Advanced Nonlinear Studies **14** (2014), xxx–xxx

Pointwise Lower Bounds for Solutions of Semilinear Elliptic Equations and Applications

Asadollah Aghajani, Alireza Mosleh Tehrani

Department of Mathematics
Iran University of Science and Technology, Tehran, Iran 16844-13114
e-mail: aghajani@iust.ac.ir, amtehrani@iust.ac.ir

Nassif Ghoussoub

Department of Mathematics
University of British Columbia, Vancouver, B.C. Canada V6T 1 Z2
e-mail: nassif@math.ubc.ca

Received on 31 July 2014
Communicated by Shair Ahmad

Abstract

We consider the semilinear elliptic problem $-\Delta u = f(x,u)$, posed in a smooth bounded domain Ω of R^N with Dirichiel data $u|_{\partial\Omega} = 0$, where $f : \Omega \times [0, a_f) \to \mathbb{R}_+$ $(0 < a_f \leqslant +\infty)$ is a function of appropriate regularity which blows up at a_f. We give pointwise lower bounds for the supersolutions under some appropriate conditions on f, and apply them to eigenvalue problem $-\Delta u = \lambda f(x,u)$, by giving upper and lower bounds for the extremal parameter λ^* and the extremal solution u^*. To demonstrate the sharpness of our results, we consider the eigenvalue problem $-\Delta u = \lambda f(u^p)$ $(p \geqslant 1)$ with Dirichlet boundary condition, and show that for every increasing, convex and superlinear C^2 function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) > 0$, $\lambda_p^* \to \frac{1}{f(0)\psi_\Omega}$ and $\|u_p^*\|_\infty \to \infty$, where ψ_Ω is the maximum of the torsion function of Ω. Also, we consider the eigenvalue problem $-\Delta u = \lambda\rho(x)f(u)$, where f is either a regular singularity such as $f(u) = e^u$, or a singular one such as $f(u) = \frac{1}{(1-u)^2}$ and give explicit estimates on λ^* and u^*, that improve and extend several results in the literature, by Payne[17], Sperb [21], Brezis-Vasquez [3], Guo-Pan-Ward [11], Ghoussoub-Guo [10], Cowan-Ghoussoub [6], and others.

1991 Mathematics Subject Classification. 35B40, 35P30, 35J91, 35B32.
Key words. semilinear elliptic problem, MEMS, nonlinear eigenvalue problem, extremal solution.

1 Introduction and preliminaries

We consider semilinear second-order elliptic equation of the form

$$\begin{cases} -\Delta u = f(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in R^N and $f : \Omega \times [0, a_f) \to \mathbb{R}_+$ $(0 < a_f \leqslant +\infty)$ is a given function which blows up at a_f. It is said that a solution of (1.1) is *classical* provided the map $\widetilde{f}(x) := f(x, u(x)) \in C(\Omega)$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Note that if f is Hölder continuous in its first variable and locally Lipschitz continuous in the second variable; by elliptic regularity theory, this is equivalent to saying that a classical solution is in $C^{2,\alpha}$ for some $\alpha > 0$ [20]. The aim of this paper is to give a nonexistence result for problem (1.1) and find pointwise lower bounds for supersolutions of (1.1) under some appropriate conditions on f. Then we apply the results to study the corresponding eigenvalue problem, i.e.,

$$\begin{cases} -\Delta u = \lambda f(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In particular, we consider the nonlinear eigenvalue problem

$$(P_{\lambda,\rho}) \qquad \begin{cases} -\Delta u = \lambda\rho(x) f(u) & x \in \Omega, \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in R^N, $0 < \lambda$, ρ is a nonnegative nonzero bounded Hölder continuous function and $f : [0, a_f) \to \mathbb{R}_+$ is a smooth, increasing, convex nonlinearity such that $f(0) > 0$ and which blows up at the endpoint of its domain. We consider two cases (according the notations of [6]) either f is a *regular nonlinearity* i.e., f is superlinear, namely $f(t)/t \to \infty$ as $t \to \infty$, and its domain is $D_f := [0, +\infty)$, or when $D_f := [0, 1)$ and $\lim_{t \nearrow a_f} f(t) = +\infty$ called a *singular nonlinearity*. Typical examples of regular nonlinearities f are e^u, $(1 + u)^p$ for $p > 1$, while singular nonlinearities include $(1 - u)^{-p}$ for $p > 1$.

It is said that a solution of $(P_{\lambda,\rho})$ is *classical* provided $\|u\|_{L^\infty} < \infty$ (resp., $\|u\|_\infty < 1$) if f is a regular (resp., singular) nonlinearity. Note that by elliptic regularity theory, this is equivalent to saying that a classical solution is in $C^{2,\alpha}$ for some $\alpha > 0$ [6]. It is known that there exist an extremal parameter $\lambda^*(\Omega, \rho) \in (0, \infty)$ depending on Ω, ρ and N, such that $(P_{\lambda,\rho})$ has a minimal classical solution $u_\lambda \in C^2(\overline{\Omega})$ if $0 < \lambda < \lambda^*(\Omega, \rho)$ while no solution exists, even in the weak sense, for $\lambda > \lambda^*(\Omega, \rho)$ (see [10] for a precise definition of weak solution). It can be shown that the map $\lambda \mapsto u_\lambda$ is increasing in λ. Its increasing pointwise limit $u^*(x) := \lim_{\lambda \to \lambda^*(\Omega,\rho)} u_\lambda(x)$ is a weak solution of $(P_{\lambda,\rho})$ for $\lambda = \lambda^*(\Omega, \rho)$ which is called the extremal solution of $(P_{\lambda,\rho})$ (see [10, 13, 14, 2]). For problem $(P_{\lambda,1})$ we refer the reader to L.E. Payne [16, 17, 18]. In this paper we are mostly interested in upper and lower bounds for the extremal parameter λ^*, as well as pointwise lower bounds for the corresponding extremal solution u^* of $(P_{\lambda,\rho})$.

The ball of radius R centred at x_0 in $\mathbb{R}^N$ will be denoted by $B_R(x_0)$. If $x_0 = 0$ and $R = 1$, then we just write B. Given a set Ω in $\mathbb{R}^N$ we let $|\Omega|$ denote its N-dimensional Lebesgue measure, while ω_N denotes the volume of the unit ball B in $\mathbb{R}^N$. The *torsion* ψ of a domain Ω is the non-negative function $\psi \in C^2(\overline{\Omega})$, that is the unique classical solution of the problem

$$\begin{cases} -\Delta u = 1 & x \in \Omega, \\ \quad u = 0 & x \in \partial\Omega. \end{cases}$$

We shall denote $\psi_\Omega := \sup_{x\in\Omega} \psi(x) = \|\psi\|_\infty$. It is a classical result [21] that whenever f is an increasing function with $f(0) > 0$, then the extremal parameter $\lambda^*(\Omega, 1)$ for problem $(P_{\lambda,1})$ satisfies the estimates:

$$\frac{1}{\psi_\Omega} \sup_{0<t<a_f} \frac{t}{f(t)} \leqslant \lambda^*(\Omega, 1) \leqslant \lambda_1(\Omega) \sup_{0<t<a_f} \frac{t}{f(t)}, \tag{1.3}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of Laplacian with Dirichlet boundary condition. Brezis-Vasquez [3], Gazzola-Malchiodi [9], Guo-Pan-Ward [11], Ghoussoub-Guo [10], Cowan-Ghoussoub [6], and others improved these estimates, at least for some specific nonlinearities f and provided upper and lower L^∞ estimates for the minimal solutions of nonlinear eigenvalue problem $(P_{\lambda,\rho})$. In this paper, we offer another approach to these problems, which will yield improvements to the known estimates on both λ^* and $\|u^*\|_\infty$.

The upper bounds will be dealt with in section 4, where we prove that

$$\lambda^*(\Omega, 1) \leqslant \frac{1}{\psi_\Omega} \int_0^{a_f} \frac{dt}{f(t)}, \tag{1.4}$$

which, in many cases, represents a sharper upper bound than (1.3). Indeed, for example in the case of $f(u) = (1-u)^{-2}$, Ghoussoub-Guo [10] used Pohozaev-type arguments, to show that for a star-shaped domain Ω, one has $\lambda^*(\Omega, 1) \leqslant \frac{(N+2)^2 P(\Omega)}{8aN|\Omega|}$, where $P(\Omega)$ is the perimeter of Ω. In particular, $\lambda^*(B, 1) \leqslant \frac{(N+2)^2}{8}$, where B is the unit ball. Formula (1.4) gives however a much better estimate, since in this case, $F(t) := \int_0^t \frac{ds}{f(s)} = \frac{1-(1-t)^3}{3}$, and $\psi_\Omega = \frac{1}{2N}$ yielding that for all $N \geqslant 3$, $\lambda^*(B, 1) \leqslant \frac{2N}{3}$.

