A convergence of computable structure theory, analysis, and randomness

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This workshop focused on the newly developing connections between computable structure theory, computable analysis, continuous logic, and algorithmic randomness. While metric structures can be studied through the model-theoretic lens of continuous logic with no additional constraints, computable structure theory has historically been centered around countable algebraic structures such as algebraically closed fields and linear orders. However, with some care, it is possible to study uncountable structures such as Banach spaces and metric spaces in this context and to develop a formal definition for an algorithmically random structure. This workshop brought researchers in these four areas together to build on recent advances in the intersection of these topics and develop new questions in and new approaches to this emerging field of study.

1 Overview

1.1 Computable structure theory

Turing machines provide the standard model of computation. That is, any physical computational device can be simulated by a Turing machine. Turing machines accept as input finite words from a fixed finite alphabet Σ and yield such words as output. However, it is entirely possible that a machine will not halt on some inputs (i.e., computer programs sometime crash). A set S of finite words is *computably enumerable* if it is the halting set of some Turing machine. A fundamental discovery is the existence of a computably enumerable set that is not computable. In other words, there is a Turing machine M so that no Turing machine can determine if M halts on an arbitrary word in Σ^*.

Thus, Turing machines establish a computability theory on the domain Σ^* of all finite words from a fixed finite alphabet Σ. However, mathematical computation takes place in a structure such as the ring of integers or the field of rational numbers. Computation is transferred to these domains by *computable presentations* wherein each element of the structure is labelled with a finite word in such a way that the induced operations and relations (including that of labelling the same element) are computable. The study of such presentations is the heart of computable structure theory. Much of computable structure theory is motivated at some level by the realization that what one can compute on a structure depends on how it is presented. For example, it is easy to show there is a computable presentation of $(\mathbb{N}, <)$ in which the successor relation is not computable. That is, there is no algorithm (Turing machine) that can determine if two words label numbers that differ by 1.

The study of computable presentations of structures is central to much of computability theory. We refer the reader to [1] and [22] for comprehensive treatments of this subject.

1.2 Computable analysis

The just-described framework of presentations only applies to countable structures. However, most scientific computation involves continuous data such as real numbers or compact subsets of the plane. Computable analysis bridges the gap between the discrete and continuous by means of computing with approximations. For example, a real number x is said to be *computable* if there is an algorithm (Turing machine) that, given $k \in \mathbb{N}$ as input, produces a $q \in \mathbb{Q}$ so that $|x - q| < 2^k$. A function $f : \mathbb{R} \to \mathbb{R}$ is computable if there is an algorithm P that satisfies the following conditions.

1. Given a rational interval I as input (that is, given the endpoints of I as input), if P halts then it produces a rational interval J that includes $f[I]$.

2. If U is a neighborhood of $f(x)$, then there is a rational interval I so that on input I P computes a rational interval J that is included in U.

Together, these criteria state that for each $x \in \mathbb{R}$, computable or incomputable, P can compute arbitrarily good approximations of $f(x)$ from sufficiently good approximations of x.

As will be seen below, these definitions can be generalized to a wide variety of Polish spaces. The recently released *Handbook of Computability and Complexity in Analysis* describes this development and many other facets of the subject [3].

1.3 Algorithmic randomness

Algorithmic randomness begins with the question "When is a sequence (finite or infinite) random?" To a first approximation, one might say "If there is no pattern in the sequence." But there is no discernible pattern in the digits of π, yet as these digits can be algorithmically calculated, one can hardly call them random. Fortunately, computability theory provides a rigorous answer to this question. Let λ denote Lebesgue measure on the Cantor space $\{0, 1\}^{\mathbb{N}}$. Informally speaking, $p \in \{0, 1\}^{\mathbb{N}}$ is random if it avoids all null sets that are effectively presented. To see what this means, call an open set $S \subseteq \{0, 1\}^{\mathbb{N}}$ *c.e. open* if there is a computably enumerable set $U \subseteq \{0, 1\}^*$ so that $S = \bigcup_{\sigma \in U}[\sigma]$ (where $[\sigma]$ is the cylinder generated by σ). A *Martin-Löf test* is a sequence $(S_n)_{n \in \mathbb{N}}$ of uniformly c.e. open sets so that $\lambda(S_n) \leq 2^{-n}$. The sequence p avoids such a test if $p \notin \bigcap_n S_n$, and p is *Martin-Löf random* if it avoids all Martin-Löf tests.

