weak model sets of maximal density, which is a new direction in the field. These systems have pure point (diffraction) spectrum, and the connection to the Besicovitch mean almost periodicity starts to become visible. The full characterisation of model sets and of weak model sets of maximal density via almost periodicity remain two important questions in this direction.

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At this conference, systems with pure point spectrum appeared in the talks of Till Hauser, Jan Mazáč and Felipe Garcia-Ramos. Additionally, Jeffrey Lagarias introduced a new aperiodic system based on the floor quotient function, and Lorenzo Sadun explained how the topology of aperiodic tilings affects the mass distributions they admit.

2.2 Systems with mixed spectra

As the case of pure point spectrum gets better understood, several people have shifted their interest towards models with mixed spectra. The Eberlein decomposition for weakly almost periodic measures provides a tool into the individual study of the pure point and the continuous spectrum, respectively. The further study of the absolutely continuous and the singular continuous diffraction spectrum becomes more subtle, as the refined Eberlein convolution is so far only established for measures with Meyer set support. Its existence in general remains an important open question in this area.

A recent new direction is the study of substitution tilings on compact alphabets. These systems generalize substitution tilings, and it is expected that many of the resulting objects have mixed (diffraction) spectra. These objects were covered in the talk of Neil Mañibo. Systems with mixed spectrum also occurred in the talk of Reem Yassawi.

3 Scientific Progress Made

The main goal of the meeting was to introduce some of the new directions in the field to all participants and encourage the collaboration between people from different subfields of aperiodic order. In this direction, the meeting was a success, and there were many discussions between the talks and at the end of the day, which will likely start many new collaborations.

Unfortunately, due to the hybrid format of the meeting, the online participants were often left out of the ongoing discussions, and some potential exchanges of ideas were missed.

4 Outcome of the Meeting

The area of Aperiodic Order is a new and fast growing area, and conferences like this are essential for its development, especially after the challenging CoVid years. While some international collaborations continued during the CoVid shutdowns, the pandemic issue made it hard for new international collaborations to start. The BIRS meeting created a great opportunity for discussions and new collaborations, which will positively impact the field in the future.