Actually, (1.4) is an improvement on (1.3) for any nonlinearity f as long as the dimension is large enough. Indeed, by using known asymptotics for $\lambda_1(\Omega)$, one can show that if

$$\bar{\bar{\lambda}}(N) := \frac{1}{\psi_\Omega} \int_0^{a_f} \frac{ds}{f(s)} \quad \text{and} \quad \bar{\lambda}(N) := \lambda_1(\Omega) \sup_{0\leqslant t\leqslant a_f} \frac{t}{f(t)},$$

then $\frac{\bar{\lambda}(N)}{N} \to \infty$ goes to infinity as the dimension $N \to \infty$, while $\frac{\bar{\bar{\lambda}}(N)}{N}$ remains bounded.

Another illustration of how our estimate (1.4) is an improvement on (1.3), we consider in section 4, semilinear second-order elliptic equations of the form

$$\begin{cases} -\Delta u = \lambda f(u^p) & x \in \Omega, \\ \quad u = 0 & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded smooth domain in R^N, $p \geqslant 1$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, convex and superlinear C^2 function with $f(0) > 0$. Denoting by λ_p^* (resp. u_p^*) the extremal parameter (resp. the extremal solution) of problem (1.5), we show by using (1.4) that

$$\lim_{p\to\infty} \lambda_p^* = \frac{1}{f(0)\psi_\Omega} \quad \text{and} \quad \lim_{p\to\infty} \|u_p^*\|_\infty = +\infty.$$

In particular, when $f(0) = 1$, then $\lim_{p\to+\infty} \lambda_p^*(B) = 2N$. Note that with the earlier result (1.3), one only arrives at

$$\frac{1}{f(0)\psi_\Omega} \leqslant \liminf_{p\to+\infty} \lambda_p^* \leqslant \limsup_{p\to+\infty} \lambda_p^* \leqslant \frac{\lambda_1(\Omega)}{f(0)}.$$

In section 3, we provide improvements for the lower bound for the critical parameter λ^*, by establishing the following estimate:

$$\sup_{0<t<\frac{\|F\|_\infty}{\psi_\Omega}} t - t^2\beta(t) \leqslant \lambda^*(\Omega, 1), \tag{1.6}$$

where

$$\beta(t) = \sup_{x\in\Omega} f'(F^{-1}(t\psi(x)))|\nabla\psi(x)|^2, \tag{1.7}$$

and $F(t) := \int_0^t \frac{ds}{f(s)},\ t > 0$. In particular, if Ω is the unit ball B, we then have

$$\sup_{0<t<\|F\|_\infty} 2Nt - 4t\alpha(t) \leqslant \lambda^*(B, 1), \tag{1.8}$$

where $\alpha(t) = \sup\limits_{0<s<F^{-1}(t)} f'(s)(t - F(s))$. As we shall see, this lower bound (1.8) is sometimes an improvement on the estimate (1.3), and yields –at least in some dimensions– the exact value of the extremal parameter for the standard nonlinearities $f(u) = e^u$, $f(u) = (1 + u)^p$ and $f(u) = (1 - u)^{-p}$ with $p > 1$. For example for $f(u) = e^u$, the previous estimate (1.3) gives that $\lambda^*(B) \geqslant \frac{2N}{e}$, while our formula above gives that

$$\lambda^*(B) \geqslant \max\Big\{\sup_{0<t<1} 2Nt - 4t^2,\ \frac{2N}{e}\Big\} = \begin{cases} \frac{2N}{e} & N = 1, 2, \\ \frac{N^2}{4} & N = 3, 4, \\ 2(N-2) & N \geqslant 5. \end{cases}$$

We also improve the lower bound on the L_∞-norm of u^* given by Cowan-Ghoussoub [6]. For example, and again in the case of $f(u) = e^u$, they show that $\|u^*\|_\infty \geqslant 1$, while our result yields that

$$\|u^*\|_\infty \geqslant -\ln(1 - \lambda^*(\Omega)\psi_\Omega),$$

which in the case of the unit ball gives that $\|u^*\|_\infty \geqslant \ln \frac{N}{2}$, hence a better lower bound when $6 \leqslant N \leqslant 9$.

We also note that Brezis-Vasquez in [3] and others established that the extremal parameter of problem $(P_{\lambda,1})$ with $f(u) = e^u$ satisfies $\lambda^*(B, 1) = 2(N - 2)$ for $N \geqslant 10$ by finding the exact singular solution u^*. Then they concluded by comparison that $\lambda^*(B, 1) > 2(N - 2)$ for $3 \leqslant N \leqslant 9$ using the fact that the extremal solution in this case must be regular. A similar reasoning is used for the case when $f(u) = (1 + u)^p$. Note that our approach yields similar results without knowing the explicit formula for the extremal solution u^* or its regularity. Also we obtained a better lower bound when $f(u) = e^u$ when $N = 2, 3$, and also for $f(u) = (1 + u)^p$ when $p \leqslant \frac{N}{N-4}$.

In summary, we have the following estimates for the unit ball in $\mathbb{R}^N$, which contains essentially all previously known results.

$$\max\Big\{\sup_{0<t<\|F\|_\infty} 2Nt - 4t\beta(t),\ 2N\sup_{0<t<a_f}\frac{t}{f(t)}\Big\} \leqslant \lambda^*(B,1),$$

and

$$\lambda^*(B,1) \leqslant \min\Big\{2N\|F\|_\infty,\ \lambda_1(B)\sup_{0<t<a_f}\frac{t}{f(t)}\Big\}.$$

Another problem that we treat in section 2 is the nonlinear eigenvalue problem $(P_{\lambda,\rho})$, when ρ can vanish somewhere on the domain Ω. We establish the following estimate without assuming that $\inf_{x\in\Omega}\rho(x) > 0$.

$$\lambda^*(\Omega,\rho) \leqslant \frac{2N\|F\|_\infty}{\sup_{x\in\Omega}\Big\{\rho_x(d_\Omega(x))d_\Omega^2(x)\Big\}}, \tag{1.9}$$

where $d_\Omega(x)$ is the distance of x to the boundary $\partial\Omega$ and $\rho_x(r) := \inf\limits_{y\in B_r(x)}\rho(y)$ defined for any $r \leqslant d_\Omega(x)$. For example, if $\rho(x) = |x|^\alpha$ and Ω is a ball of radius R, we obtain that

$$\lambda^*(B_R,|x|^\alpha) \leqslant \frac{2N}{\alpha^\alpha}\Big(\frac{\alpha+2}{R}\Big)^{(\alpha+2)}\int_0^{a_f}\frac{dt}{f(t)}.$$

In particular, we consider the problem

$$(M_{\lambda,\rho,p}) \qquad \begin{cases} -\Delta u = \lambda\dfrac{\rho(x)}{(1-u)^p} & x\in\Omega,\\ u = 0 & x\in\partial\Omega,\end{cases}$$

where $p \geqslant 1$, which has often been used to model Micro-Electro-Mechanical devices (MEMS). Here $\lambda > 0$ is proportional to the applied voltage, $0 < u(x) < 1$ denotes the deflection of an underlying membrane and $\rho(x)$ is the so-called permittivity profile [7, 11]. By $\lambda^*(\Omega,\rho,p)$ we mean the extremal parameter of problem $(M_{\lambda,\rho,p})$. We improve on some of the recently obtained results by Guo-Pan-Ward [11], Ghoussoub-Guo [10], Cowan-Ghoussoub [6], and others about upper and lower L^∞ estimates for the minimal solutions of nonlinear eigenvalue problem $(M_{\lambda,\rho,2})$. See also [4, 5, 15] and the references cited therein.

