Extremal Value Theory of a Stationary $S\alpha S$ Random Fields

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Stationary $S\alpha S$ random fields driven by conservative flows.

$\Phi = \{\phi_t, t \in \mathbb{N}_0^d\}$ is a conservative $\mathbb{N}_0^d$-action on measurable space $(E, \mathcal{E}, \mu)$.

- $\phi_{\mathbf{0}}$ is the identity map on S.
- $\phi_{u+v} = \phi_u \circ \phi_v$ for all $u, v \in \mathbb{N}_0^d$.

Assume that the action is measure preserving. Take $f \in L_\alpha(\mu)$ and a $S\alpha S$ random measure M controlled by μ:

$$X_t := \int_E f \circ \phi_t dM, \quad t \in \mathbb{N}_0^d$$

$\{X_\cdot\}$ is a stationary $S\alpha S$ random fields, and such random fields are assumed to have long memory.

Rosiński, J(2000). Decomposition of stationary α-stable random fields, Annals of Probability.

To study the long memory properties, we mainly focus on two types of limit theorems.

$$\lim_{n\to\infty} \frac{1}{b_n} \max_{k\in nB} X_t \quad B \in \mathcal{B}([0,\infty)^d) \quad \text{weak convergence of sup-measures}$$
$$\lim_{n\to\infty} \frac{1}{b_n} \max_{\mathbf{0}\leqslant t\leqslant \lfloor nt \rfloor} X_t \quad t \in [0,\infty)^d \quad \text{functional convergence}$$

$\mathbb{E} = \mathbb{R}^d, \mathbb{R}^d_+, [0,1]^d$, (generally LCHS spaces). $m : \mathcal{G} \to [0,\infty]$ is called a sup measure if

- $m(\emptyset) = 0$
- $m(\cup_\gamma G_\gamma) = \sup_\gamma m(G_\gamma)$ for arbitrary collections $\{G_\gamma \in \mathcal{G} : \gamma\}$.

$\{m_n\} \overset{\text{vague}}{\longrightarrow} m$ if and only if

$$\limsup_{n\to\infty} m_n(K) \leqslant m(K) \quad \forall\, K \in \mathcal{K} \quad \liminf_{n\to\infty} m_n(G) \geqslant m(G) \quad \forall\, G \in \mathcal{G}$$

$\mathcal{M}$ is compact and metrizable under sup-vague topology.

Random sup-measure is a random elements in $\mathcal{M}$. In particular, if $\mathcal{M}(G)$ is a cont rv for any open rectangle G, then weak-convergence $M_n \Rightarrow M$ is equivalent to fdd convergence.

$$(m_n(B_1), \cdots, m_n(B_k)) \Rightarrow (m(B_1), \ldots, m(B_k)), \quad \text{open disjoint rectangles } B_i$$

Sup-derivatives and sup-integral:

$$d^\vee m(t) := \inf_{G: t \in G} m(G), \quad G \in \mathcal{G} \quad \text{uniquely determines } m, \quad d^\vee m \text{ is usc}$$

$$i^\vee f(B) := \sup_{t \in B} f(t) \quad \text{uniquely specifies a sup-measure, } f \text{ is usc.}$$

Markov chains and product Markov chains.

$\{x_n, n \geqslant 0\}$ is an irreducible and null recurrent MC on $\mathbb{Z}$,with invariant measure $(\pi_i, i \in \mathbb{Z})$, $\pi_0 = 1$. On the path space

- $(E, \mathcal{E}) = (\mathbb{Z}^{\mathbb{N}_0}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}_0}))$.
- $\mu(\cdot) = \sum_{i \in \mathbb{Z}} \pi_i P_i(\cdot)$.
- $P_i(\cdot)$ is the law of $\{x_n : n \geqslant 0, x_0 = i\}$.

left shift operator $\quad T(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$

Proposition (Harris and Robbins 1953)

$\{x_n\}$ *irreducible and null-recurrent* $\Leftrightarrow$ T *conservative and ergodic.*

For a conservative action with

- $A = \{x_0 = 0\}$, $\mu(A) = 1$.
- $b_n^\alpha = \mu\left(\bigcup_{k=1}^n T^{-k}A\right) \in \mathrm{RV}_\beta, \quad \beta \in (0,1)$

