Limit theorems for long-range dependent processes based on random partitions

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(Université de Tours, Institut Denis Poisson)

joint work with
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and
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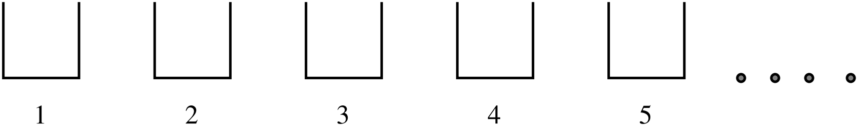
Self-Similarity, Long-Range Dependence, and Extremes
BIRS-CMO Oaxaca, June 2018

Outline

Infinte urn model

$(Y_n)_{n\geq 1}$ i.i.d. with values in $\mathbb{N} = \{1, 2, \ldots\}$.



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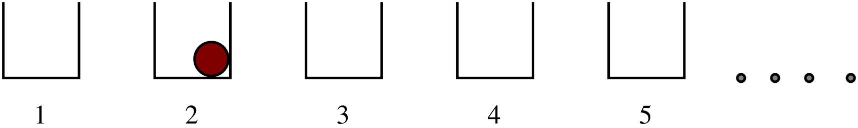

$Y_1 = 2$

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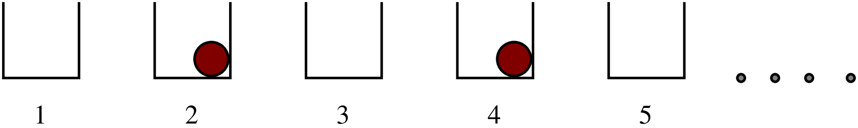

$Y_1 = 2$, $Y_2 = 4$

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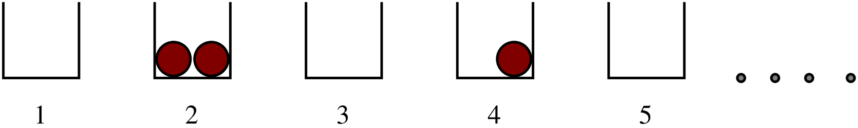

$Y_1 = 2,\ Y_2 = 4,\ Y_3 = 2$

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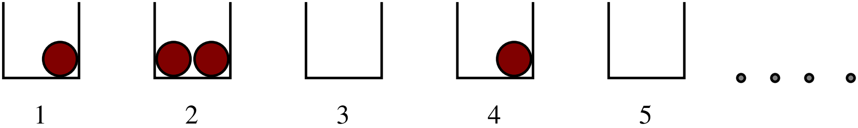

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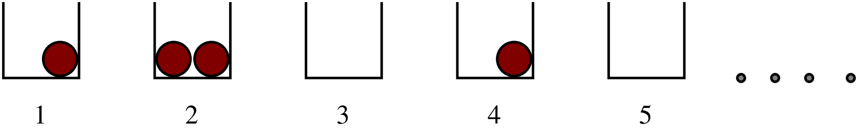

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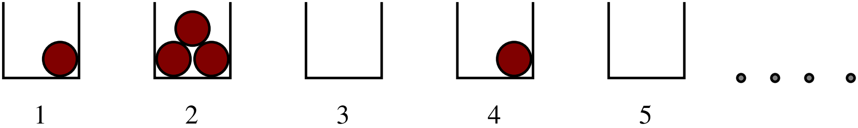

$Y_1 = 2, \; Y_2 = 4, \; Y_3 = 2, \; Y_4 = 1, \; Y_5 = 100, \; Y_6 = 2$

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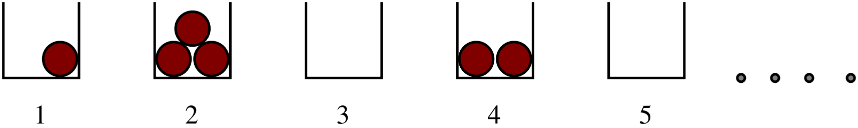

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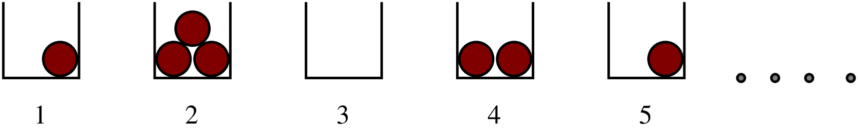

$Y_1 = 2,\ Y_2 = 4,\ Y_3 = 2,\ Y_4 = 1,\ Y_5 = 100,\ Y_6 = 2,\ Y_7 = 4,\ Y_8 = 5$

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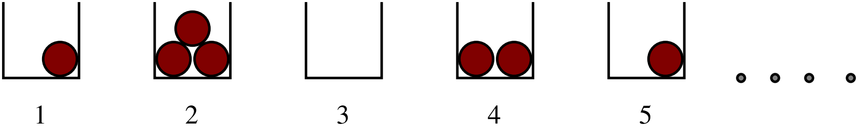

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$\longrightarrow$ Random partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ as $\{1, 3, 6\}$, $\{2, 7\}$, $\{4\}$, $\{5\}$, $\{8\}$.