Note for example that estimate (1.9) yields in this case that

$$\lambda^*(\Omega,\rho,2) \leqslant \frac{2N}{3\sup_{x\in\Omega}\Big\{\rho_x(d_\Omega(x))d_\Omega^2(x)\Big\}} =: \bar{\bar{\lambda}}(N), \tag{1.10}$$

which is worth comparing to the following upper bound obtained in [10]:

$$\lambda^*(\Omega,\rho,2) \leqslant \min\Big\{\frac{4\lambda_1(\Omega)}{27\inf_{x\in\Omega}\rho(x)}, \frac{\lambda_1(\Omega)}{\int_\Omega\rho\phi dx}\Big\} := \bar{\lambda}(N), \tag{1.11}$$

where ϕ is the normalized positive eigenfunction correspond to $\lambda_1(\Omega)$ with $\int_\Omega \phi dx = 1$. Again, one can see that in the case of a unit ball B, $\frac{\bar{\bar{\lambda}}}{N}$ remains bounded as $N \to \infty$, while $\frac{\bar{\lambda}}{N} \to \infty$. Actually, if $\inf_{x\in\Omega} \rho(x) > 0$, then (1.10) yields for a general domain Ω, that

$$\lambda^*(\Omega, \rho, 2) \leqslant \frac{2N}{3r_\Omega^2 \inf_{x\in\Omega} \rho(x)},$$

where $r_\Omega := \sup\limits_{x\in\Omega} d_\Omega(x)$ is the Chebyshev radius of Ω. This means that (1.10) is better than (1.11) whenever

$$\frac{\lambda_1(\Omega)}{N} \geqslant \frac{1}{r_\Omega^2} \max\left\{\frac{9}{2}, \frac{2\sup_{x\in\Omega} \rho(x)}{3\inf_{x\in\Omega} \rho(x)}\right\}.$$

In section 5, we collect corresponding explicit estimates for the standard nonlinearities $f(u) = e^u$, $f(u) = (1+u)^p$ and $f(u) = (1-u)^{-p}$ with $p > 1$.

2 Pointwise lower bound for the supersolutions of problem (1.1)

In this section, first we obtain a pointwise lower bound for the supersolutions of problem (1.1) i.e.,

$$\begin{cases} -\Delta u \geqslant f(x,u) & x \in \Omega, \\ u \geqslant 0 & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $f : \Omega \times [0, a_f) \to \mathbb{R}_+$ ($0 < a_f \leqslant +\infty$) is a function such that $m_{f,\Omega}$ (defined below) is an increasing C^1-function with $m_{f,\Omega}(0) > 0$. Here we have denoted

$$m_{f,\Omega}(t) := \inf_{x\in\Omega} f(x,t) \quad \text{and} \quad M_{f,\Omega}(t) := \sup_{x\in\Omega} f(x,t) \quad \textit{for all } t \in [0, a_f). \tag{2.2}$$

Proposition 2.1 *Let $u \in C^2(\Omega)$ be a solution of* (2.1)*, then*

$$F(u(x)) \geqslant \psi(x) \quad \textit{for all } x \in \Omega,$$

where ψ is the torsion function and $F(t) = \int_0^t \frac{ds}{m_{f,\Omega}(s)}$ for $t \in [0, a_f)$.

Proof. By a simple computation we have

$$\begin{aligned} \Delta F(u) &= F''(u)|\nabla u|^2 + F'(u)\Delta u \\ &= \frac{-m'_{f,\Omega}(u)}{m^2_{f,\Omega}(u)}|\nabla u|^2 + \frac{\Delta u}{m_{f,\Omega}(u)} \\ &\leqslant \frac{-m'_{f,\Omega}(u)}{m^2_{f,\Omega}(u)}|\nabla u|^2 - 1 \leqslant -1 = \Delta\psi. \end{aligned}$$

Thus $\Delta(F(u(x)) - \psi(x)) \leqslant 0$ for all $x \in \Omega$. On the other hand, $F(u(x)) - \psi(x) \geqslant 0$ for all $x \in \partial\Omega$, hence, by the maximum principle we must have $F(u(x)) \geqslant \psi(x)$ for all $x \in \Omega$, as claimed. □ An immediate application of the above is that every solution $u \in C^2(\Omega)$ of $(P_{\lambda,1})$ satisfies the following

$$F^{-1}(\lambda\psi(x)) \leqslant u(x) \quad \text{for all } x \in \Omega,$$

where F is defined by $F(t) = \int_0^t \frac{ds}{f(s)}$.

Now, consider the problem

$$\begin{cases} -\Delta u \geqslant \rho(x) f(u) & x \in \Omega, \\ u \geqslant 0 & x \in \partial\Omega. \end{cases} \tag{2.3}$$

where Ω is a bounded domain in R^n, ρ is a nonnegative function and $f : [0, a_f) \to [0, \infty)$ $(0 < a_f \leqslant \infty)$ is a nondecreasing C^1 regular or singular function such that $f(0) > 0$. Here, if one would like to apply Theorem 2.1, one would have to impose the condition $\inf_{x\in\Omega} \rho(x) > 0$. To get around this assumption, which is not desirable in MEMS models, we consider the following notion. We define for any $r < d_\Omega(x) := \text{dist}(x, \partial\Omega)$, the function

$$\rho_x(r) := \inf_{y \in B_r(x)} \rho(y). \tag{2.4}$$

The following theorem gives pointwise lower bounds for solutions of (2.3).

Theorem 2.1 *Let $u \in C^2(\Omega)$ be a solution of* (2.3)*. Then*

$$u(y) \geqslant F^{-1}\Big(\rho_x(d_\Omega(x)) \tfrac{d_\Omega(x)^2 - |y-x|^2}{2N}\Big) \quad \textit{for all } x, y \in \Omega \textit{ such that } |y - x| < d_\Omega(x), \tag{2.5}$$

where $F(t) := \int_0^t \frac{ds}{f(s)}$. In particular, we have

$$u(x) \geqslant F^{-1}\Big(\rho_x(d_\Omega(x)) \tfrac{d_\Omega(x)^2}{2N}\Big) \quad \textit{for all } x \in \Omega. \tag{2.6}$$

If $\rho(x) = |x|^\alpha$, $\alpha > 0$, we then have

$$u(x) \geqslant F^{-1}\Big((|x| - d_\Omega(x))^\alpha \frac{d_\Omega(x)^2}{2N}\Big) \;\; if \;\; |x| > d_\Omega(x). \tag{2.7}$$

Proof. Since $B_{d_\Omega(x)}(x) \subseteq \Omega$ for all $x \in \Omega$, it follows from (2.3) that for all $y \in B_{d_\Omega(x)}(x)$ we have

$$\begin{aligned} \Delta F(u(y)) &= \frac{-f'(u(y))}{f^2(u(y))} |\nabla u|^2 + \frac{\Delta u(y)}{f(u(y))} \\ &\leqslant -\rho(y) \\ &\leqslant -\rho_x(d_\Omega(x)). \end{aligned} \tag{2.8}$$

Now consider the auxiliary function $w(y) = \frac{d_\Omega(x)^2 - |y-x|^2}{2N}$, which satisfies $\Delta w = -1$ in $B_{d_\Omega(x)}(x)$ and $w = 0$ on $\partial B_{d_\Omega(x)}(x)$. From (2.8) we get that

$$\Delta\Big(F(u(y)) - \rho_x(d_\Omega(x)) w(y)\Big) \leqslant 0 \;\text{ in }\; B_{d_\Omega(x)}(x),$$

and

$$F(u(y)) - w(y) \geqslant 0 \ \text{ on } \ \partial B_{d_\Omega(x)}(x).$$

It follows from the maximum principle that $F(u(y)) - \rho_x(d_\Omega(x))w(y) \geqslant 0$ in $B_{d_\Omega(x)}(x)$, that is

$$F(u(y)) \geqslant \rho_x(d_\Omega(x))\frac{d_\Omega(x)^2 - |y-x|^2}{2N} \qquad \text{for all } y \in B_{d_\Omega(x)}(x), \tag{2.9}$$

which proves (2.5). Taking $y = x$ in (2.9) gives (2.6).