There are other notions of randomness such as Schnorr randomness, but most of them are defined similarly and thus our definition above gives a sense of randomness notions in general. The text of Downey and Hirschfeldt provides a very comprehensive treatment of algorithmic randomness [8].

1.4 Continuous logic

Continuous logic is the model theory of metric structures. Unlike classical model theory, there are continuum-many truth values; usually the interval $[0, 1]$. The value 0 represents truth, and the value 1 represents falsehood. The usual sentential connectives are replaced by three: subtraction from 1, multiplication by $\frac{1}{2}$, and bounded subtraction. The universal and existential quantifiers are replaced by supremum and infimum. The equality sign is replaced by a symbol for the metric.

Continuous logic has undergone considerable development in recent decades. In particular, it has been used to develop the model theory of operator algebras [16].

1.5 Connections between these areas

Algorithmic randomness and computable analysis have long been linked. Many theorems from analysis hold on a conull set; researchers in algorithmic randomness define classes of random reals as those that are, in some effective sense, "large with respect to measure." The points for which effectivized versions of theorems such as Birkhoff's ergodic theorem and the Lebesgue differentiation hold have been characterized in terms of randomness [2, 4, 10, 14]. Algorithmic randomness and computable structure theory have also been linked, though not for quite as long [13], and the question of defining a random structure has been considered as well [17, 18].

The very natural connection between computable analyis and computable structure theory has only been recognized fairly recently. One merely needs to expand the definition of computable presentation to metric structures. These are structures that consist of a complete metric space together with collections of operations and functionals; for example Banach spaces, C^* algebras, etc. A computable presentation of a metric structure consists of specifying a dense sequence with respect to which the operations and functionals can be computably approximated with arbitrary precision.

Computable structure theory is the foundation for computable model theory. It is hoped to establish effective metric structure theory as the foundation for a computable model theory of metric structures. In fact, computability theory has already been used to produce a negative solution of the Connes Embedding Problem by showing that the universal theory of the hyperfinite II-1 factor is undecidable.

2 Introductory talks

We began on Monday with an introductory talk in each of the four main areas represented by the participants: continuous model theory, computable structure theory, algorithmic randomness, and computable analysis.

Isaac Goldbring gave us ”A primer in continuous model theory”:

> In this talk, we give an introduction to modern continuous model theory. We use the metric ultraproduct construction to motivate the notion of a continuous language and the appropriate notion of structure for such languages. We give some examples of metric structures of interest in this workshop, such as Banach spaces, Hilbert spaces, and C*-algebras. We then move on to discuss compactness and completeness for continuous logic. Time permitting, we discuss some of the more nuanced aspects of continuous model theory, such as generalized formulas, definable sets, and the metric on type spaces.

Wesley Calvert gave us ”How to think about computable structures”:

> In the very early days of computable structure theory there were only specific algebraic questions about specific structures that were presumed to be well known (e.g. does every explicitly given field have a unique computable algebraic closure). Later, this line of thought was abstracted into computable model theory, which could perhaps have looked something like model theory — except that, as Millar famously remarked, there were ”too many counterexamples.” Over time what took hold was the study of a new category, taking the putative pathologies (e.g. distinctions between classically isomorphic structures) as features of a new and interesting mathematical realm. Apparently niche interests like infinitary logics, admissible set theory, and alpha-jump priority constructions found a natural home in this discipline. This talk will outline the kinds of questions asked in computable structure theory, and the style of arguments used.

Dan Turetsky gave us ”Broad Swathes of Randomness”:

> I will give an overview of algorithmic randomness; this will cover definitions, central and illustrative results, and applications of randomness to other areas of computability theory. My intention is to give a picture of the field and show some of its potential for crossover with other areas.

Finally, Alexander (Sasha) Melnikov gave us ”Computable dualities”:

> I will talk about several recent results that explicitly relate computable algebra with computable topology via various sorts of effective dualities.

Ample time for questions and discussion was provided after each talk.