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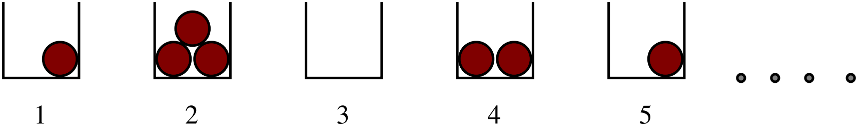

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Bahadur (1960), Karlin (1968), Gnedin, Hansen & Pitman (2007)

Infinte urn model

- $K_{n,\ell} = \sum_{i=1}^{n} \mathbb{1}_{\{Y_i=\ell\}}$.
- Occupancy process: $Z(n) = \sum_{\ell\geq 1} \mathbb{1}_{\{K_{n,\ell}>0\}}$.
- Odd-occupancy process: $U(n) = \sum_{\ell\geq 1} \mathbb{1}_{\{K_{n,\ell} \text{ odd }\}}$.

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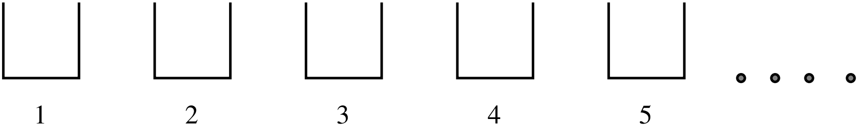

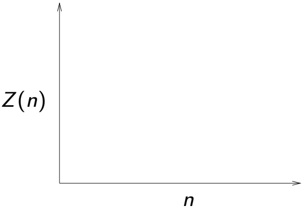

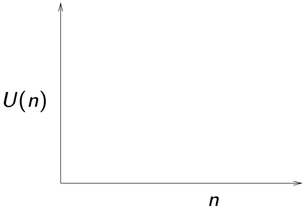

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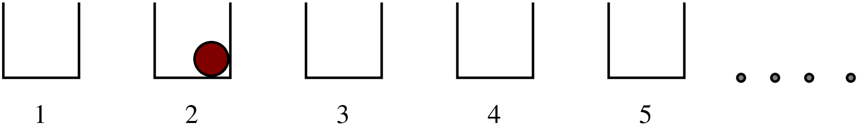

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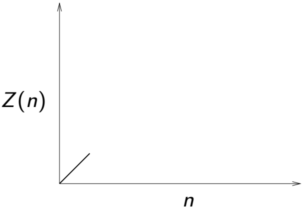

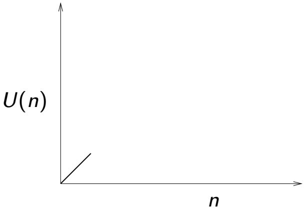

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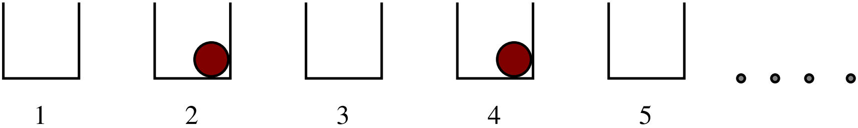

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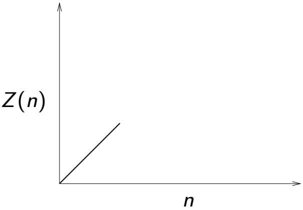

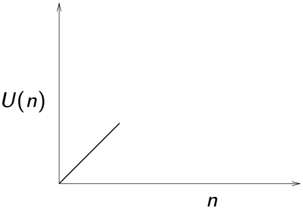

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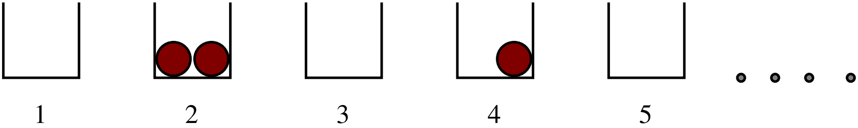

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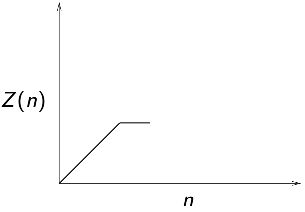

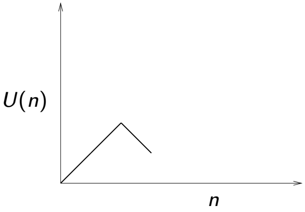

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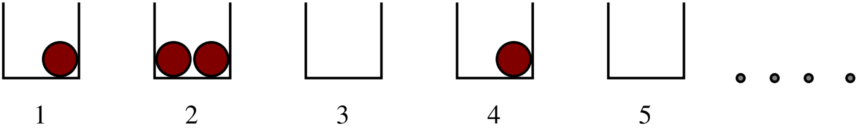