Now if $\rho(x) = |x|^\alpha$, $\alpha \geqslant 0$ then for $x \in \Omega$ such that $|x| > d_\Omega(x)$ we have

$$\rho_x(r) := \inf_{y \in B_r(x))} |y|^\alpha = (|x| - d_\Omega(x))^\alpha$$

By substituting it for ρ_x in (2.6), one gets (2.7)

Remark 2.1 *Note that estimate* (2.5) *is* (*strictly*) *better than* (2.6)*. For example let* $\Omega = B_R(0)$ *and for simplicity take* $\rho \equiv 1$*. Then* (2.6) *gives the estimate* $u(x) \geqslant F^{-1}(\frac{(R-|x|)^2}{2N})$ *for all* $x \in \Omega$*. However, by taking* $x = 0$ *in* (2.5) *gives* $u(y) \geqslant F^{-1}(\frac{R^2-|y|^2}{2N})$ *for all* $y \in \Omega$*. But it is easy to see that* $(R-|x|)^2 \leqslant R^2 - |x|^2$ *for all* $|x| < R$*.*

3 Improved upper bound for the extremal parameter

The following immediate corollary of Theorem 2.1 gives an upper bound for the extremal parameter λ^* of the nonlinear eigenvalue problem (1.2).

Corollary 3.1 *Let* $u \in C^2(\Omega)$ *be a solution of the non-linear eigenvalue problem* (1.2) *and* $x_0 \in \Omega$ *such that* $\psi(x_0) = \psi_\Omega$*. Then, we have*

$$\lambda \leqslant \frac{1}{\psi_\Omega}\int_0^{u(x_0)} \frac{ds}{m_{f,\Omega}(s)} \leqslant \frac{1}{\psi_\Omega}\int_0^{a_f} \frac{ds}{m_{f,\Omega}(s)}.$$

In particular, the extremal parameter of problem $(P_{\lambda,1})$ *satisfies*

$$\frac{1}{\psi_\Omega}\sup_{0<t<a_f}\frac{t}{f(t)} \leqslant \lambda^*(\Omega,1) \leqslant \frac{1}{\psi_\Omega}\int_0^{a_f}\frac{ds}{f(s)}. \tag{2.1}$$

Proof. By Proposition 2.1 we have

$$F(u(x)) \geqslant \lambda\psi(x) \qquad \text{for all } x \in \Omega, \tag{2.2}$$

where $F(t) = \int_0^t \frac{ds}{m_{f,\Omega}(s)}$. This implies that

$$\lambda \leqslant \frac{1}{\psi_\Omega}\int_0^{u(x_0)} \frac{ds}{m_{f,\Omega}(s)} \leqslant \frac{1}{\psi_\Omega}\int_0^{a_f} \frac{ds}{m_{f,\Omega}(s)},$$

as desired.

Remark 3.1 The upper estimate in (2.1) can be much better, especially in high dimensions, than the previously known one, namely that

$$\lambda^*(\Omega,1) \leqslant \lambda_1(\Omega) \sup_{0\leqslant t\leqslant a_f} \frac{t}{f(t)},$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian on $H_0^1(\Omega)$. Actually, if

$$\bar{\bar{\lambda}}(N) := \frac{1}{\psi_\Omega}\int_0^{a_f} \frac{ds}{f(s)} \quad \text{and} \quad \bar{\lambda}(N) := \lambda_1(\Omega) \sup_{0\leqslant t\leqslant a_f} \frac{t}{f(t)},$$

then $\frac{\bar{\lambda}(N)}{N} \to \infty$ goes to infinity as the dimension $N \to \infty$, while $\frac{\bar{\bar{\lambda}}(N)}{N}$ remains bounded. This follows immediately from the fact that for any bounded domain Ω in $\mathbb{R}^N$ the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian satisfies

$$\lambda_1(\Omega) \geqslant \frac{4N\pi^2}{N+2}\,(\frac{1}{\omega_N|\Omega|})^{\frac{2}{N}}. \tag{2.3}$$

The next theorem illustrates the remarkable usefulness of (2.1).

Theorem 3.1 *Let $\Omega \subseteq \mathbb{R}^N$ ($N \geqslant 2$) be a bounded smooth domain, and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, convex and superlinear C^2-function such that $f(0) > 0$. Then*

$$\lim_{p\to\infty} \lambda_p^* = \frac{1}{f(0)\psi_\Omega} \qquad \textit{and} \qquad \lim_{p\to\infty} \|u_p^*\|_\infty = +\infty,$$

where λ_p^* and u_p^* are the extremal parameter and extremal solution of problem (1.5) *respectively.*

Proof. Take $f_p(t) := f(t^p)$ for $p \geqslant 1$. It is easy to see that there exists a unique $t_p > 0$ such that

$$\frac{t_p}{f_p(t_p)} = \sup_{t>0}\frac{t}{f_p(t)} \qquad \text{for all } p \geqslant 1. \tag{2.4}$$

Indeed, the function $g_p(t) := \frac{t}{f_p(t)}$ satisfies $g_p(0) = 0$ and $g_p(t) \to 0$ as $t \to +\infty$ for $p \geqslant 1$, which means that there exists $t_p > 0$ such that $g_p(t_p) = \sup_{t>0} g_p(t)$. To prove uniqueness, note that

$$g_p'(t) = \frac{f_p(t) - tf_p'(t)}{f_p^2(t)} := \frac{h_p(t)}{f_p^2(t)}, \tag{2.5}$$

and since $h_p'(t) = -tf_p''(t) \leqslant 0$ for $t \geqslant 0$, then t_p is unique. Now, we show that $t_p \to 1$ as $p \to +\infty$. Since $g_p'(t_p) = 0$, by (2.5) we have

$$f(t_p^p) - p\, t_p^p f'(t_p^p) = 0 \qquad \text{for all } p \geqslant 1. \tag{2.6}$$

From the convexity of f we have $f(t) - f(0) \leqslant tf'(t)$ for $t \geqslant 0$. So from (2.6) we get

$$0 \leqslant 1 - \frac{f(0)}{f(t_p^p)} \leqslant \frac{1}{p} \qquad \text{for all } p \geqslant 1. \tag{2.7}$$

Taking the limit as p tends to infinity in (2.7) we have $f(t_p^p) \to f(0)$ as $p \to \infty$, it follows that $t_p^p \to 0$ as $p \to \infty$. Now, if $\lim\limits_{p\to\infty} t_p \neq 1$, then we can find a subsequence $\{t_{p_j}\}$ such that for $j \in \mathbb{N}$, $1 < \alpha < t_{p_j}$ or $t_{p_j} < \beta < 1$ for some $\alpha, \beta \in \mathbb{R}$. The first case implies that $t_{p_j}^{p_j} \to +\infty$ as $j \to \infty$ which is a contradiction, and the later case implies that $0 \leqslant p_j t_{p_j}^{p_j} \leqslant p_j \beta^{p_j} \to 0$ as $j \to \infty$. Then by (2.6) we must have $f(t_{p_j}^{p_j}) \to 0 < f(0)$ as $j \to \infty$ which contradicts the fact that $f(t_p^p) \to f(0)$ as $p \to \infty$. Hence, we showed that $t_p \to 1$ as $p \to +\infty$. Now from (2.4) we get

$$\lim_{p\to\infty} \sup_{t>0} \frac{t}{f_p(t)} = \lim_{p\to\infty} \frac{t_p}{f(t_p^p)} = \frac{1}{f(0)}. \tag{2.8}$$

On the other hand

$$\lim_{p\to\infty} \frac{1}{f_p(t)} = \begin{cases} \dfrac{1}{f(0)} & \text{if } 0 \leqslant t < 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Taking $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\zeta(t) = 1/f(0)$ for $t \in [0,1]$ and $\zeta(t) = 1/f_2(t) = 1/f(t^2)$ for $t \in (1, +\infty)$, then $\zeta \in L^1(\mathbb{R}_+)$ and $1/f_p(t) \leqslant \zeta(t)$ for $p \geqslant 2$. Now, by the Lebesgue dominated convergence theorem,

$$\lim_{p\to\infty} \int_0^\infty \frac{ds}{f_p(s)} = \frac{1}{f(0)}. \tag{2.9}$$

Now, estimate (2.1) guarantees that

$$\frac{1}{\psi_\Omega} \sup_{t>0} \frac{t}{f_p(t)} \leqslant \lambda_p^* \leqslant \frac{1}{\psi_\Omega} \int_0^{u_p^*(x_0)} \frac{dt}{f_p(t)} \leqslant \frac{1}{\psi_\Omega} \int_0^{+\infty} \frac{dt}{f_p(t)}. \tag{2.10}$$

Taking the limit as p tends to infinity in (2.10) and using (2.8) and (2.9), it follows that

$$\lim_{p\to\infty} \lambda_p^* = \frac{1}{f(0)\psi_\Omega} \qquad \text{and} \qquad \lim_{p\to\infty} u_p^*(x_0) = +\infty.$$

Next, we consider problem $(P_{\lambda,\rho})$ and establish some estimates on the extremal parameter $\lambda^*(\Omega,\rho)$ using Theorem 2.1.