The processes $X_t := \int_E 1_A \circ T^t dM$ has limit theorems:

Theorem

$$\frac{1}{b_n} \max_{k \in nB} X_t \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \eta^{\alpha,\beta}(B) \quad B \in \mathcal{B}([0,1])$$

- $\beta \in (0, 1/2)$, $d^\vee \eta^{\alpha,\beta}(t) = \sum_{j=1}^\infty \Gamma_j^{-1/\alpha} 1_{\{t \in V_j + R_j\}} \quad t \in [0,1]$.
- $\beta \in [1/2, 1)$, $d^\vee \eta^{\alpha,\beta}(t) = \bigvee_{j=1}^\infty \Gamma_j^{-1/\alpha} 1_{\{t \in V_j + R_j\}} \quad t \in [0,1]$.

Let $L_{1-\beta}$ be the standard $(1-\beta)$-subordinator

$$R_1 := \overline{\{L_{1-\beta}(t), t \geqslant 0\}} \subset [0,\infty) \quad (1-\beta) - \text{stable regenerative set}$$

$P(V_j \leqslant x) = x^\beta, \quad x \in [0,1]$ indep of R_j, $\{V_j, R_j\}$ iid family and $\{\Gamma_j\}$ arrival time of unit Poisson on $(0,\infty)$.

Lacaux, C and Samorodnitsky, G(2016). Time-changed extremal process as random sup measure, Bernoulli.
Samorodnitsky, G and Wang,Y (2017) Extremal value theory for long range dependent infinitely divisible processes.

d iid copies Markov chains $(x_n^{(1)}), \dots, (x_n^{(d)})$, has path spaces $(E_1, \mathcal{E}_1, \mu_1), \dots, (E_d, \mathcal{E}_d, \mu_d)$. T_i is the left shift operator on E_i.

$$x_k := (x_{k_1}^{(1)}, \dots, x_{k_d}^{(d)}), \quad k \in \mathbb{N}_0^d$$

Path space of (x_k) and the shift operator are

- $(E, \mathcal{E}, \mu) = (\prod_{i=1}^d E_i, \prod_{i=1}^d \mathcal{E}_i, \prod_{i=1}^d \mu_i)$
- $T^k(x) = ((T_1)^{k_1} x_1, \dots, (T_d)^{k_d} x_d)$

Proposition

The action $\mathcal{T}$ on the σ-finite (infinite) measure space $(E, \mathcal{E}, \mu)$ is conservative, ergodic and measure-preserving.

Consider the random fields

- $A := \{x_{\mathbf{0}} = \mathbf{0}\}$, M is a $S\alpha S$ random measure on $(E, \mathcal{E})$ controlled by μ.
- $X_k := \int_E (1_A \circ T^k)(x) M(dx), \quad k \in \mathbb{N}_0^d$

The scaling constant in limit theorems is

$$b_n = \mu\left(\bigcup_{k=\mathbf{1}}^{\mathbf{n}} T^{-k} A\right)^{1/\alpha}$$

$$= \prod_{i=1}^{d} b_n^{(i)} \in \mathrm{RV}_{d\beta/\alpha} \quad \text{assuming} \;\; b_n^{(1)} = \cdots = b_n^{(d)} \in \mathrm{RV}_{\beta/\alpha}$$

where $b_n^{(i)}$ is the scaling constant for process $X^{(i)}$ generated by $(E_i, \mathcal{E}_i, \mu_i, T_i)$.