3 Group formation and problem selection

After these introductory talks on Monday, we began Tuesday morning with an open problem session. We requested that each problem proposed have two properties:

- it should be related to at least two of the general areas described above, and
- it should be specifically enough stated that a group could reasonably be expected to make progress on it over the course of a week.

The participants identified 12 problems, several with constituent subproblems developed by other participants after a more general problem was stated.

After a tea break, we reconvened to form problem groups that would work together on Tuesday afternoon. We did this “AIM-style”: we began with approval voting to identify a short list of problems that the participants would be actively interested in working on and then moved on to choosing our own groups. We began with four problems under consideration.

At the beginnings of Wednesday morning, Thursday morning, and Thursday afternoon, each group reported on their progress to the workshop at large. In addition to reporting on the main ideas, we asked each group to decide whether they would like to continue working on their current problem and, if so, whether they would like to borrow someone with a particular kind of expertise from one of the other groups. We are pleased to say that this ”borrowing” was a regular occurrence. If the group had no interest in pursuing their current problem, the participants would either disperse to existing groups or choose a new problem from the list to work on. Occasionally, participants would decide to spend the next session working with a different group while their usual group continued working.

Three problems were discussed intensely over the week; two others were discussed but were found to be trivial upon closer inspection. Each group remaining on Thursday afternoon spent Friday morning writing summaries of their work and planning their next steps. Indeed, there was some friendly competition over which group was going to meet soonest after the workshop concluded.

4 Final group reports

The following are the reports of the work that was done on the three problems that were discussed most intensely.

4.1 Presentations of C^* algebras

The group considered two questions: the computable presentability of uniformly hyperfinite (UHF) algebras and the computability of the Gelfand transform.

The following result is fundamental to understanding UHF algebras.

Fact: *If m, n, t are positive integers so that $n = tm$, then the map*

$$A \mapsto \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}$$

is an isometric embedding of $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$.

This leads to the definition of uniformly hyperfinite algebras.

Definition 1. A C^*-algebra A is *uniformly hyperfinite* if there is a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers so that $n_k | n_{k+1}$ for each k and so that A is isomorphic to the direct limit $M_{n_0}(\mathbb{C}) \to M_{n_1}(\mathbb{C}) \to \ldots$.

UHF algebras are characterized by their supernatural numbers. These are defined as follows. Let $(p_j)_{j\in\mathbb{N}}$ be the increasing enumeration of the prime numbers.

Definition 2. Suppose A is a UHF algebra. Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of positive integers so that $n_k|n_{k+1}$ for each k and so that A is isomorphic to the direct limit $M_{n_0}(\mathbb{C}) \to M_{n_1}(\mathbb{C}) \to \ldots$. For every $j \in \mathbb{N}$, let

$$f(j) = \sup\{m \in \mathbb{N} \,:\, \exists k\; p^m|n_j\}.$$

f is the *supernatural number of* A.

It is well known that two UHF's are isomorphic if and only if they have the same supernatural number. It is natural to conjecture that a UHF has a computable presentation if and only if it has a computable supernatural number. However, at least one direction of this conjecture is false. In particular, the group demonstrated the following.

Proposition 1. *There is a computably presentable UHF algebra A so that the supernatural number of A is Turing equivalent to* $\mathbf{Tot}$.

Thus, the computability of the supernatural number is too strong a condition to characterize the computable presentability of a UHF algebra. The following definition from computable analysis provides a possible alternative.

Definition 3. Let $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. f is *lower semi-computable* if there is a computable $g : \mathbb{N}^2 \to \mathbb{N}$ so that for every $n \in \mathbb{N}$, $(g(n,s))_{s\in\mathbb{N}}$ is nondecreasing and $\lim_s g(n,s) = f(n)$.

The following are fairly straightforward consequences of the definitions.

Proposition 2. *Suppose A is a UHF algebra. Then the following are equivalent.*

1. *The supernatural number of A is lower semi-computable.*
2. *There is a computable sequence $(n_k)_{k\in\mathbb{N}}$ so that A is isomorphic to the directed limit $M_{n_0}(\mathbb{C}) \to M_{n_1}(\mathbb{C}) \to \ldots$ and $n_k|n_{k+1}$.*

Corollary 1. *If the supernatural number of a UHF algebra A is lower semi-computable, then A is computably presentable.*

However, we are left with the following.