Theorem 3.2 (i) *The extremal parameter $\lambda^*(\Omega,\rho)$ of problem $(P_{\lambda,\rho})$ satisfies the following*

$$\lambda^*(\Omega,\rho) \leqslant \frac{2NF(\|u^*\|_\infty)}{\sup\limits_{x\in\Omega}\Big\{\rho_x(d_\Omega(x))d_\Omega^2(x)\Big\}} \leqslant \frac{2N\|F\|_\infty}{\sup\limits_{x\in\Omega}\Big\{\rho_x(d_\Omega(x))d_\Omega^2(x)\Big\}}, \tag{2.11}$$

where $F(t) := \int_0^t \frac{ds}{f(s)}$, d_Ω is the distance to the boundary $\partial\Omega$, and $\rho_x(r) := \inf\limits_{y\in B_x(r)} \rho(y)$.

(ii) *If $\alpha > 0$, then*

$$\lambda^*(\Omega, |x|^\alpha) \leqslant \frac{2NF(\|u^*\|_\infty)}{\sup\limits_{|x|>d_\Omega(x)} \Big\{(|x| - d_\Omega(x))^\alpha d_\Omega(x)^2\Big\}}. \tag{2.12}$$

(*iii*) *In particular,*

$$\lambda^*(\Omega,1) \leqslant \frac{2NF(\|u^*\|_\infty)}{r_\Omega^2} \leqslant \frac{2N\|F\|_\infty}{r_\Omega^2}, \tag{2.13}$$

where $r_\Omega := \sup_{x\in\Omega} d_\Omega(x)$ *is the Chebyshev radius of* Ω*, and in the case of a ball of radius* R*, we have*

$$\lambda^*(B_R,|x|^\alpha) \leqslant 2N\frac{(\alpha+2)^{(\alpha+2)}}{\alpha^\alpha}F(\|u^*\|_\infty)R^{-(2+\alpha)}. \tag{2.14}$$

Proof. By Theorem 2.1 and the fact that F is increasing we have

$$\|F\|_\infty \geqslant F(\|u^*\|_\infty) \geqslant F(u^*(x)) \geqslant \lambda^*(\Omega,\rho)\rho_x(d_\Omega(x))\frac{d_\Omega(x)^2}{2N} \qquad x\in\Omega,$$

which proves (2.11). Taking $\rho \equiv 1$ in (2.11) gives (2.13). (ii) is straightforward. In the case Ω is a ball of radius R, we have $d_\Omega(x) = R - |x|$ for all $x \in \Omega$. By elementary calculus it can be easily checked that for all $R > 0$ and $\alpha > 0$, the function $g(t) = (2t-R)^\alpha(R-t)^2$ defined on $[R/2,R]$ takes its maximum in $t = \frac{\alpha+1}{\alpha+2}R$. It then follows that

$$\sup_{|x|<d_\Omega(x)} (|x| - d_\Omega(x))^\alpha d_\Omega(x)^2 = \frac{\alpha^\alpha R^{(\alpha+2)}}{(\alpha+2)^{(\alpha+2)}}.$$

Remark 3.2 Suppose $f(u) = \frac{1}{(1-u)^2}$, then the estimate

$$\lambda^*(\Omega,\rho,2) \leqslant \frac{2N}{3\sup_{x\in\Omega}\left\{\rho_x(d_\Omega(x))d_\Omega^2(x)\right\}} := \bar{\bar{\lambda}}(N), \tag{2.15}$$

is better than the following one obtained by Guo-Pan-Ward [11]

$$\lambda^*(\Omega,\rho,2) \leqslant \min\Big\{\frac{4\lambda_1(\Omega)}{27\inf_{x\in\Omega}\rho(x)}, \frac{\lambda_1(\Omega)}{\int_\Omega \rho\phi dx}\Big\} := \bar{\lambda}(N), \tag{2.16}$$

where ϕ is the normalized positive eigenfunction correspond to $\lambda_1(\Omega)$ with $\int_\Omega \phi dx = 1$. Again, this is most obvious in higher dimensions, since $\frac{\bar{\bar{\lambda}}}{N}$ remains bounded as $N \to \infty$, while $\frac{\bar{\lambda}}{N}$ goes to infinity.

Indeed, first note that $\bar{\lambda} \geqslant \min\{\frac{4\lambda_1(\Omega)}{27\inf_{x\in\Omega}\rho(x)}, \frac{\lambda_1(\Omega)}{\sup_{x\in\Omega}\rho(x)}\}$. From the fact that $|\Omega| \leqslant (\frac{diam(\Omega)}{2})^N\omega_N$ and using the lower estimate (2.3) on $\lambda_1(\Omega)$, we can deduce that $\frac{\bar{\lambda}}{N} \to \infty$ as $N \to \infty$. This proves that the upper bound (2.15) considerably improves (2.16) for large N.

Remark 3.3 Note that, since F is increasing then for a regular nonlinearity f we have $\|F\|_\infty = \int_0^\infty \frac{ds}{f(s)}$ while for a singular nonlinearity $\|F\|_\infty = \int_0^1 \frac{ds}{f(s)}$. Also, By considering the smallest ball containing Ω and the largest ball contained in Ω, it is not hard to see that

$$\frac{8N}{\mathrm{diam}(\Omega)^2} \leqslant \frac{1}{\psi_\Omega} \leqslant \frac{2N}{r_\Omega^2}.$$

The above then yields the following explicit bounds for $\lambda^*(\Omega,\rho)$, in terms of the size of the domain. If $D_f = [0,\infty)$, then

$$\max\Big\{\frac{8N}{\mathrm{diam}(\Omega)^2}, \frac{2}{r_\Omega^2}\Big\} \sup_{t>0} \frac{t}{f(t)} \leqslant \lambda^*(\Omega, 1) \leqslant \frac{2N}{r_\Omega^2} \int_0^\infty \frac{ds}{f(s)}.$$

In particular,

$$\frac{2N}{R^2} \sup_{t>0} \frac{t}{f(t)} \leqslant \lambda^*(B_R(0), 1) \leqslant \frac{2N}{R^2} \int_0^\infty \frac{ds}{f(s)}.$$

4 Improved lower bounds for the extremal parameter

In this section we obtain two different lower bounds for the extremal parameter of problem (1.2), which will improve (1.3). First, we show how the latter estimate can be obtained

Lemma 4.1 *Let $f : \Omega\times\mathbb{R}_+ \to \mathbb{R}_+$ be a β-Hölder continuous in its first variable and locally Lipschitz continuous in the second variable. If $M_{f,\Omega}$ (defined in (2.2)) is increasing and $M_{f,\Omega}(0) > 0$, then problem (1.2) has a positive classical solution provided $0 \leqslant \lambda \leqslant \frac{1}{\psi_\Omega} \sup_{t>0} \frac{t}{M_{f,\Omega}(t)}$. This means that*

$$\frac{1}{\psi_\Omega} \sup_{t>0} \frac{t}{M_{f,\Omega}(t)} \leqslant \lambda^*. \tag{3.1}$$

Proof. Obviously, $u \equiv 0$ is an allowable subsolution and therefore it suffices to find a supersolution. To this end, we consider $\overline{u} = \alpha\psi$ to be a supersolution for (1.1) where ψ is the torsion function and α is a positive real number to be determined later. Clearly, $\overline{u} \geqslant 0$ on $\partial\Omega$. Since the function $M_{f,\Omega}$ is increasing, then it is just sufficient to have

$$\Delta\overline{u} + \lambda M_{f,\Omega}(\alpha\psi_\Omega) = -\alpha + \lambda M_{f,\Omega}(\alpha\psi_\Omega) \leqslant 0. \tag{3.2}$$

