Heavy Tails in a Robust Newsvendor Model

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- Let $\alpha = 1 - c/p \in (0, 1)$ denote the *critical ratio*, then the solution is

$$q^* = F^{-1}(\alpha).$$

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- Quantity of interest for now: $\mathbb{E}_F[D - q]^+$.

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$$\mathcal{F}_{1,2} = \left\{ F \in \mathbb{M}(\mathbb{R}_+) : \int_0^\infty \mathrm{d}F(w) = 1, \int_0^\infty w \mathrm{d}F(w) = m_1, \int_0^\infty w^2 \mathrm{d}F(w) = m_2 \right\}$$

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- Given an order quantity $q > m_2/2m_1$, the worst-case demand distribution for $\sup_{F \in \mathcal{F}_{1,2}} \mathbb{E}_F[D - q]_+$ is two-point:

$$D_q^* = \begin{cases} q - \sqrt{q^2 - 2m_1 q + m_2}, & \text{w.p. } \frac{1}{2}\left(1 + \frac{q - m_1}{\sqrt{q^2 - 2m_1 q + m_2}}\right) \\ q + \sqrt{q^2 - 2m_1 q + m_2}, & \text{w.p. } \frac{1}{2}\left(1 - \frac{q - m_1}{\sqrt{q^2 - 2m_1 q + m_2}}\right) \end{cases}$$

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- Support points and probabilities depend on q.
- For $0 \le q \le m_2/2m_1$, the worst-case distribution is $D^*_{m_2/2m_1}$.

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- The worst-case bound is:

$$\sup_{F \in \mathcal{F}_{1,2}} \mathbb{E}_F[D-q]_+ = \begin{cases} \frac{1}{2}\left(\sqrt{q^2 - 2m_1 q + m_2} - (q - m_1)\right), & \text{if } q > \frac{m_2}{2m_1} \\ m_1 - \frac{q m_1^2}{m_2}, & \text{if } 0 \leq q \leq \frac{m_2}{2m_1} \end{cases}$$

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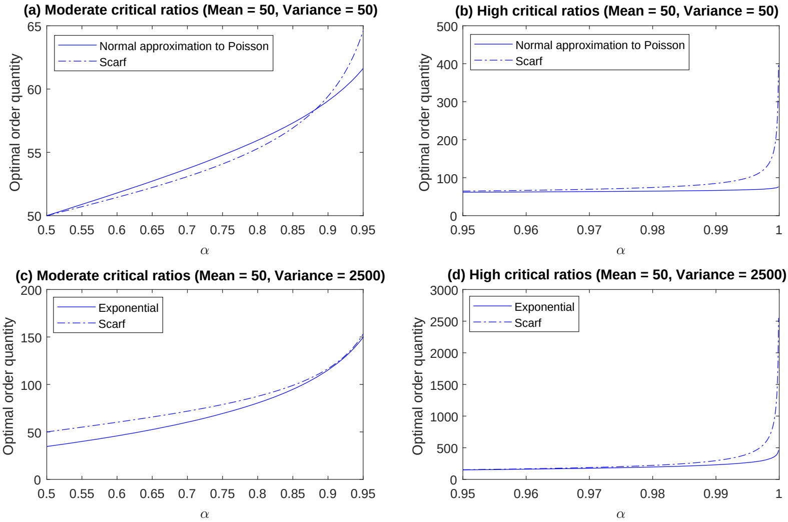
- The optimal order quantity has a closed form solution:

$$q^* = \begin{cases} m_1 + \frac{\sqrt{m_2 - m_1^2}}{2} \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}}, & \text{if } \frac{m_2 - m_1^2}{m_2} < \alpha < 1 \\ 0, & \text{if } 0 \le \alpha \le \frac{m_2 - m_1^2}{m_2} \end{cases}$$

Scarf's Model (1958): Observations

- Scarf (1958): A Poisson approximation works well for a wide range of values; albeit for very high critical ratios, the model prescribes higher order quantities than the Poisson distribution.
- Gallego and Moon (1993): Optimal order quantities from Scarf's model and optimal order quantity for normally distributed demands and concluded numerically that for a large range of critical ratios, the loss in profit is not significant.
- Wang, Glynn and Ye (2015): *"In the distributionally robust optimization approach, the worst-case distribution for a decision is often unrealistic. Scarf (1958) shows that the worst-case distribution in the newsvendor context is a two-point distribution. This raises the concern that the decision chosen by this approach is guarding under some overly conservative scenarios, while performing poorly in more likely scenarios. Unfortunately, these drawbacks seem to be inherent in the model choice and cannot be remedied easily."*

Scarf's Model



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- F^* dominates $F \in \mathcal{F}$ in an increasing convex order sense Müller and Stoyan (2002)).
- F^* may not have an explicit characterization and might not even lie in the original set of distributions $\mathcal{F}$.