Question 1. *If A is a computably presentable UHF algebra, does it follow that the supernatural number of A is lower semi-computable?*

The group then considered the computability of the Gelfand transform. This transform represents a unital commutative C^* algebra A as $C^*(X)$ for a suitably chosen compact metric space X. The group obtained the following, which is currently being prepared for submission.

Theorem 1. *If an Abelian C^* algebra A is computably presentable, then there is a metric space X with a computably compact presentation so that A is isomorphic to $C^*(X)$.*

The converse of Theorem 1 is already known to be true [9]. The proof of Theorem 1 is uniform in the sense that it yields a procedure that, given an index of a presentation of A, yields an index of a computably compact presentation of X.

4.2 Random structures

The initial question this group sought to address was this: Can you adapt Gromov-style randomness (for groups) to random Banach spaces? If there is a good notion, would it have to contain a classical sequence space like ℓ^p or C_0 as Banach thought would be the case? The subquestions presented in the open problem session were "What is a random compact Polish space, and what would the right measure be?" and "Is it more appropriate to consider genericity?"

We discussed "Gromov-style" typicality in the sense of Franklin, Ho, and Knight [11] for Banach spaces and concluded that it wasn't a useful approach: the idea of *limiting density* relies on the ability to consider larger and larger proportions of the structures under discussion in a reasonable way based on their presentations, and we did not see a plausible way to carry that out.

We began with the question of what the space of compact Polish spaces should look like and identified it with the space $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ of compact and overt subsets of the Hilbert cube. This approach is equivalent to coding a compact Polish space as the completion of a metric on $\mathbb{N}$ together with a witness of total boundedness.

Avoiding any particular rational hyperplane in $[0,1]^\omega$ is a dense c.e. open property for elements $(\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$, which implies that any 1-generic compact Polish space is totally disconnected. A 1-generic compact Polish space also has no isolated points, as for any $x \in [0,1]^\omega$ and $0 < r$ it is a dense c.e. open property for $A \in (\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ that either $\overline{B}(x,r) \cap A = \emptyset$ or $|B(x,r) \cap A| > 1$.

This reasoning establishes that a 1-generic compact Polish space is homeomorphic to 2^ω. We can even say a bit more and observe that 2^ω is relatively categorical as a compact Polish space, meaning that from a name for some $A \in (\mathcal{K} \wedge \mathcal{V})([0,1]^\omega)$ such that A is homeomorphic to 2^ω we can compute a homeomorphism from 2^ω to A. It is important here that the compact information is provided since 2^ω is not relatively computably categorical as a Polish space.

Being connected is a computably closed property of compact Polish spaces, so the space of connected compact Polish spaces can be identified with a Polish space again. It then makes sense to ask about 1-generic connected compact Polish spaces. It was suggested that such a generic space should be the pseudo-arc.

To define the space of separable Banach spaces, we make use of the fact that any separable Banach space can be obtained by equipping the vector space $c_{00}(\mathbb{Q})$ of finitely supported sequences of rational sequences with a suitable seminorm and then completing it. Being a seminorm is an effectively closed property of some $p \in \mathbb{R}_{\geq 0}^{c_{00}(\mathbb{Q})}$, which gives us a Polish space of representatives for separable Banach spaces.

We then turned our attention to the question of what a *generic Banach space* would look like. One of the options proposed was C[Cantor set], in part because it is universal.

The next day, we were directed to the Gurarij space. The article [15] contains a proof that the Gurarij space is a generic Banach space in the sense of genericity for a forcing condition. Their arguments should translate rather directly into showing that a 1-generic element of the space of separable Banach spaces mentioned above is a Gurarij space.

4.3 Supernormal numbers

A well-known result characterizes normal numbers by a condition regarding compressibility by finite-state transducers. In view of this result, we first relativize the notion of normality to a given enumeration and then strengthen the notion of normality by requiring a number to be normal to every enumeration.