Choose $\alpha > 0$ such that $\frac{\alpha\psi_\Omega}{M_{f,\Omega}(\alpha\psi_\Omega)} = \sup_{t>0} \frac{t}{M_{f,\Omega}(t)}$. For this α, (3.2) holds, provided

$$\lambda \leqslant \frac{1}{\psi_\Omega} \frac{\alpha\psi_\Omega}{M_{f,\Omega}(\alpha\psi_\Omega)} = \frac{1}{\psi_\Omega} \sup_{t>0} \frac{t}{M_{f,\Omega}(t)},$$

as desired. □

As an example, consider the problem

$$\begin{cases} -\Delta u = \lambda|x|^\alpha e^u & x \in B, \\ u = 0 & x \in \partial B, \end{cases} \tag{3.3}$$

where $\alpha \geqslant 0$ is a real number. Here we have

$$f(x,t) = |x|^\alpha e^t,\ \Omega = B_1(0),\ M_{f,\Omega}(t) = \sup_{x\in B_1(0)} |x|^\alpha e^t = e^t,\ \sup_{t>0} \frac{t}{e^t} = \frac{1}{e} \text{ and } \frac{1}{\psi_\Omega} = 2N.$$

Then by Lemma 4.1 we have $\frac{2}{e}N \leqslant \lambda^*$. □

In the following we use again the sub- (super-) solution approach to give another lower bound for the extremal parameter λ^* associated to the nonlinear eigenvalue problem $(P_{\lambda,\rho})$. For any bounded domain Γ in R^N, we denote by $\lambda_1(\Gamma)$ the first eigenvalue of $-\Delta$ on $H_0^1(\Gamma)$ and by φ_Γ corresponding positive eigenfunction normalized with $\sup_\Gamma \varphi_\Gamma = 1$. We associate to a given domain $\Omega \subseteq \mathbb{R}^N$ and a given nonlinearity f, the following parameter

$$\nu_{\Omega,f} = \sup\Big\{\lambda_1(\Gamma)J_f(\inf_\Omega \varphi_\Gamma);\ \Gamma \text{ domain } of\ \mathbb{R}^N,\ \Gamma \supseteq \overline{\Omega}\Big\}, \tag{3.4}$$

where $J_f(t) = \frac{A_t}{f(A_t)}$, A_t being the unique solution of the below equation

$$f(tA) = tf(A),\ 0 < t < 1,\ 0 < A < a_f.$$

Theorem 4.1 *Consider the eigenvalue problem $(P_{\lambda,\rho})$. Then*

$$\lambda^*(\Omega,\rho) \geqslant \frac{\nu_{\Omega,f}}{\sup_\Omega \rho(x)},$$

where $\nu_{\Omega,f}$ is defined by (3.4).

Proof. To prove the theorem we need to construct a supersolution of $(P_{\lambda,\rho})$ for every $\lambda < \nu_{\Omega,f}(\sup_\Omega \rho(x))^{-1}$. For a bounded domain $\Gamma \supseteq \overline{\Omega}$ with smooth boundary, let $(\lambda_1(\Gamma), \varphi_\Gamma)$ be the first eigenpair normalized in such a way that

$$\sup_\Gamma \varphi_\Gamma = 1,\ \text{and}\ \inf_\Omega \varphi_\Gamma := s_1 > 0.$$

We construct a supersolution in the form $\bar{u} = A\varphi_\Gamma$ where $0 < A < a_f$ is a scalar to be chosen later. We require that

$$\Delta\bar{u} + \lambda\rho(x)f(\bar{u}) = -A\lambda_1(\Gamma)\varphi_\Gamma + \lambda\rho(x)f(A\varphi_\Gamma) \leqslant 0\ \ in\ \ \Omega,$$

which can be satisfied as long as

$$\lambda \leqslant \frac{1}{\sup_\Omega \rho(x)}\beta(A,\Gamma,f),\ \text{where}\ \beta(A,\Gamma,f) := \lambda_1(\Gamma)\inf\Big\{\frac{As}{f(As)};\ s \in [s_1(\Gamma),1]\Big\}.$$

Hence we have

$$\lambda^*(\Omega,\rho) \geqslant \frac{1}{\sup_\Omega \rho(x)}\ \sup\Big\{\beta(A,\Gamma,f);\ 0 < A < a_f,\ \Gamma \supseteq \overline{\Omega}\Big\}.$$

Therefore, it remains to show that

$$\nu_{\Omega,f} = \sup\Big\{\beta(A,\Gamma,f);\ 0 < A < a_f,\ \Gamma \supseteq \overline{\Omega}\Big\}.$$

First, define $g(t) = \frac{t}{f(t)}$ for $0 \leqslant t < a_f$. By the properties of f we have $0 = g(0) = \lim_{t\to a_f} g(t)$, and we claim that there is unique point $0 < t_0 < a_f$ such that $g(t_0) = \max_{0\leqslant t<a_f} g(t)$, g is increasing on $[0,t_0]$ and decreasing on $[t_0,a_f)$. To see this note that we have

$$g'(t) = \frac{h(t)}{f^2(t)},\ where\ h(t) := f(t) - tf'(t),$$

and since $h(0) = f(0) > 0$, $h'(t) = -tf''(t) \leqslant 0$ and because of the convexity and superlinearity of f, $\lim_{t\to a_f} h(t) = -\infty$, thus we get the claim. This shows that $\inf_{s\in[s_1,1]} g(As) = \min\{g(s_1A), g(A)\}$. For a fixed $s_1 < 1$ take $T(A) = g(s_1A) - g(A)$ on $[0, a_f]$, then T is a continuous function that is negative near zero and positive near a_f so there is $0 < A_{s_1} < a_f$ such that $T(A_{s_1}) = 0$. It is obvious that $A_{s_1} > t_0 > s_1A_{s_1}$ and also is unique. Indeed $T(A) = 0$ if and only if $g(A) = g(s_1A)$ or $s_1f(A) - f(s_1A) = 0$ which has a unique solution from the fact that $(s_1f(A) - f(As_1))' = s_1(f'(A) - f'(s_1A)) > 0$ on $(0, a_f)$. Thus we have

$$\inf_{s\in[s_1,1]} g(As) = \begin{cases} g(As_1) & 0 < A \leqslant A_{s_1}, \\ g(A) & A_{s_1} \leqslant A < a_f. \end{cases}$$

From the fact that $A_{s_1} > t_0 > s_1A_{s_1}$ and $g(t)$ is increasing on $[0, t_0]$ and decreasing on $[t_0, a_f)$ it follows that

$$\sup_{0<A<a_f} \inf_{s\in[s_1,1]} g(As) = g(A_{s_1}) = \frac{A_{s_1}}{f(A_{s_1})} = J_f(s_1) = J_f(\inf_\Omega \varphi_\Gamma),$$

which completes the proof. □

Remark 4.1 If $f(u) = \frac{1}{(1-u)^p}$, then it is easy to see that $J_f(t) = \frac{t(1-\sqrt[p]{t})(1-t)^p}{(1-t\sqrt[p]{t})^{p+1}}$ for $0 < t < 1$, which yields that for $p = 2$, $J_f(t) = \frac{t(1+t+2\sqrt{t})}{(t+1+\sqrt{t})^3}$, hence Theorem 4.1 is a direct extension of a similar result obtained by Ghoussoub-Guo in the case of a quadratic MEMS nonlinearity [10].

For $f(u) = e^u$, we have $J_f(t) = \frac{t^{\frac{1}{1-t}}}{1-t} \ln \frac{1}{t}$ for $0 < t < 1$, and thus the above proposition yields that:

$$\lambda^*(\Omega, \rho) \geqslant (\sup_\Omega \rho(x))^{-1} \sup \left\{\lambda_1(\Gamma)s(\Gamma);\ \Gamma \supset \bar{\Omega}\right\},$$

where $s(\Gamma) := J_{e^u}(\inf_\Omega \varphi_\Gamma)$.

While the above estimate may have an interesting theoretical value, it is unfortunately not suitable for explicit computations. We will therefore consider problem $(P_{\lambda,\rho})$ with $\rho \equiv 1$ and give a lower bound for the extremal parameters in term of the torsion function ψ of Ω. It should be interesting to find a relation between such a torsion function and the quantity $\sup_{\Gamma\supset\bar{\Omega}} \lambda_1(\Gamma)s(\Gamma)$ defined above.

Theorem 4.2 *Consider the semilinear elliptic equation $(P_{\lambda,1})$, then*

$$\lambda^*(\Omega, 1) \geqslant \max\Big\{ \sup_{0<\alpha<\frac{\|F\|_\infty}{\psi_\Omega}} \alpha - \alpha^2\beta(\alpha)\ ,\ \frac{1}{\psi_\Omega} \sup_{0<t<a_f} \frac{t}{f(t)}\Big\}, \tag{3.5}$$

where $\beta(t) := \sup_{x\in\Omega} f'(F^{-1}(t\psi(x)))|\nabla\psi(x)|^2$, and $F(t) := \int_0^t \frac{ds}{f(s)},\ t > 0$.