Revisiting Scarf's Model

- The distribution F^* from Scarf's model

$$F^*(d) = \mathbb{P}(D^* \leq d) = \begin{cases} \frac{1}{2}\left(1 + \frac{d-m_1}{\sqrt{d^2-2m_1d+m_2}}\right), & \text{if } d > \frac{m_2}{2m_1} \\ 1 - \frac{m_1^2}{m_2}, & \text{if } 0 \leq d \leq \frac{m_2}{2m_1} \end{cases}$$

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- F^* defines a censored student-t random variable with d.o.f 2:

$$D^* = \begin{cases} \tilde{t}_2\left(m_1, (m_2 - m_1^2)/2\right), & \text{if } \tilde{t}_2\left(m_1, (m_2 - m_1^2)/2\right) > \frac{m_2}{2m_1} \\ 0, & \text{otherwise} \end{cases}$$

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- This characterizes a "heavy tail optimality" property of Scarf's model.

Related DRN Models

Our focus is on sets where demand might take any value in $[0, \infty)$ and the high service level regime.

- Closed form solution: Ben-Tal and Hochman (1972, 1976) - mean, mean absolute deviation, Natarajan, Sim and Uichanco (2017) - mean, variance and semivariance.

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- Bertsimas and Popescu (2002, 2005) and Lasserre (2002) - SDP techniques to compute the worst-case bounds given:

$$\mathcal{F}_{1,2,\dots,n} = \left\{ F \in \mathbb{M}(\mathbb{R}_+) : \int_0^\infty \mathrm{d}F(w) = 1, \int_0^\infty w^i \mathrm{d}F(w) = m_i,\ i = 1, 2, \dots, n \right\}$$

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- While some attempts has been made to solve this problem analytically for $n = 3$ and $n = 4$, the tight worst-case bounds have complicated expressions involving roots of cubic and quartic equations (see Jansen, Haezendonck, and Goovaerts (1986) Zuluaga, Peña and Du (2009)).

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- Engelke and Ivanovs (2017) provide robust asymptotic bounds on tail probabilities in multivariate extremes.

Related DRN Models

- A related moment model that has been studied in option pricing but remains unexplored in the newsvendor model (see Grundy (1991)):

$$\mathcal{F}_n = \left\{ F \in \mathbb{M}(\mathbb{R}_+) : \int_0^\infty \mathrm{d}F(w) = 1, \int_0^\infty w^n \mathrm{d}F(w) = m_n \right\}$$

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- For $0 \leq q \leq (n-1)m_n^{1/n}/n$, the worst-case demand distribution is degenerate with the mass at the point $m_n^{1/n}$.
- The distribution F^* in this model defines a Pareto random variable:

$$D^* = \text{Pareto}\left(\frac{(n-1)m_n^{1/n}}{n}, n \right)$$

Model

- Consider an ambiguity set defined by a known value of the first and the nth moment:

$$\mathcal{F}_{1,n} = \left\{ F \in \mathbb{M}(\mathbb{R}_+) : \int_0^\infty \mathrm{d}F(w) = 1, \int_0^\infty w \mathrm{d}F(w) = m_1, \int_0^\infty w^n \mathrm{d}F(w) = m_n \right\}$$

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- For rational values of n, the DRN problem under this ambiguity set can be solved as a SDP.
- We develop lower and upper bounds that are approximately optimal for large values of q, which helps us characterize F^*.

Main Results

Proposition

Given an ambiguity set $\mathcal{F}_{1,n}$, there exists a positive quantity $\underline{q}(m_1, m_n, n)$ which depends on the parameters m_1, m_n and n, such that:

$$\sup_{F \in \mathcal{F}_{1,n}} \mathbb{E}_F[D - q]_+ \quad \geq \quad \frac{(m_n - m_1^n)}{n^n q^{n-1}}(n-1)^{n-1}, \quad \forall q \geq \underline{q}(m_1, m_n, n).$$

Main Results

Proposition

Consider the ambiguity set $\mathcal{F}_{1,n}$.