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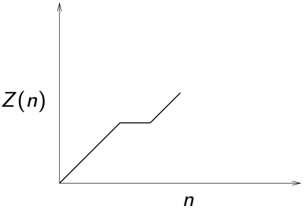

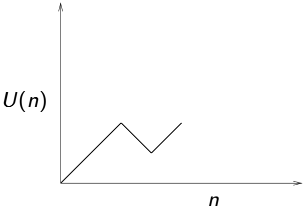

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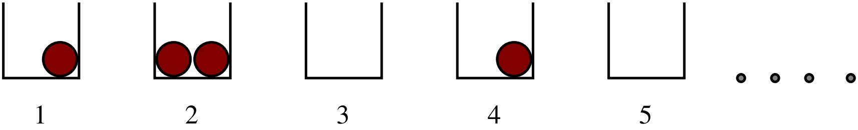

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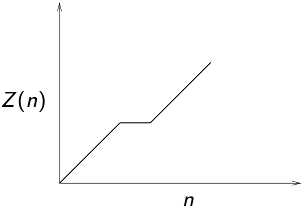

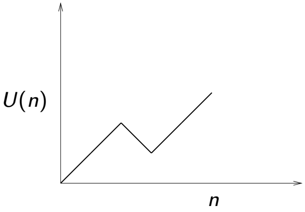

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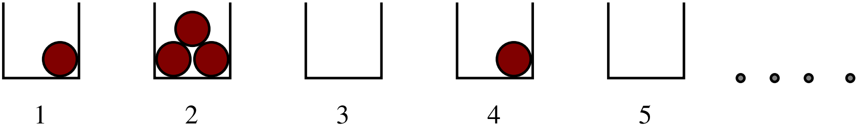

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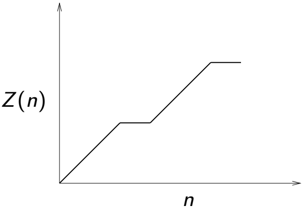

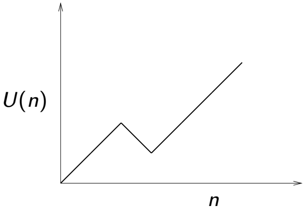

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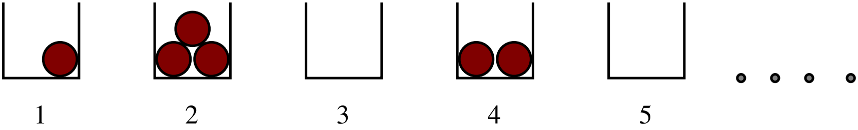

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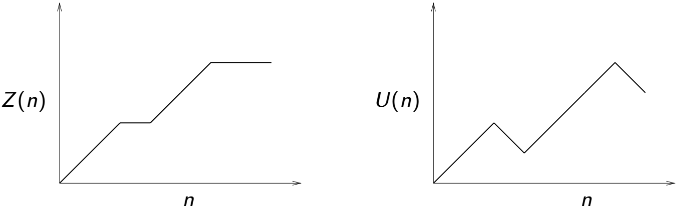

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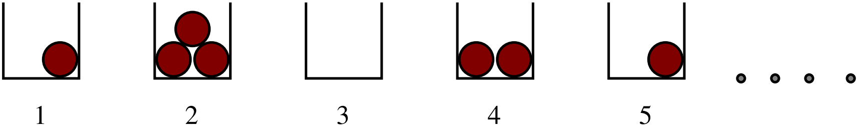

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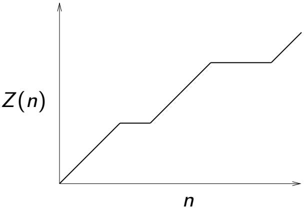

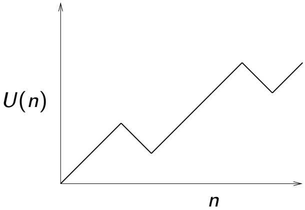

Infinte urn model

Let $p_k = \mathbb{P}(Y_1 = k)$, $k \geq 1$.

Assumptions:

- (p_k) is nonincreasing and $p_k > 0$ for all $k \geq 1$.
- Regular variation: $\max\{k \geq 1 \mid p_k \geq 1/t\} = t^{\beta} L(t)$, for some $\beta \in (0, 1)$ and L slowly varying function.