Define a random sup-measure,

$$\mathscr{M}_n(B) := \max_{k \in nB} X_k \quad B \in \mathcal{B}([0,1]^d)$$

Main theorem

Theorem (Convergence of random sup-measure)

For $0 < \alpha < 2, 0 < \beta < 1$, *in space SM with the sup vague topology.*

$$\frac{1}{b_n}\mathscr{M}_n(\mathbf{X}) \Longrightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \eta^{\alpha,\beta} \quad n \to \infty$$

$$\eta^{\alpha,\beta}(t) \triangleq \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} 1_{\{t \in \prod_{i=1}^d (V_j^{(i)} + R_j^{(\beta,i)})\}}, \quad t \in [0,1]^2$$

$$\eta^{\alpha,\beta}(B) \triangleq \sup_{t \in B} \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} 1_{\{t \in \prod_{i=1}^d (V_j^{(i)} + R_j^{(\beta,i)})\}}, \quad B \in \mathcal{B}([0,1]^2)$$

Random variables on RHS are all independent and have same distribution in the $d = 1$ *settings. Note that:*

$$\eta^{\alpha,\beta}(t) \leqslant \lfloor \beta^{-1} \rfloor \bigvee_{j=1}^{\infty} \Gamma_j^{-1/\alpha} 1_{\{t \in \prod_{i=1}^d (V_j^{(i)} + R_j^{(\beta,i)})\}}$$

1. Due to the fact that the limiting random fields is self-similar, we only consider the convergence in $[0,1]^d$.
2. Functional convergence in J_1 topology are natural corollaries of random sup-measure convergence.
3. We focus on the situation $d = 2$. General cases are similar.

Start with series representation and truncation.

$$\left(X_k : \mathbf{0} \leqslant k \leqslant \mathbf{n}\right) \overset{d}{=} \left(b_n C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} 1_{\{T^k(U_j^{(n)})_{\mathbf{0}}=\mathbf{0}\}}, \mathbf{0} \leqslant k \leqslant \mathbf{n} \right)$$

$$\mathscr{M}_{\ell,n}(B) \triangleq b_n C_\alpha^{1/\alpha} \max_{k \in nB} \sum_{j=1}^{\ell} \epsilon_j \Gamma_j^{-1/\alpha} 1_{\{T^k(U_j^{(n)})_{\mathbf{0}}=\mathbf{0}\}}, \quad B \in \mathcal{B}([0,1]^2)$$

- $\{\epsilon_j\}$ is a iid, Rademacher rvs.
- $\{\Gamma_j\}$ are the arrival times of a unit rate Poisson process on $(0, \infty)$.
- For each n, $\{U_j^{(n)} : j \geqslant 1\}$ are iid E-valued (paths of chains) random elements with same law η_n

$$\frac{\mu\left(\cdot \cap \{x \in E : x_t = \mathbf{0} \text{ for some } \mathbf{0} \leqslant t \leqslant \mathbf{n}\}\right)}{\mu\left(\{x \in E : x_t = \mathbf{0} \text{ for some } \mathbf{0} \leqslant t \leqslant \mathbf{n}\}\right)}$$

Proposition

For any $\delta > 0$,

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\max_{\mathbf{0}\leqslant k\leqslant \mathbf{n}} C_\alpha^{1/\alpha} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{\{T^k(U_j^{(n)})_{\mathbf{0}}=\mathbf{0}\}} \right| > \delta \right) = 0$$

Together with

$$\frac{1}{b_n}\mathscr{M}_{\ell,n} \Rightarrow \eta_\ell^{\alpha,\beta}, \quad \eta_\ell^{\alpha,\beta} \uparrow \eta^{\alpha,\beta}$$

is sufficient to prove weak convergence.

We need to consider simultaneous return time to 0 of several Markov chains. Because we want to pick up "positive" summands in the series representation.

- $Z_{j.n} := \frac{1}{n}\{\mathbf{0} \leqslant k \leqslant \mathbf{n} : T^k(U_j^{(n)})_{\mathbf{0}} = \mathbf{0}\}$, this is a random closed set in $[0,1]^2$.
- $i = 1, 2$, $Z_{j,n}^{(i)} = \frac{1}{n}\{0 \leqslant k \leqslant n : T^k(U_j^{(i,n)})_0 = 0\}$, $U^{(n)} = (U^{(1,n)}, U^{(2,n)})$
- $Z_{j,n} = Z_{j,n}^{(1)} \times Z_{j,n}^{(2)}$.