(a) *Given $n \in (2, \infty)$, there exists a positive quantity $\overline{q}(m_1, m_n, n)$ which depends on the parameters m_1, m_n and n, such that:*

$$\sup_{F \in \mathcal{F}_{1,n}} \mathbb{E}_F[D - q]_+ \leq \frac{(m_n - m_1^n)}{n^n q^{n-1} - n^2 m_1^{n-1}(n-1)^{n-1}}(n-1)^{n-1} \quad \forall q \geq \overline{q}(m_1, m_n, n).$$

(b) *Given $n \in (1, 2)$, for any $\epsilon > 0$, there exists a positive quantity $\overline{q}(m_1, m_n, n, \epsilon)$ which depends on the parameters m_1, m_n, n and ϵ such that:*

$$\sup_{F \in \mathcal{F}_{1,n}} \mathbb{E}_F[D - q]_+ \leq \frac{(m_n - m_1^n)}{n^n q^{n-1} - n^2 m_1^{n-1}(n-1)^{n-1-\epsilon}}(n-1)^{n-1} \quad \forall q \geq \overline{q}(m_1, m_n, n, \epsilon).$$

Outline of Proof: Lower Bound

- Primal feasible - Consider a three point random variable D with a distribution defined as follows:

$$D = \begin{cases} \frac{qn}{n-1}, & \text{w.p. } \frac{(m_n - m_1^n)}{n^n q^n}(n-1)^n \\ d, & \text{w.p. } p \\ 0, & \text{w.p. } 1 - p - \frac{(m_n - m_1^n)}{n^n q^n}(n-1)^n \end{cases}$$

- Find the values d and p as a function of q such that moment constraints for m_1 and m_n are met.
- Show that for large values of q, the value d is less than q and the corresponding p defines a valid probability measure.

Outline of Proof: Upper Bound

- Dual feasible - The dual formulation is:

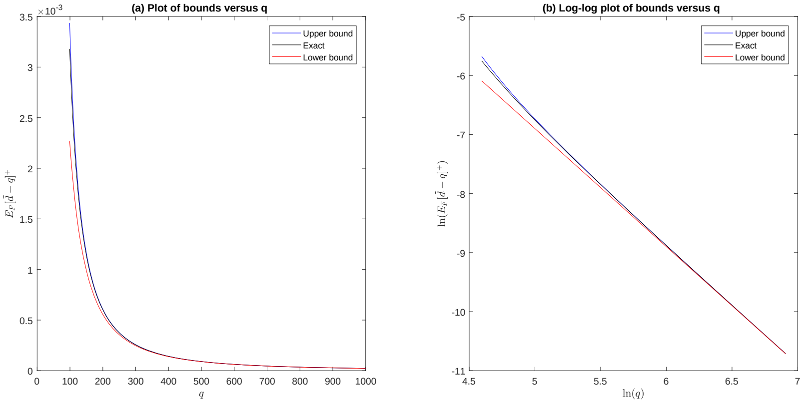
$$\begin{array}{lll} \inf & y_0 + y_1 m_1 + y_n m_n & \\ \text{s.t.} & y_0 + y_1 d + y_n d^n \geq 0, & \forall d \geq 0, \\ & y_0 + y_1 d + y_n d^n \geq d - q, & \forall d \geq 0, \end{array}$$

- Set:

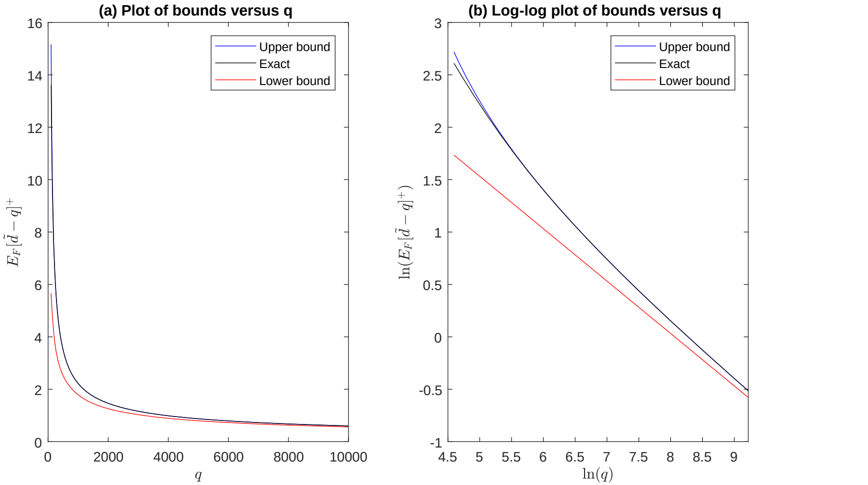
$$\begin{aligned} y_0 &= \frac{(n-1) m_1^n (n-1)^{n-1}}{n^n (q^{n-1} - K)}, \\ y_1 &= \frac{-n m_1^{n-1} (n-1)^{n-1}}{n^n (q^{n-1} - K)}, \\ y_n &= \frac{(n-1)^{n-1}}{n^n (q^{n-1} - K)}, \end{aligned}$$