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Central Limit Theorem (Karlin, 1968)

For $\sigma_n = (\Gamma(1-\beta) n^\beta L(n))^{1/2}$,

$$\frac{Z(n) - \mathbb{E}Z(n)}{\sigma_n} \Rightarrow c_1 \mathcal{N}(0,1)$$

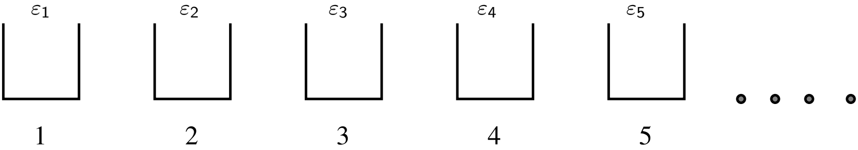
$$\frac{U(n) - \mathbb{E}U(n)}{\sigma_n} \Rightarrow c_2 \mathcal{N}(0,1)$$

where $c_1 = (2^\beta - 1)^{1/2}$ *and* $c_2 = 2^{\beta-1}$.

Outline

Randomization

$(\varepsilon_n)_{n\geq 1}$ i.i.d. Rademacher random variables.



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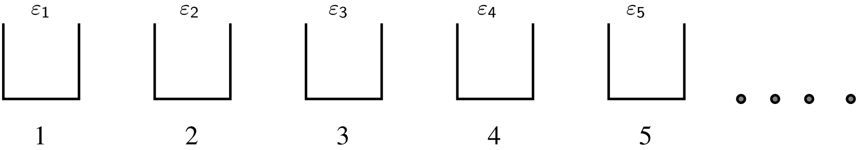
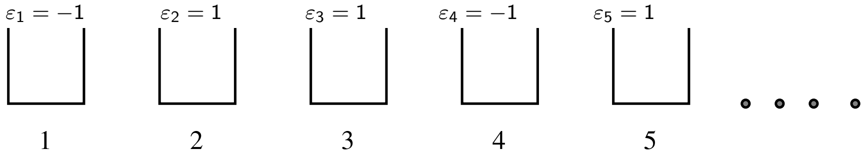
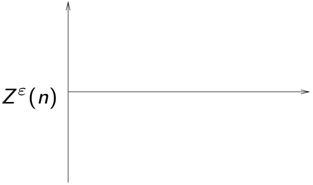


Randomized Occupancy Process (ROP): $Z^{\varepsilon}(n) = \sum_{\ell\geq 1} \varepsilon_\ell \mathbb{1}_{\{K_{n,\ell}>0\}}$

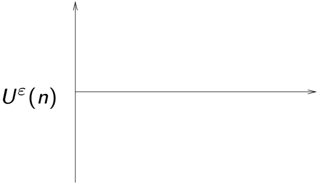
Randomized Odd-Occupancy Process (ROOP): $U^{\varepsilon}(n) = \sum_{\ell\geq 1} \varepsilon_\ell \mathbb{1}_{\{K_{n,\ell} \text{ odd }\}}$

Randomization



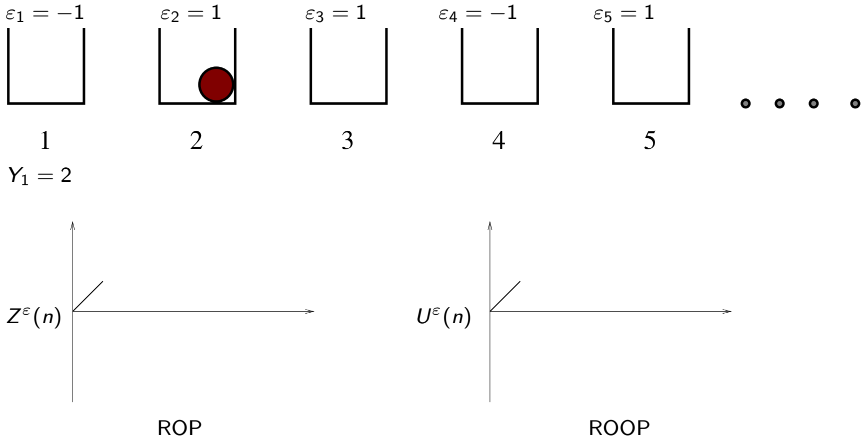


ROP



ROOP

Randomization



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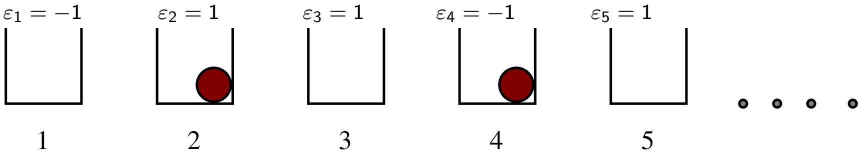

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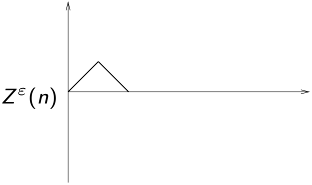