Once the chain visits zero, the time interval between succeeding visits are iid random variables. In C. Lacaux and G. Samorodnitsky 2016

$$Z_{j,n}^{(1)} = \frac{1}{n}\{\text{first return time} + \text{range of an} \uparrow \text{random walk }\}$$
$$\to (V_j^{(1)} + R_j^{(1)}) \cap [0,1] := \tilde{R}_j^{(1)} \cap [0,1]$$
$$\text{where} \quad x \in [0,1], \mathbb{P}(V_j^{(1)} \leqslant x) = x^\beta$$
$$R_j^{(1)} \text{iid} \quad (1-\beta) - \text{stable regenerative set}$$

- For $S \subset \mathbb{N}$, $\hat{I}_{S,n} := \cap_{j \in S}, Z_{j,n} = \hat{I}^{(1)}_{S,n} \times \hat{I}^{(2)}_{S,n}$.
- $S \subset \mathbb{N}$, $I^{(1)}_S := \cap_{j \in S} \tilde{R}^{(1)}_j$
- $I_S = I^{(1)}_S \times I^{(2)}_S$

Theorem (G. Samorodnitsky and Y. Wang 2017)

$$(\hat{I}^{(1)}_{S,n})_{S \subset \{1,\ldots,\ell\}} \Rightarrow (I^{(1)}_S)_{S \subset \{1,\ldots,\ell\}} \quad n \to \infty \quad in\ \mathcal{F}([0,1])^{2^\ell}$$

Corollary

$$(\hat{I}_{S,n})_{S \subset \{1,\ldots,\ell\}} \Rightarrow (I_S)_{S \subset \{1,\ldots,\ell\}} \quad n \to \infty \quad in\ \mathcal{F}([0,1]^2)^{2^\ell}$$

$$\hat{I}^*_{S,n} \triangleq \hat{I}_{S,n} \cap \left(\bigcup_{j \in \{1,\dots,\ell\} - S} Z_{j,n} \right)^c$$

return time to 0 by chains indexed by S exclusively

$B \subset [0,1]^2$ is an open rectangle

$$H_n(B) := \bigcup_{S \subset \{1,\dots,\ell\}} \left(\left\{ \hat{I}_{S,n} \cap B \neq \emptyset \right\} \bigcap \left\{ \hat{I}^*_{S,n} \cap T = \emptyset \right\} \right)$$

Proposition

$$\lim_{n \to \infty} \mathbb{P}\{H_n(B)\} = 0$$

Intuition: Consider the limit, intersections of stable subordinators.

Take $B_1, \dots, B_m$ disjoint open rectangles in $[0,1]^2$. For each i, on complement of $H_n(B_i)$

$$\mathscr{M}_{\ell,n}(B_i) = \max_{k \in nB_i} b_n \sum_{j=1}^{\ell} \epsilon_j \Gamma_j^{-1/\alpha} 1_{\{T^k(U_j^{(n)})_{\mathbf{0}}=\mathbf{0}\}}$$

$$= C_\alpha^{1/\alpha} \max_{S \subset \{1,\dots,\ell\}} 1_{\{\hat{I}_{S,n} \cap B_i \neq \emptyset\}} \sum_{j \in S} \epsilon_j \Gamma_j^{-1/\alpha}$$

$$= C_\alpha^{1/\alpha} \max_{S \subset \{1,\dots,\ell\}} 1_{\{\hat{I}_{S,n} \cap B_i \neq \emptyset\}} \sum_{j \in S} 1_{\{\epsilon_j = 1\}} \Gamma_j^{-1/\alpha}$$

$$\frac{1}{b_n} \mathscr{M}_{\ell,n} \Rightarrow C_\alpha \max_{S \subset \{1,\dots,\ell\}} 1_{\{I_S \cap B_i \neq \emptyset\}} \sum_{j \in S} 1_{\{\epsilon_j = 1\}} \Gamma_j^{-1/\alpha}$$

$$= C_\alpha \max_{t \in B_i} \sum_{j=1}^{\ell} 1_{\{\epsilon_j = 1\}} \Gamma_j^{-1/\alpha} 1_{\{t \in \tilde{R}_j\}}$$

$$\stackrel{d}{=} C_\alpha^{1/\alpha} 2^{-1/\alpha} \sum_{j \in S} \Gamma_j^{-1/\alpha} 1_{\{t \in \tilde{R}_j\}} := \eta_\ell^{\alpha,\beta} \qquad \eta_\ell^{\alpha,\beta} \uparrow \eta^{\alpha,\beta} \text{a.s.}$$