- Choose K carefully for each of the cases with $n \in (2, \infty)$ and $n \in (1, 2)$ which ensures dual feasibility for large values of q.

Numerical ($n = 3$, $m_1 = 50$, $m_3 = 125150$)



Numerical ($n = 3/2,\ m_1 = 50,\ m_{3/2} = 500, \epsilon = 0.2$)



Regular Variation

- A function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying at infinity (Bingham, Goldie and Teugels (1998), Resnick (2008)) if for all $t > 0$, we have:

  $$\lim_{x\to\infty} \frac{u(tx)}{u(x)} = t^{\beta},$$

  for some $\beta \in \mathbb{R}$. We denote this function as $u \in \mathcal{RV}_\beta$.
- A nonnegative random variable with distribution function F is regularly varying if $\overline{F} := 1 - F \in \mathcal{RV}_{-\beta}$ for some $\beta \geq 0$.
- Examples: Pareto, Cauchy, Burr, Log-gamma, Student-t.

Tail behavior

Theorem

Given an ambiguity set $\mathcal{F}_{1,n}$ and fixed parameters m_1, m_n and n, as $q \to \infty$

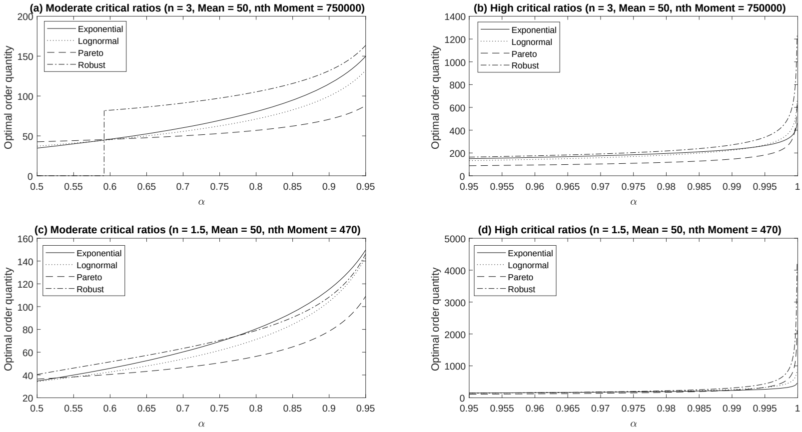
$$\mathbb{E}_{F^*}[D-q]_+ = \sup_{F \in \mathcal{F}_{1,n}} \mathbb{E}_F[D-q]_+ \quad \in \quad \mathcal{RV}_{-(n-1)}.$$

Moreover (under differentiation, Landau (1916))

$$\overline{F}^* \quad \in \quad \mathcal{RV}_{-n},$$

with $\mathbb{E}_{F^*}[D^{*k}] < \infty$ for all $0 \le k < n$ and $\mathbb{E}_{F^*}[D^{*k}] = \infty$ if $k \ge n$.