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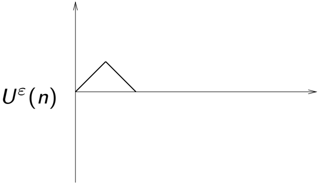

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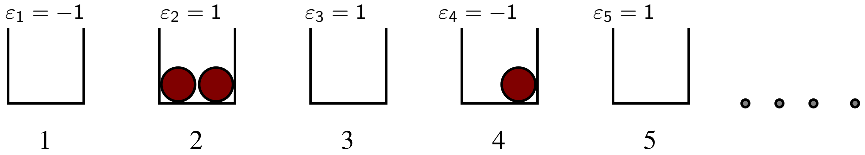

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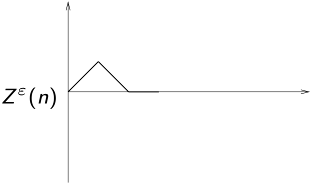

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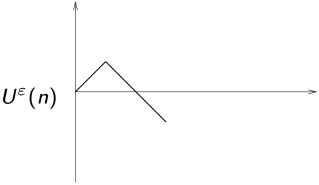
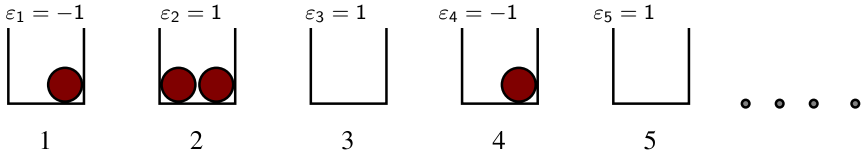
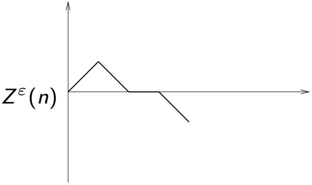


ROOP

Randomization



$Y_1 = 2,\ Y_2 = 4,\ Y_3 = 2,\ Y_4 = 1,$



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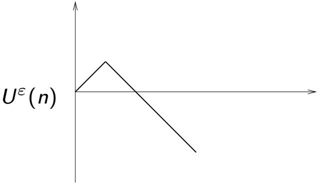

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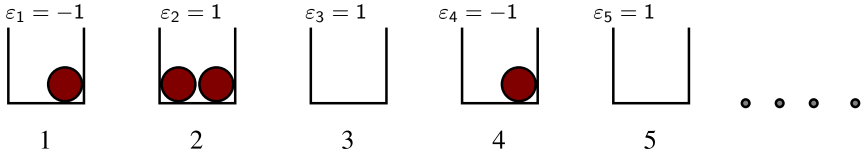
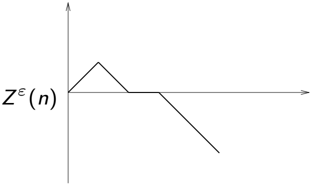


$Y_1 = 2,\ Y_2 = 4,\ Y_3 = 2,\ Y_4 = 1,\ Y_5 = 100$



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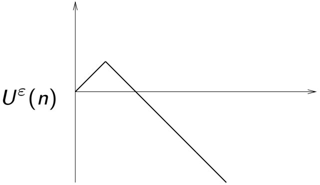

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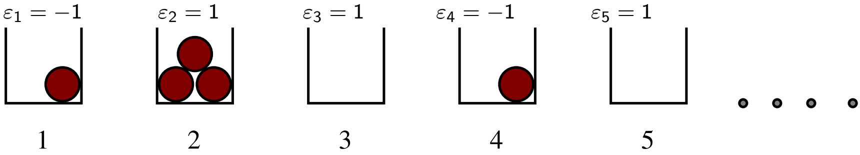
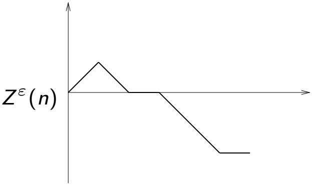


$Y_1 = 2,\ Y_2 = 4,\ Y_3 = 2,\ Y_4 = 1,\ Y_5 = 100,\ Y_6 = 2$



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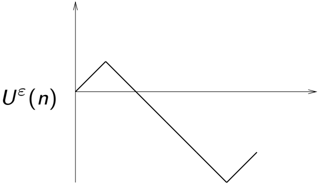

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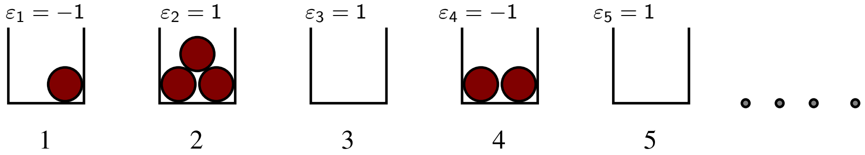