Corollary

$$\mathscr{M}_n(|\mathbf{X}|)(B) := \max_{k:k/n\in B} |X_k|, \quad B \in \mathcal{B}([0,1]^d)$$

$$\frac{1}{b_n}\mathscr{M}_n(\mathbf{X}) \Longrightarrow C_\alpha^{1/\alpha}\eta^{\alpha,\beta} \quad n \to \infty$$

$$\frac{1}{b_n}\mathscr{M}_{\ell,n}(|\mathbf{X}|)(B) = C_\alpha^{1/\alpha} \max_{S\subset\{1,\dots,\ell\}} 1_{\{\hat{I}_{S,n}\cap B_i\neq\emptyset\}} \sum_{j\in S} \epsilon_j \Gamma_j^{-1/\alpha}$$

$$\Rightarrow C_\alpha^{1/\alpha}\Big(\max_{S\subset\{1,\dots,\ell\}} 1_{\{\hat{I}_{S,n}\cap B\neq\emptyset\}} \sum_{j\in S} 1_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha}$$

$$\bigvee \max_{S\subset\{1,\dots,\ell\}} 1_{\{\hat{I}_{S,n}\cap B\neq\emptyset\}} \sum_{j\in S} 1_{\{\epsilon_j=-1\}}\Gamma_j^{-1/\alpha}\Big)$$

Taking the maximum of two iid rvs cancells out $2^{-1/\alpha}$.

Convergence of partial maxima processes

Theorem

Assume that $0 < \alpha < 2, 0 < \beta < 1$.

$$\left(\frac{1}{b_n} \max_{\mathbf{1} \leqslant k \leqslant \lfloor nt \rfloor} X_k, t \in [0,1]^2 \right) \Longrightarrow (C_\alpha^{1/\alpha} 2^{-1/\alpha} \eta([0,t]), t \in [0,1]^2)$$

in $D([0,1]^2)$ *with Skorohod* J_1*-topology.*

Again, it suffices to show J_1 convergence for the truncated partial maxima random fields.

Fix $S \subset \{1, \ldots, \ell\}$, and $i = 1, 2$

$$Y_{S,n}^{(i)}(t) := 1_{\{\hat{I}_{S,n}^{(i)} \cap [0,t_i] \neq \emptyset\}}, \quad t_i \in [0,1]$$
$$Y_S^{(i)}(t) := 1_{\{I_S^{(i)} \cap [0,t_i] \neq \emptyset\}}, \quad t_i \in [0,1]$$

$\hat{I}_{S,n}^{(i)} \Rightarrow I_S^{(i)}$ implies $\min \hat{I}_{S,n}^{(i)} \Rightarrow \min I_S^{(i)}$ So

$$(Y_{S,n}^{(i)}(t_i), t_i \in [0,1]) \overset{J_1}{\Longrightarrow} (Y_S^{(1)}(t_i), t_i \in [0,1]) \quad \text{in } D[0,1]$$

$$(Y_{S,n}(t), t \in [0,1]) \overset{J_1}{\Longrightarrow} (Y_S(t), t \in [0,1]) \quad \text{in } D([0,1]^2)$$

where

$$Y_{S,n}(t) := 1_{\{\hat{I}_{S,n} \cap [0,t] \neq \emptyset\}} = \prod_{i=1}^{2} Y_{S,n}^{(i)}(t_i), \quad t \in [0,1]^2$$

$$Y_S(t) := 1_{\{I_S \cap [0,t] \neq \emptyset\}} = \prod_{i=1}^{2} Y_S^{(i)}(t_i), \quad t \in [0,1]^2$$

