On the computability of Mandelbrot-like sets

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Joint with D. Coronel and M. Yampolsky.

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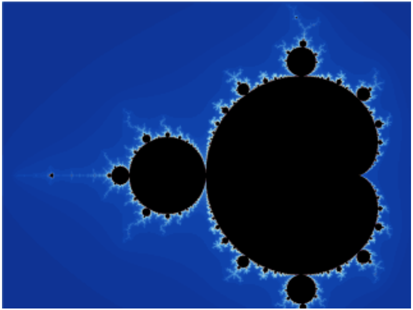
Open Question

Is the Mandelbrot set computable ?

Can we trust this ?



$\mathcal{M} = \{c \in \mathbb{C} : \sup_n |f^n(0)| \leq 2\}$, where $f(z) = z^2 + c$.

Computability of Closed Sets

A remainder

Definition

$K \subset \mathbb{C}$ is computable if there is a Turing Machine M which, on input n, outputs a finite set $M(n) = K_n$ of *rational* points such that

$$d_H(K, K_n) \leq 2^{-n}$$

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1. K has a recursively enumerable complement (we say it is **upper-computable**)
2. there is a uniformly computable sequence of points $\{z_n\}_n \subset K$, which is dense in K (we say it is **lower-computable**).

What is known about it ?

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- this conjecture is probably the most important in Complex Dynamics.

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- However, the boundary $\partial\mathcal{M}$ may still be non computable !

Dynamics of the quadratic family

The big picture

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Let $z_0 \in \mathbb{C}$ be an "initial condition", and let

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Goal: to understand what kind of behaviours may the sequence z_n exhibit, depending on the parameter c.

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- that is, ∞ is "an attracting fixed point", and one can define its basin of attraction

$$B_c(\infty) = \{z_0 \in \mathbb{C} : z_n \to \infty\}.$$

- Thus, the interesting dynamics happens on the complement of $B_c(\infty)$. This is the so called **filled Julia set**, denoted by K_c.

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Therefore, the Mandelbrot set *really* is the **conectedness locus** of the family f_c:

$$\mathcal{M} = \{c \in \mathbb{C} : K_c \text{ is connected } \},$$

and its boundary $\partial\mathcal{M}$ is the **bifurcation locus**.

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- F_c is made by an unbounded connected component ($B_c(\infty)$), and the interior of K_c.
- How is the dynamics in the interior of K_c?

Local dynamics around periodic cycles

A point z is periodic if $f^p(z) = z$ for some p. Its **period** is the minimal such p, and its **multiplier** is

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Using the fact that $\overline{\mathbb{C} \setminus \mathcal{M}} \cap \overline{H} = \partial\mathcal{M}$, Hertling showed that $\partial\mathcal{M}$ is lower-computable.

Our contribution

Let $\lambda \in \mathbb{C}$ be fixed and such that $|\lambda| = 1$. Consider the one-parameter cubic family

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Define the filled Julia set K_c and the connectedness locus $\mathcal{M}_\lambda$ by

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Theorem (Coronel, R., Yampolsky)

There exists a computable λ such that the interior of $\mathcal{M}_\lambda$ is **not r.e.** *In particular, the bifurcation locus $\partial\mathcal{M}_\lambda$ is* **not computable**.

Computability of Julia sets

Known results

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Remark: note that, in particular, there is c such that K_c is computable but its interior is not r.e.

Proof for part b): the empty interior case

Proposition

J_c *is always lower-computable.*

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Proposition

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Proof:

- It follows from the following fact proved by Fatou:

$$J(f_c) = \overline{\{\text{repelling periodic orbits}\}}$$

- compute all periodic points and output only those for which $\lambda > 1$.

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- but since J_c has no interior, we have $J_c = K_c$,
- and as we saw, K_c is *always* upper-computable,
- which finishes the proof.

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- Note that $\mathcal{M}_\lambda$ has little copies of these Julia sets
- Show that one can make them non computable
- The rest $\mathcal{M}_\lambda$ can be "trimmed", so it can't have a computable interior.

THANKS !