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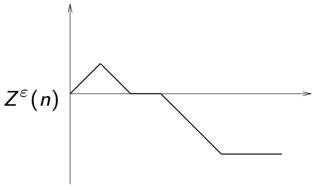

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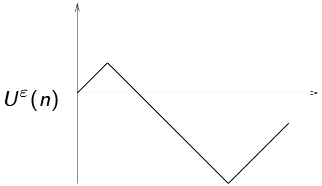

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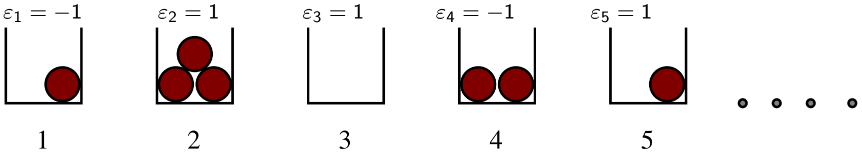

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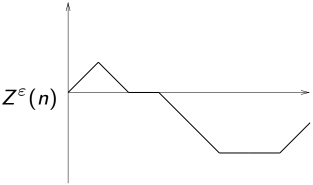

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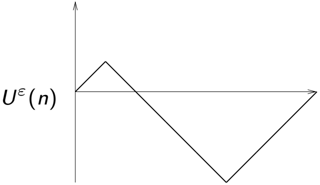

ROOP

Correlated random walks

ROP: $Z^{\varepsilon}(n) = \sum_{\ell \geq 1} \varepsilon_\ell \mathbb{1}_{\{K_{n,\ell} > 0\}} = \sum_{i=1}^{n} X_i \quad \text{with } X_i = \varepsilon_{Y_i} \mathbb{1}_{\{K_{i,Y_i} = 1\}}.$

ROOP: $U^{\varepsilon}(n) = \sum_{\ell \geq 1} \varepsilon_\ell \mathbb{1}_{\{K_{n,\ell} \text{ odd}\}} = \sum_{i=1}^{n} X_i \quad \text{with } X_i = \varepsilon_{Y_i} (-1)^{K_{i,Y_i}+1}.$

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Functional CLT (D., Wang, 2016)

For $\sigma_n = (\Gamma(1-\beta)n^{\beta}L(n))^{1/2}$,

$$\left\{\frac{Z^{\varepsilon}(\lfloor nt \rfloor)}{\sigma_n}\right\}_{t\in[0,1]} \Rightarrow \left\{\mathbb{B}(t^{\beta})\right\}_{t\in[0,1]} \quad \textit{(time-changed Brownian motion)}$$

$$\left\{\frac{U^{\varepsilon}(\lfloor nt \rfloor)}{\sigma_n}\right\}_{t\in[0,1]} \Rightarrow c_\beta \left\{\mathbb{B}^{\beta/2}(t)\right\}_{t\in[0,1]} \quad \textit{(fractional Brownian motion)}$$

in $D([0,1])$.

Here $c_\beta = 2^{(\beta-1)/2}$.

Outline

Heavy-tailed randomization

Assume $(\varepsilon_n)_{n\geq 1}$ are symmetric i.i.d. random variables in the DoA of a symmetric α-stable law, $\alpha \in (0,2)$:

$$\lim_{x\to\infty} \frac{\mathbb{P}(|\varepsilon_1| > x)}{x^{-\alpha}} = C_\varepsilon \in (0,\infty).$$

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Theorem (D., Samorodnitsky, Wang)

For $b_n = (\Gamma(1-\beta)n^\beta L(n))^{1/\alpha}$,

$$\left\{\frac{Z^\varepsilon(\lfloor nt\rfloor)}{b_n}\right\}_{t\in[0,1]} \Rightarrow \sigma_\varepsilon \left\{\mathbb{Z}^\alpha(t^\beta)\right\}_{t\in[0,1]} \quad \textit{(time-changed S}\alpha\textit{S Lévy process)}$$

in $D([0,1])$.

Here $\sigma_\varepsilon^\alpha = C_\varepsilon \int_0^\infty x^{-\alpha}\sin x\,dx$.

Heavy-tailed randomization

Let

$$\mathbb{U}^{\alpha,\beta}(t) := \int_{\mathbb{R}_+\times\Omega'} \mathbb{1}_{\{N(rt)(\omega')\ \mathrm{odd}\}} m_{\alpha,\beta}(dr, d\omega'), \quad t \geq 0,$$

where $m_{\alpha,\beta}$ is a SαS random measure on $\mathbb{R}_+ \times \Omega'$ with control measure

$$\Gamma(1-\beta)^{-1}\beta r^{-\beta-1} dr \times \mathbb{P}'(d\omega'),$$

and N is a standard Poisson process defined on the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$.

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If $\alpha \geq 1$, open question.