Numerical: Optimal Order Quantity



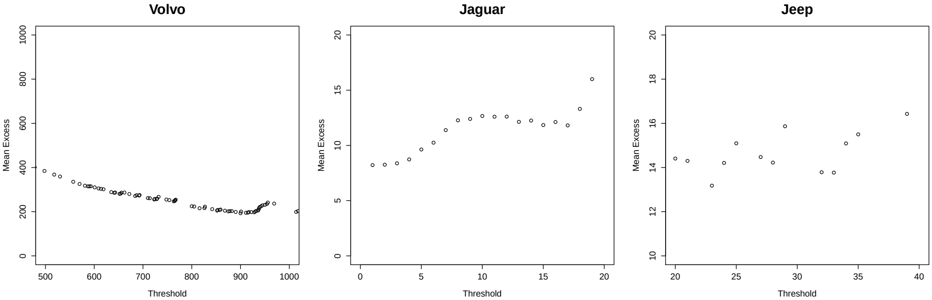
Evidence of Heavy-tailed Demand

Reference	Data	Demand
Clauset, Shalizi & Newman (2009)	Two data sets out of 24 - Number of calls, books sold	Power law is a good fit in 17 sets.
Gaffeo, Scorcu & Vici (2008)	Italian books	Power law: parameter < 2
Chevalier & Goolsbee (2003)	Sales of books on Amazon.com	Pareto: Parameter 1.2
Bimpkis & Markakis (2016)	626 products from North American retailer	Pareto: Parameter 1.38
Natarajan, Sim & Uichanco (2017)	Spare parts (IBM)	Best fit is often a heavy-tailed distribution

Norway monthly car sales: Jan 2007 – Jan 2017

- Published by *Opplysningsrådet for Veitrafikken (OFV)*.
- Obtained from `https://www.kaggle.com`.
- Monthly sales of cars by Make and Model.
- We observe monthly sales for: Volvo, Jaguar and Jeep.
- # of months observed: 121, 117, 109.
- Partition into: training set, test set.

Mean Excess Plot



Volvo

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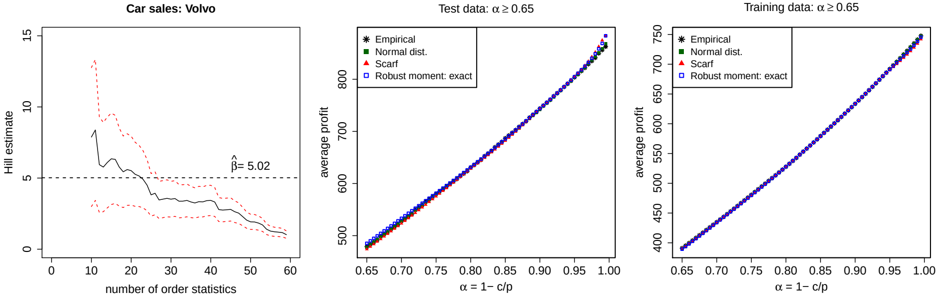

Jaguar

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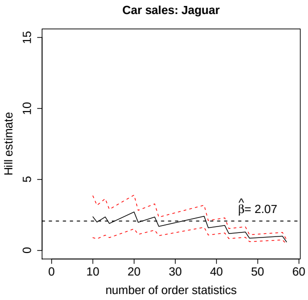

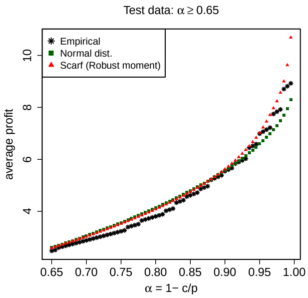

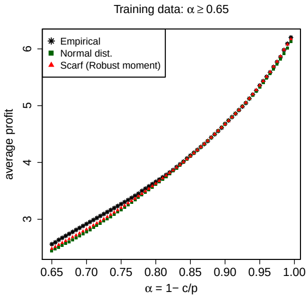

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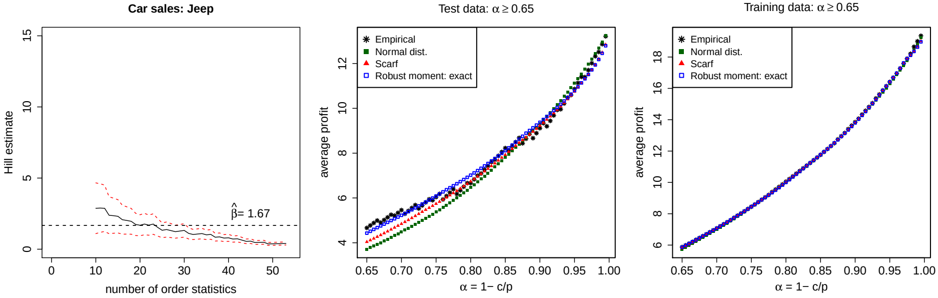

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- `https://arxiv.org/abs/1806.05379`.

Thank you