Proof. Take an $\alpha \in (0, \frac{\|F\|_\infty}{\psi_\Omega})$ and define $\overline{u}(x) = F^{-1}(\alpha\psi(x))$ for $x \in \Omega$. It is evident that $\overline{u} \in C^2(\Omega) \cap C^1(\partial\Omega)$. We show that $\bar{u}$ is a supersolution of $(P_{\lambda,1})$ for $\lambda = \alpha - \alpha^2\beta(\alpha)$. To do this we compute $\Delta\bar{u}(x)$. Note that if we take $y = F^{-1}(\alpha t)$ then it is easy to see that $y' = \alpha f(y)$ and $y'' = \alpha^2 f(y) f'(y)$. So

$$\begin{aligned}\Delta\bar{u}(x) &= [\alpha^2 f'(\bar{u})|\nabla\psi(x)|^2 - \alpha] f(\bar{u}) \\ &\leqslant (\alpha^2 \sup_{x\in\Omega} f'(F^{-1}(\alpha\psi(x)))|\nabla\psi(x)|^2 - \alpha) f(\bar{u}) \\ &= -(\alpha - \alpha^2\beta(\alpha)) f(\bar{u}).\end{aligned}$$

In other words, $\Delta\bar{u}(x) + (\alpha - \alpha^2\beta(\alpha)) f(\bar{u}) \leqslant 0$, and since we have $\bar{u}(x) = 0,\ x \in \partial\Omega$, this shows that $\bar{u}$ is a supersolution of $(P_{\lambda,1})$ for $\lambda = \alpha - \alpha^2\beta(\alpha)$. On the other hand, $\underline{u} = 0$ is an allowable subsolution, thus problem $(P_{\lambda,1})$ with $\lambda = \alpha - \alpha^2\beta(\alpha)$ has a classical solution and hence

$$\lambda^*(\Omega, 1) \geqslant \alpha - \alpha^2\beta(\alpha).$$

Taking the supremum on $\alpha \in (0, \frac{\|F\|_\infty}{\psi_\Omega})$ and combining it with (1.3), we obtain (3.5). Now, we consider some special cases of Theorem 4.2 and give some explicit lower bounds for the extremal parameter of problem $(P_{\lambda,1})$.

Corollary 4.1 *Consider the semilinear elliptic equation* $(P_{\lambda,1})$ *on the unit ball B, In particular if and assume the function* $f'(t)(\alpha - F(t))$ *is decreasing on* $(0, a_f)$. *Then, we have*

$$\lambda^*(B, 1) \geqslant \max\Big\{ \sup_{0<\alpha<\|F\|_\infty} 2N\alpha - 4\alpha^2 f'(0)\ ,\ 2N \sup_{0<t<a_f} \frac{t}{f(t)} \Big\}. \tag{3.6}$$

Proof. When $\Omega = B$ we have $\psi(x) = \frac{1-|x|^2}{2N}$, then from (3.5) we get

$$\lambda^*(B, 1) \geqslant \sup_{0<\alpha<2N\|F\|_\infty} \alpha - \alpha^2\beta(\alpha),\ where\ \ \beta(\alpha) := \sup_{x\in B} f'(F^{-1}(\alpha\frac{1-|x|^2}{2N}))\frac{|x|^2}{N^2}. \tag{3.7}$$

Taking $t := F^{-1}(\alpha\frac{1-|x|^2}{2N})$, make the change $\alpha \to 2N\alpha$ in (3.7), and use that the function $f'(t)(\alpha - F(t))$ is decreasing on $(0, F^{-1}(\alpha))$, to obtain that $\beta(\alpha) = f'(0)(\alpha - F(0)) = f'(0)\alpha$, which proves (3.6).

5 Some applications

In this section, we apply our results to the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda f(u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a smooth bounded domain in R^N and $f(u) = (1-u)^{-p}$, $f(u) = e^u$ or $f(u) = (1+u)^p$ for $p > 1$. Problem (3.1) has been extensively studied in the literature because of wide applications to physical models (see for example [3, 8, 9]).

Consider the singular nonlinearity $f(u) = (1-u)^{-p}$ for $0 \leqslant u < 1$, with $p > 1$. In this case, we have

$$F(t) = \frac{1-(1-t)^{p+1}}{p+1} \quad \text{and} \quad F^{-1}(t) = 1 - (1-(p+1)t)^{\frac{1}{p+1}}.$$

Also, $\sup\limits_{0<t<1} \frac{t}{f(t)} = \frac{p^p}{(p+1)^{(p+1)}}$. We can therefore deduce the following result

Corollary 5.1 *The extremal parameter and extremal solution of* (3.1) *with* $f(u) = (1-u)^{-p}$ *satisfy:*

$$\max\Big\{\sup_{0<t<\frac{1}{(p+1)\psi_\Omega}} t - t^2\beta(t)\ ,\ \frac{p^p}{(p+1)^{(p+1)}}\frac{1}{\psi_\Omega}\Big\} \leqslant \lambda^*(\Omega),$$

$$\lambda^*(\Omega) \leqslant \min\Big\{\frac{\lambda_1(\Omega)p^p}{(p+1)^{(p+1)}},\ \frac{1}{(p+1)\psi_\Omega}\Big\},$$

and

$$1 - \sqrt[p+1]{1-(p+1)\lambda^*(\Omega)\psi(x)} \leqslant u^*(x) \quad \textit{for all } x\in\Omega, \tag{3.2}$$

where ψ *is the torsion function, and* β *is as defined in* (1.7)*. In particular, if* Ω *is the unit ball, then*

$$\tau(N,p) \leqslant \lambda^*(B) \leqslant \frac{2N}{p+1}, \tag{3.3}$$

and

$$1 - \sqrt[p+1]{1-\frac{(p+1)\tau(N,p)}{2N}} \leqslant u^*(x) \quad \textit{for all } x\in\Omega, \tag{3.4}$$

where $\tau(N,p)$ *is defined in* (3.5) *below.*

Proof. It suffices to note that

$$\begin{aligned}\tau(N,p) &= \max\Big\{\sup_{0<\alpha<\frac{1}{p+1}} 2N\alpha - 4p\alpha^2,\ \frac{2Np^p}{(p+1)^{p+1}}\Big\}\\ &= \begin{cases} \frac{2Np^p}{(p+1)^{p+1}} & N \leqslant 8(\frac{p}{p+1})^{p+1},\\ \frac{N^2}{4p} & 8(\frac{p}{p+1})^{p+1} < N \leqslant \frac{4p}{p+1},\\ \frac{2}{p+1}(N-\frac{2p}{p+1}) & N > \frac{4p}{p+1}.\end{cases}\end{aligned} \tag{3.5}$$

For the regular case $f(u) = e^u$ and $f(u) = (1+u)^p$ by a similar argument as above we can prove that

Corollary 5.2 *The extremal parameter and extremal solution of* (3.1) *with* $f(u) = e^u$ *satisfy*

$$\max\Big\{\sup_{0<t<\frac{1}{\psi_\Omega}} t - t^2\beta(t)\ ,\ \frac{1}{e\psi_\Omega}\Big\} \leqslant \lambda^*(\Omega) \leqslant \min\Big\{\frac{\lambda_1(\Omega)}{e}\ ,\ \frac{1}{\psi_\Omega}\Big\},$$

and

$$-\ln\big(1-\lambda^*(\Omega)\psi(x)\big) \leqslant u^*(x) \quad \textit{for all } x\in\Omega. \tag{3.6}$$

In particular, in the case of the unit ball, we have

$$\tau(N) \leqslant \lambda^*(B) \leqslant 2N. \tag{3.7}$$

and

$$-\ln\left(1-\frac{\tau(N)}{2N}\right) \leqslant u^*(x) \quad \textit{for all } x\in\Omega, \tag{3.8}$$

where $\tau(N)$ *is defined in* (3.9) *below.*

Proof. Note that

$$\lambda^*(B) \geqslant \tau(N) = \max\Big\{\sup_{0<\alpha<1} 2N\alpha-4\alpha^2,\ \frac{2N}{e}\Big\} = \begin{cases} \frac{2N}{e} & N=1,2,\\ \frac{N^2}{4} & N=3,4,\\ 2(N-2) & N\geqslant 5.\end{cases} \tag{3.9}$$