On the complement $H_n([0,1]^2)$ and apply continuous mapping theorem,

$$\left(\frac{1}{b_n} M_n(t), t \in [0,1]^2\right) \quad t = (t_1, t_2), [0,t] := [0,t_1] \times [0,t_2]$$

$$= \left(C_\alpha^{1/\alpha} \max_{S \subset \{1,\dots,\ell\}} 1_{\{\hat{I}_{S,n} \cap [0,t] \neq \emptyset\}} \sum_{j \in S} \epsilon_j \Gamma_j^{-1/\alpha}, \quad t \in [0,1]^2 \right)$$

$$\overset{J_1}{\Rightarrow} \left(C_\alpha^{1/\alpha} \max_{S \subset \{1,\dots,\ell\}} 1_{\{I_S \cap [0,t] \neq \emptyset\}} \sum_{j \in S} 1_{\{\epsilon_j = 1\}} \Gamma_j^{-1/\alpha}, \quad t \in [0,1]^2 \right)$$

$$\overset{d}{=} \left(\left(\frac{C_\alpha}{2}\right)^{1/\alpha} \eta_\ell^{\alpha,\beta}([0,t]), \quad t \in [0,1]^2 \right)$$

Corollary

Assume that $0 < \alpha < 2, 0 < \beta < 1$.

$$\left(\frac{1}{b_n} \max_{\mathbf{1} \leqslant k \leqslant \lfloor nt \rfloor} |X_k|, t \in [0,1]^2\right) \Longrightarrow (C_\alpha^{1/\alpha} \eta([0,t]), t \in [0,1]^2)$$

in $D([0,1]^2)$ *with Skorohod* J_1*-topology.*

Random fields driven by additive simple random walks: in progress

1. $\{S_{\cdot}^{(1)}\}$ and $\{S_{\cdot}^{(2)}\}$, two independent simple symmetric random walk from origin.
2. $S_k \triangleq S_{k_1}^{(1)} + S_{k_2}^{(2)} + S_{\mathbf{0}}$, let $S_{\mathbf{0}} \in \mathbb{Z}$ with counting measure π.

$(E, \mathcal{E})$, path space of $(S_k, k \in \mathbb{N}_0^2)$.

$$\mu := \sum_{n=-\infty}^{\infty} \mathbb{P}_n \quad \text{is a } \sigma\text{-finite measure}$$

$$x \in E, x = x_1 \oplus x_2, k = (k_1, k_2) \in \mathbb{N}_0^d, T^k x = (T_1)^{k_1} x_1 + (T_2)^{k_2} x_2$$

$(E, \mathcal{E}, \mu, \mathcal{T})$ is a conservative, ergodic and measure-preserving system.

A denotes the set $\{S_{\mathbf{0}} = 0\}$.

$$X_k := \int_E 1_A \circ \theta^k dM = \int_E 1_{\{S_k=0\}} dM, \quad k \in \mathbb{Z}_+^2$$

In this model

$$b_n^\alpha \sim c\sqrt{n} \quad \text{for some constant } c$$

A random sup-measure

$$\mathscr{M}_n(\mathbf{X})(B) = \max_{k:k/n \in B} |X_k|, \quad B \in \mathcal{B}([0,\infty)^2)$$

Theorem (in progress)

Assume $0 < \alpha < 2$,

$$\frac{1}{b_n}\mathscr{M}_n(|\mathbf{X}|) \Longrightarrow C_\alpha^{1/\alpha}\eta \quad as\ n \to \infty$$

in space $SM[0,1]^2$

$$\eta(t) = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} 1_{\{t \in W_j\}}$$

$\{W_j\}_{j \geqslant 1}$, *iid rsc's on* $[0,1]^2$, $\forall\ G \in \mathcal{G}([0,1]^2)$,

$$\mathbb{P}\{W_1 \cap G \neq \emptyset\} = \frac{1}{c_2}\int_{-\infty}^{\infty} \mathbb{P}\left\{ \mathcal{Z}\Big(\left\{ B_\cdot^{(1)} + B_\cdot^{(2)} + t \right\}_{[0,1]^2} \Big) \cap G \neq \emptyset \right\} dt$$

$B^{(1)}$ *and* $B^{(2)}$, *independent linear standard Brownian motions started from the origin.*

Unsolved Problem:
Whether zeros of additive simple random walk on $\mathbb{Z}^d, d > 1$, after proper scaling, converges weakly to zeros of additive Brownian motion.

Thanks