Properties : $\mathbb{U}^{\alpha,\beta}$ is β/α-self-similar with non-ergodic stationary increments (Samorodnitsky, 2005)

Idea of the proof

Let $d \geq 1$ and $\boldsymbol{\delta} \in \Lambda_d = \{0,1\}^d \setminus \{(0,\dots,0)\}$.
Consider the multiparameter odd-occupancy process

$$M^{\boldsymbol{\delta}}(\mathbf{n}) := \sum_{k=1}^{\infty} \mathbb{1}_{\{K_{\mathbf{n},k}=\boldsymbol{\delta} \mod 2\}} = \sum_{k=1}^{\infty} \prod_{j=1}^{d} \mathbb{1}_{\left\{K_{n_j,k}=\delta_j \mod 2\right\}}, \quad \mathbf{n} \in \mathbb{N}^d.$$

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Then

$$\lim_{n\to\infty} \frac{M^{\boldsymbol{\delta}}(\lfloor n\mathbf{t} \rfloor)}{n^{\beta} L(n)} = \int_0^{\infty} \mathbb{P}\left(\vec{N}(r\mathbf{t}) = \boldsymbol{\delta} \mod 2\right) \beta r^{-\beta-1} dr \quad \text{in probability.}$$

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Moreover,

$$\left\{ \frac{M^{\boldsymbol{\delta}}(\lfloor n\mathbf{t} \rfloor) - \mathbb{E} M^{\boldsymbol{\delta}}(\lfloor n\mathbf{t} \rfloor)}{(n^{\beta} L(n))^{1/2}} \right\}_{\mathbf{t}\in[0,1]^d} \Rightarrow \left\{ \mathbb{M}^{\boldsymbol{\delta}}(\mathbf{t}) \right\}_{\mathbf{t}\in[0,1]^d}$$

in $D([0,1]^d)$, where $\mathbb{M}^{\boldsymbol{\delta}}$ is a centered Gaussian random field with

$$\mathrm{Cov}(\mathbb{M}^{\boldsymbol{\delta}}(\mathbf{t}), \mathbb{M}^{\boldsymbol{\delta}}(\mathbf{s})) = \int_0^{\infty} \mathrm{Cov}\left(\mathbb{1}_{\{\vec{N}(r\mathbf{t}) = \boldsymbol{\delta} \mod 2\}}, \mathbb{1}_{\{\vec{N}(r\mathbf{s}) = \boldsymbol{\delta} \mod 2\}}\right) \beta r^{-\beta-1} dr.$$

Outline

Related models for extremes

Infinite urn model $(Y_i)_{i\geq 1}$ with positive heavy-tailed randomization $(\varepsilon_k)_{k\geq 1}$.

Empirical random sup-measures on $[0,1]$

$$M_n(A) = \max_{i/n\in A} X_i, \quad A \subset [0,1],$$

with

$$\begin{aligned}
& X_i = \varepsilon_{Y_i} \quad \text{(occupancy)},\\
\text{or} \quad & X_i = \varepsilon_{Y_i} \mathbb{1}_{\{K_{i,Y_i} \text{ odd}\}} \quad \text{(odd-occupancy)},\\
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They all have the same associated extremal process

$$M_n(t) = M_n([0,t]) = \max_{i=1,\dots,\lfloor nt\rfloor} X_i, \quad t\in[0,1].$$

In the sequel, $X_i = \varepsilon_{Y_i}$.

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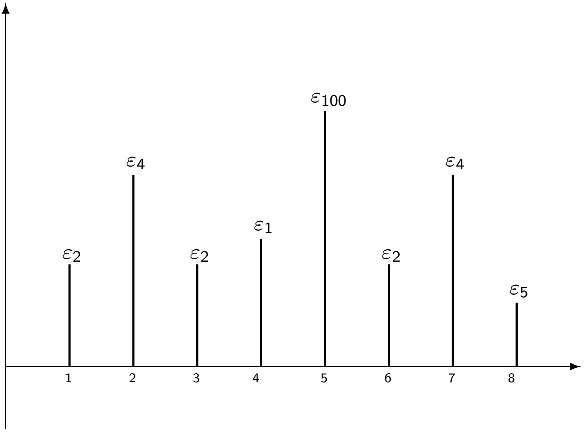

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Theorem (D., Wang)

For $b_n = (C_\varepsilon \Gamma(1-\beta) n^\beta L(n))^{1/\alpha}$*,*

$$\frac{1}{b_n} M_n \Rightarrow \mathcal{M}_{\alpha,\beta},\ \text{as } n \to \infty,$$

in $\mathrm{SM}([0,1])$*, where* $\mathcal{M}_{\alpha,\beta}$ *is the Karlin random sup-measure on* $[0,1]$*:*

$$\mathcal{M}_{\alpha,\beta}(A) := \sup_{\ell\geq 1} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbb{1}_{\{N_\ell(x_\ell A)\neq 0\}}, \quad A \subset [0,1],$$

where $\{(\Gamma_\ell, x_\ell)\}_{\ell\geq 1}$ *is an enumeration of a Poisson process on* $\mathbb{R}_+ \times \mathbb{R}_+$ *with intensity* $d\gamma \times \Gamma(1-\beta)^{-1}\beta x^{-\beta-1}dx$ *and* $\{N_\ell\}_{\ell\geq 1}$ *are i.i.d. standard Poisson processes on* $\mathbb{R}_+$*.*

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$\mathcal{M}_{\alpha,\beta}$ has been considered as example of Choquet-RSM in Molchanov & Strokorb (2016).