Corollary 5.3 *The extremal parameter and extremal solution of* (3.1) *with* $f(u)=(1+u)^p$, $(p>1)$ *satisfy:*

$$\max\Big\{\sup_{0<t<\frac{1}{(p+1)\psi_\Omega}} t-t^2\beta(t)\,,\ \frac{p^p}{(p-1)^{(p-1)}}\frac{1}{\psi_\Omega}\Big\} \leqslant \lambda^*(\Omega),$$

$$\lambda^*(\Omega) \leqslant \min\Big\{\frac{p^p\lambda_1(\Omega)}{(p-1)^{(p-1)}}\,,\ \frac{1}{(p-1)\psi_\Omega}\Big\},$$

and

$$\frac{1}{\sqrt[p-1]{1-(p-1)\lambda^*(\Omega)\psi(x)}}-1 \leqslant u^*(x) \quad \textit{for all } x\in\Omega. \tag{3.10}$$

In particular

$$\tau(N,p) \leqslant \lambda^*(B) \leqslant \frac{2N}{p-1},$$

where $\tau(N,p)$ *is defined in* (3.11) *below.*

Proof. Note that $\lambda^*(B)$ is larger than

$$\begin{aligned}\tau(N,p) &= \max\Big\{\sup_{0<t<\frac{1}{p-1}} 2Nt-4pt^2,\ \frac{2N(p-1)^{p-1}}{p^p}\Big\}\\ &= \begin{cases} \frac{2N(p-1)^{p-1}}{p^p} & N\leqslant 8(1-\frac{1}{p})^{p-1},\\ \frac{N^2}{4p} & 8(1-\frac{1}{p})^{p-1} < N \leqslant \frac{4p}{p-1},\\ \frac{2}{p-1}(N-\frac{2p}{p-1}) & N>\frac{4p}{p-1}.\end{cases}\end{aligned} \tag{3.11}$$

Remark 5.1 The above three corollaries improve on existing results in many ways. We have seen this in the upper and lower bounds for λ^*. One can also note the improvements in the lower L^∞ bounds for u^*. For example, Ghoussoub-Cowan had proved in [6] the following:

(i) If $f(u) = (1-u)^{-p}$, then $\frac{1}{p+1} \leqslant \|u^*\|_\infty$.

(ii) If $f(u) = e^u$, then $1 \leqslant \|u^*\|_\infty$.

(iii) If $f(u) = (1+u)^p$, then $\frac{1}{p-1} \leqslant \|u^*\|_\infty$.

Note that (3.2), (3.6) and (3.10) give better estimates, at least in certain dimensions. For example, if Ω is the unit ball, our results above yield:

(a) If $f(u) = (1-u)^{-2}$, then $1 - \sqrt[3]{\frac{4}{3N}} \leqslant \|u^*\|_\infty$, which gives a better lower bound than (i) when $5 \leqslant N \leqslant 7$.

(b) If $f(u) = e^u$, and $3 \leqslant N \leqslant 9$, then $\ln \frac{N}{2} \leqslant \|u^*\|_\infty$, which gives a better lower bound than (ii) when $6 \leqslant N \leqslant 9$.

(c) If $f(u) = (1+u)^p$, and $p > \frac{N}{N-2}$, then $\sqrt[1-p]{\frac{2p}{(p-1)N}} - 1 \leqslant \|u^*\|_\infty$, which gives a better lower bound than (iii) when $N > 2(\frac{p}{p-1})^p$.

Remark 5.2 By combining our results with those of Cowan-Ghoussoub [6], one can obtain stronger upper bound for λ^*. Namely,

- If $f(u) = \frac{1}{(1-u)^2}$ and $3 \leqslant N \leqslant 7$, then there exists $\delta_N > 0$ such that
$$\frac{2(3N-4)}{9} \leqslant \lambda^*(B) \leqslant \frac{2N(1-e^{-3\delta_N})}{3}.$$
Note that if $N \geqslant 8$, then $\lambda^*(B) = \frac{2(3N-4)}{9}$ and $u^*(x) = 1 - |x|^{\frac{2}{3}}$.

- If $f(u) = e^u$ and $3 \leqslant N \leqslant 9$, then there exists $\theta_N > 0$ such that
$$2(N-2) \leqslant \lambda^*(B) \leqslant 2N(1-e^{-\theta_N}).$$
Note that for $N \geqslant 10$, we have $\lambda^* = 2(N-2)$ and $u^*(x) = -2\ln|x|$.

- If $f(u) = (1+u)^p$ and $3 \leqslant N \leqslant 4$, then there exists $\kappa_N > 0$ such that
$$\frac{2}{p-1}(N - \frac{2p}{p-1}) \leqslant \lambda^*(B) \leqslant \frac{2N}{p-1}(1 - \frac{1}{(1+\kappa_N)^{p-1}}).$$

References

[1] H. Amann, Nonlinear Elliptic Equations with Nonlinear Boundary Conditions, New Developments in Differential Equations, North Holland Math. Studies **21** (1976), 43–63. North-Holland Publ., Amsterdam.

[2] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa,*Blow up for $u_t - \Delta u = g(u)$ revisited* , Adv. Differential Equations **1** (1996), 73–90.

[3] H. Brezis, L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443–469.

[4] X. Cabré, A. Capella, *Regularity of radial minimizers and extremal solutions of semi-linear elliptic equations*, J. Funct. Anal. **238** (2006), 709–733.

[5] C. Cowan, P. Esposito, N. Ghoussoub, *Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains*, Discrete Contin. Dyn. Syst. **28** (2010), 1033–1050.

[6] C. Cowan, N. Ghoussoub, *Estimates on pull-in distances in microelectromechanical systems models and other nonlinear eigenvalue problems*, SIAM J. Math. Anal. **42** (2010), 1949–1966.

[7] C. Cowan, N. Ghoussoub, *Regularity of the extremal solution in a MEMS model with advection, Methods Appl. Anal.* **15** (2008), no. 3, 355–362.

[8] P. Esposito, N. Ghoussoub, Y.J. Guo, Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS, volume 20 of Courant Lecture Notes in Mathematics. Courant Institute of Mathematical Sciences, New York, 2010

[9] F. Gazzola, A. Malchiodi, *Some remarks on the equation $-\Delta u = \lambda(1+u)^p$ for varying λ, p and varying domains*, Comm. Partial Differential Equations **27** (2002), no. 3-4, 809–845.

[10] N. Ghoussoub, Y. Guo, *On the partial differential equations of electrostatic MEMS devices: stationary case*, SIAM J. Math. Anal. **38** (2006), 1423–1449

[11] Y. Guo, Z. Pan, M.J. Ward, *Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties*, SIAM J. Appl. Math. **66** (2005), no. 1, 309–338.

[12] P. Li, S.T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983), 309–318.

[13] Y. Martel, Uniqueness of weak extremal solutions of nonlinear elliptic problems, J. Differential Equations **16** (1974), 103–125.

[14] F. Mignot and J-P. Puel, Sur une classe de problemesnon lineaires avec non linearite positive, croissante, convexe, Comm. Partial Differential Equations **5** (1980), 791–836.

[15] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations*, C. R. Acad. Sci. Paris Ser. I Math., **330** (2000), 997–1002.

[16] L.E. Payne, *Bounds for the maximum stress in the Saint Venant torsion problem*, Indian J. Mech. Math., Special Issue, (1968), 51–59.

[17] L.E. Payne, *Bounds for the solutions of a class of quasilinear elliptic boundary value problems in terms of the torsion function*, Proc. Roy. Soc. Edinburgh Sect. A **88** (1981), 251–265.

[18] L.E. Payne, *Some remarks on maximum principles*, J. Analyse Math. **30** (1976), 421–433.

[19] J.A. Pelesko, *Mathematical modeling of electrostatic MEMS with tailored dielectric properties*, SIAM J. Appl. Math., **62** (2002), 888–908.

[20] P. Quittner, P. Souplet, Superlinear Parabolic Problems Blow-up, Global Existence and Steady States, Birkäuser Advanced Texts, Basel, 2007.

[21] R.P. Sperb, Maximum Principles and Their Applications, Academic Press, New York, 1981.