Definition 4. We define C_r^f and $C_{r,D}^f$:

$$C_r^f(x) := \min\{|\sigma| : |f(\sigma) - x| < 2^{-r}\}$$

$$C_{r,D}^f(x) := \min\{C_D(\sigma) : |f(\sigma) - x| < 2^{-r}\}$$

Definition 5. Given a function f, we say that a real x is strongly f-normal ($x \in \mathrm{SNorm}^f$) if and only if for every finite state machine D there is a constant k such that for all n, we have

$$C_{n,D}^f(x) \geq n - K(n) - k.$$

Definition 6. Given a prefix-free function f, we say that a real x is weakly f-normal $x \in \mathrm{WNorm}^f$) if and only if there is a constant k such that for all n,

$$C_n^f(x) \geq n - K(n) - k.$$

We proved the following:

Theorem 2. *The following properties of a real x are equivalent.*

1. *For all upper semi-computable functions f, the real x is strongly f-normal.*

2. *For all upper semi-computable functions f, the real x is weakly f-normal.*
3. *The real x is strongly f-normal for some universal upper semi-computable function f.*
4. *The real x is weakly f-normal for some universal upper semi-computable function f.*

We then considered supernormal reals:

Definition 7. We say that a real x is supernormal if and only if it satisfies one (equivalently, all) of the equivalent conditions of Theorem 2.

Our results on these reals follow.

Proposition 3. *If a real A is supernormal, then A is Martin-Löf random.*

Proposition 4. *Every 2-random real is supernormal.*

Proposition 5. $\overline{\Omega}$ *is not supernormal.*

Proposition 6. Ω *is supernormal.*

5 Organizational matters

5.1 Diversity, equity, inclusion, and belonging

The organizers set out from the beginning to create a workshop that not only included participants from underrepresented groups but also actively welcomed them. When we made the ”long list” of participants, we sought balance along many different dimensions, including the participant's gender, ethnicity, type of employing institution, and geographical location. If we felt we were deficient in one area, we asked an appropriate colleague for suggestions. Then, when we sent invitations, we sought to maintain the balance we had established and had a reasonable amount of success in doing so.

Before the workshop, we established an unofficial mentoring program for the graduate students who would be present. Due to the pandemic, they were less acquainted with the other participants in the workshop than we could have expected five years ago. We introduced them to each other by e-mail, and then, after confirming that this would be acceptable to them, introduced them to some more senior workshop participants by e-mail in advance to give them a greater sense of belonging.

5.2 Virtual participation

Based on advance polling of the participants who planned to attend virtually, we decided to establish a group Overleaf project and a Dropbox folder. The Dropbox folder would contain slides from the introductory talks and papers that the participants found helpful, while the progress from the week would be recorded in the Overleaf project. We felt that the presence of Zoom would render tools such as a Slack workspace pointless.

It seems that virtual participation in the working groups worked as well as the schedules of the virtual participants permitted. Indeed, some of the virtual participants mentioned that this was the smoothest integration of virtual participation into a workshop that they had ever experienced; we would particularly like to thank the BIRS staff who provided our tech support. At the beginning of each morning and afternoon working session, each group would leave a Zoom link in the main Overleaf document, and a member of the group would keep the Zoom link open. We would assign groups to different spaces in the Juniper Hotel based on not only the group's size but also the type of AV tech available in each space; the group working on presentations of C^* algebras had the most virtual participants and thus kept the main room that was set up for talks.

It should be noted that this ability to accommodate virtual participation was necessary for the workshop's success. There were 16 participants on site and 11 virtual participants. Of the virtual participants, four initially intended to attend in person but could not. One participant had to change his participation from on site to virtual last minute due to health issues, one due to local responsibilities, and two more due to the unexpected inability to get a Canadian visa in a reasonable amount of time.

6 Outcome of the workshop

The feedback from the meeting participants was overwhelmingly positive both in terms of the meeting's organization and structure and the scientific content. While all of the participants were well versed in at least one of the four areas represented at this meeting and some were experts in two, none of the participants were familiar with all of them. This resulted in very productive and open conversations.

Furthermore, the structure of the meeting seems to have enabled the groups to work together very effectively. The introductory talks on the first day gave everyone a common framework and provided a familiar setting for the virtual participants to start in. The regular updates and corresponding constant self-evaluations of each of the working groups made it easy for the groups to decide whether they were still being productive, whether they needed to "borrow" another participant, or whether it would make sense for them to disband and work on another topic.