Karlin random sup-measure

For all $z > 0$,

$$\mathbb{P}(\mathcal{M}_{\alpha,\beta}(A) \le z) = \exp\left(-\frac{\theta_\beta(A)}{z^\alpha}\right) \quad \text{with} \quad \theta_\beta(A) := \mathrm{Leb}(A)^\beta.$$

The function θ_β is the extremal coefficient of $\mathcal{M}_{\alpha,\beta}$.

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Another representation:

$$\mathcal{M}_{\alpha,\beta}(A) \stackrel{d}{=} \sup_{\ell \ge 1} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbb{1}_{\{\mathcal{R}_\ell^{(\beta)} \cap A \ne \emptyset\}}, \quad A \subset [0,1],$$

with $(\mathcal{R}_\ell^{(\beta)})_{\ell \ge 1}$ i.i.d. random sets such that

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Remark:

$$\mathcal{R}^{(\beta)} \stackrel{d}{=} \bigcup_{i=1}^{Q_\beta} \{U_i\},$$

with (U_i) be i.i.d. uniformly distributed over $(0,1)$ and Q_β an $\mathbb{N}$-valued random variable such that

$$\mathbb{P}(Q_\beta = k) = \frac{\beta(1-\beta)(2-\beta)\cdots(k-1-\beta)}{k!}, \quad k \in \mathbb{N}.$$

Karlin Random Sup-Measure

The Karlin RSM can be compared with:

$$\mathcal{M}_\alpha^{is}(\cdot) = \sup_{\ell \in \mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbb{1}_{\{U_\ell \in \cdot\}} \quad \text{(independently scattered RSM)}$$

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where $(S_\ell^{(\beta)})$ are i.i.d. random closed sets of $[0,1]$, each consisting of a randomly shifted β-stable regenerative set. (Lacaux & Samorodnitsky, 2016)

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The max-increment process $\{\mathcal{M}((t,t+1])\}_{t\in\mathbb{R}}$ of
- $\mathcal{M}_{\alpha}^{is}$ is mixing,
- $\mathcal{M}_{\alpha,\beta}^{sr}$ is ergodic but not mixing,
- $\mathcal{M}_{\alpha,\beta}$ is not ergodic.

Idea of the proof of the theorem

For each $n \in \mathbb{N}$, $\ell \geq 1$, let

$$R_{n,\ell} := \{i \in \{1, \dots, n\} : Y_i = \ell\}$$

and consider the point process

$$\xi_n := \sum_{\ell \geq 1,\, K_{n,\ell} \neq 0} \delta_{\left(\frac{\varepsilon_\ell}{b_n}, \frac{R_{n,\ell}}{n}\right)},$$

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Theorem

For $b_n = (C_\varepsilon \Gamma(1-\beta) n^\beta L(n))^{1/\alpha}$,

$$\xi_n \Rightarrow \xi := \sum_{\ell=1}^{\infty} \delta_{\left(\Gamma_\ell^{-1/\alpha}, \mathcal{R}_\ell^{(\beta)}\right)}, \text{ as } n \to \infty,$$

in $\mathfrak{M}_+((0,\infty) \times \mathcal{F}([0,1]))$.

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The joint convergence of the $\dfrac{\widehat{R}_{n,k}}{n}$ uses Poissonization technique.

Let N be a standard Poisson process on $\mathbb{R}_+$, and $\tau_1, \tau_2, \dots$ its arrival times.

At time n:

$$\widetilde{K}_{n,\ell} = \#\{i \,:\, Y_i = \ell \text{ and } \tau_i \leq n\} \text{ and } \widetilde{Z}(n) = \#\{\ell : \widetilde{K}_{n\ell} \neq 0\}$$

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Let $\tilde{\ell}_{n,k}$ the index s.t. $\varepsilon_{n,k} = \varepsilon_{\tilde{\ell}_{n,k}}$ and $\widetilde{R}_{n,k} = \{\tau_i \le n \,:\, Y_i = \tilde{\ell}_{n,k}\}$.

Then

$$d_H\left(\widetilde{R}_{n,k}/n, \widehat{R}_{n,k}/n\right) \to 0, \quad \text{as } n \to \infty.$$