This workshop was beset by practical challenges. A similar proposal by the first two organizers was accepted for a BIRS workshop in May 2020, but the meeting was cancelled due to the pandemic. Later, when they were offered the opportunity to hold the workshop virtually, they declined given the interactive events they anticipated and chose to reapply instead, adding the third organizer. Then, at this March 2023 workshop, two of the four participants who gave introductory talks had to change their participation from on site to virtual, and two of the three organizers developed health issues that precluded them from attending the meeting on site (indeed, one of them had to withdraw from participation entirely). Nonetheless, the meeting was a success.

References

[1] C. J. Ash and J. Knight, *Computable structures and the hyperarithmetical hierarchy*, Studies in Logic and the Foundations of Mathematics, vol. 144, North-Holland Publishing Co., Amsterdam, 2000.

[2] L. Bienvenu, A. R. Day, M. Hoyrup, I. Mezhirov, and A. Shen, A constructive version of Birkhoff's ergodic theorem for Martin-Löf random points, *Inform. and Comput.*, **210** (2012), 21–30.

[3] V. Brattka and P. Hertling, eds., *Handbook of Computability and Complexity in Analysis*, Springer, Cham, 2021.

[4] V. Brattka, J. S. Miller, and A. Nies, Randomness and differentiability, *Trans. Amer. Math. Soc.*, **368** (2016), 581–605.

[5] T. Brown and T.H. McNicholl, Analytic computable structure theory and L^p-spaces part 2, *Arch. Math. Logic*, **59** (2020), 427–443.

[6] J. Clanin, T.H. McNicholl, and D. M. Stull, Analytic computable structure theory and L^p spaces, *Fund. Math.*, **244** (2019), 255–285.

[7] R.G. Downey and A.G Melnikov, Computably compact metric spaces. To appear in the *Bulletin of Symbolic Logic*.

[8] Rodney G. Downey and Denis R. Hirschfeldt, *Algorithmic Randomness and Complexity*, Springer, 2010.

[9] A. Fox, Computable presentations of C*-algebras. To appear in the *Journal of Symbolic Logic*.

[10] J. N.Y. Franklin, N. Greenberg, J. S. Miller, and K.M. Ng, Martin-Löf random points satisfy Birkhoff's ergodic theorem for effectively closed sets, *Proc. Amer. Math. Soc.*, **140** (2012), 3623–3628.

[11] J. N.Y. Franklin, M.-C. Ho, and J. Knight, Free structures and limiting density. Submitted.

[12] J. N.Y. Franklin and T. H. McNicholl, Degrees of and lowness for isometric isomorphism, *J. Log. Anal.*, **12** (2020), Paper No. 6.

[13] J. N.Y. Franklin and R. Solomon, Degrees that are low for isomorphism, *Computability*, **3** (2014), 73–89.

[14] P. Gács, M. Hoyrup, and C. Rojas, Randomness on computable probability spaces—a dynamical point of view, *Theory Comput. Syst.*, **48** (2011), 465–485.

[15] J. Garbulińska and W. Kubiś, Remarks on Gurariĭ spaces, *Extracta Mathematicae*, **26**, (2011), 235–269.

[16] I. Goldbring, ed., *Model Theory of Operator Algebras*, De Gruyter, Berlin, 2023.

[17] B. Khoussainov, A quest for algorithmically random infinite structures, In *Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, Article No. 56, ACM, 2014.

[18] B. Khoussainov, A quest for algorithmically random infinite structures, II, In *Logical foundations of computer science*, Lecture Notes in Comput. Sci., **9537**, 159–173, Springer, 2016.

[19] T. H. McNicholl, Computable copies of ℓ^p, *Computability*, **6** (2017), 391 – 408.

[20] A. G. Melnikov, Computably isometric spaces, *J. Symbolic Logic*, **78** (2013), 1055–1085.

[21] A.G. Melnikov and A. Nies, The classification problem for compact computable metric spaces. In *The Nature of Computation*, Lecture Notes in Computer Science, **7921**, 320–328, Springer, Heidelberg, 2013.

[22] Antonio Montalbán, *Computable Structure Theory—Within the aArithmetic*, Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Ithaca, NY, 2021.