A positivity conjecture for unitary VOAs

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Part 1: Unitary VOAs

Unitary vertex operator algebras

- VOA: $v \in V \quad \longmapsto \quad Y(v,z) \in \mathrm{End}(V)[[z^{\pm 1}]]$

- A unitary VOA is one with an invariant inner product:

$$\big\langle Y(v,z)u_1, u_2 \big\rangle = \big\langle u_1, Y(\tilde{v}, \tfrac{1}{\bar{z}})u_2 \big\rangle$$

  for some involution $v \mapsto \tilde{v}$.

- This encodes covariance of the 3-point functions with respect to orientation reversing conformal transformations.

- Examples include VOAs from affine Lie algebras at positive integer level, Virasoro VOAs with $c \in \{1 - \frac{6}{m(m+1)}\} \cup [1, \infty)$, the Heisenberg VOA, lattice VOAs, the Moonshine VOA, etc.

Unitary modules

- V-module: $\quad v \in V \quad \mapsto \quad Y^M(v,z) \in \mathrm{End}(M)[[z^{\pm 1}]]$

- A unitary V-module is one with an invariant inner product:

$$\langle Y^M(v,z)a_1, a_2\rangle_M = \langle a_1, Y^M(\tilde{v}, \tfrac{1}{\bar{z}})a_2\rangle_M$$

  for the same involution $v \mapsto \tilde{v}$.

- Equivalently, $\langle \cdot\,, \cdot \rangle_M$ induces an isomorphism $M' \cong M^\dagger$, where M' is the contragredient (dual) module, and $M^\dagger$ is the complex conjugate

- For unitary VOAs from affine Lie algebras and the Virasoro algebra, this reduces to the usual unitarity condition. E.g.

$$\langle L_n a_1, a_2\rangle = \langle a_1, L_{-n} a_2\rangle$$

Unitary modules?

It is widely believed that:

Conjecture

If V is a rational unitary VOA, then every V-module admits a unitary structure.

Even if it is easy to find an invariant Hermitian form, it is usually hard to prove that the form is positive directly.

E.g. for unitary minimal models Vir_c with $c < 1$, this conjecture was proven by finding all irreducible modules inside affine Lie algebras (the GKO coset construction).

The conjecture generally fails badly for non-rational VOAs (Heisenberg, Vir_c with $c \geq 1$, etc.)

Intertwining operators

Intertwining operators $\mathcal{Y} \in \binom{K}{M\,N}$:

$$a \in M \quad \longmapsto \quad \mathcal{Y}(a, z) \;\in\; \mathrm{Hom}(N, K)\{z\}$$

$\binom{K}{M\,N}$ wants to be $\mathrm{Hom}(M \boxtimes N, K)$ for an as-yet-undefined tensor product of modules $M \boxtimes N$.

Tensor products

More precisely: fix a category $\mathcal{C}$ of some flavor of V-modules.

The tensor product $M \boxtimes N$, if it exists, is the object in $\mathcal{C}$ representing the functor $\mathcal{C} \to \mathrm{Vec}$ given by:

$$K \mapsto \binom{K}{M\,N}.$$

That is, we must have a distinguished

$$\mathcal{Y}_{\boxtimes} \in \binom{M \boxtimes N}{M \quad N} \cong \mathrm{End}(M \boxtimes N)$$

such that for every $K \in \mathcal{C}$, any $\mathcal{Y} \in \binom{K}{M\,N}$ factors uniquely as

$$\mathcal{Y} = f \circ \mathcal{Y}_{\boxtimes}$$

through a homomorphism $f : M \boxtimes N \to K$.

Problem: For a given VOA, find a category of modules $\mathcal{C}$ such that $M \boxtimes N$ always exists, and makes $\mathcal{C}$ into a tensor category.

You may also need to restrict to $\binom{K}{M\,N}_{nice} \subset \binom{K}{M\,N}$

Theorem (Huang-Lepowsky)

If V is 'strongly rational,' the category of (strong) V-modules is a modular tensor category.

The proof of associativity requires a multi-step construction of $M \boxtimes N$, taking the contragredient module of a certain subspace of $(M \otimes N)^*$.

Unitarity of tensor products

It is widely believed:

Conjecture

If M and N are unitary V-modules, then $M \boxtimes N$ has a natural unitary structure.

This is an essential ingredient in obtaining a unitary tensor category of unitary modules.

The challenge is that positivity is hard to prove after the fact (think coset construction).

Warning: The obvious inner product on $(M \otimes N)^*$ arising in the Huang-Lepowsky construction is not invariant.

Regardless, there should be a unitary tensor category whose modules look like direct integrals of simples.

Unitary constructions; working backwards

- Consider a unitary $M \boxtimes N$, with its $\mathcal{Y}_{\boxtimes} \in \binom{M\boxtimes N}{M\ \ N}$.

  When $|z| < 1$, we expectrequire $\mathcal{Y}_{\boxtimes}(a, z)b \in \mathcal{H}_{M\boxtimes N}$.

- If W is an inner product space, there is an equivalence

  $\{$ maps $T : W \to \mathcal{H}$ $\}$ $\quad\longleftrightarrow\quad$ $\{$ semidefinite inner products on W $\}$.

$\rightarrow$ Starting with a map T, you have an inner product:

$$\langle a, b \rangle_T := \langle T^* T a, b \rangle_W$$

$\leftarrow$ Starting with an inner product $\langle \cdot\, , \cdot \rangle_{new}$, you have:

$$\mathcal{H} = \overline{W}^{\langle\ ,\ \rangle_{new}}, \qquad T = \text{'identity'}$$

- So we get $\langle \cdot\, , \cdot \rangle_{\boxtimes, z}$ on $M \otimes N$ from $a \otimes b \mapsto \mathcal{Y}_{\boxtimes}(a, z)b$.

Positivity conjectures

Conjecture (Positivity conjecture)

The form on $M \otimes N$ given by

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{\boxtimes, z} := \left\langle Y^N\left(\mathcal{Y}(\tilde{a}_2, \bar{z}^{-1} - z)a_1,\, z\right)b_1, b_2 \right\rangle_N$$

is positive semidefinite.

- where M and N are unitary V-modules, $0 < |z| < 1$,
- $\mathcal{Y} \in \binom{V}{M^\dagger\ M}$, where $M^\dagger$ is the complex conjugate module,
- and $a \mapsto \tilde{a}$ is a certain explicit involution.

Conjecture (Strong positivity conjecture)

There is a canonical unitary V-module structure on a dense subspace of $\overline{M \otimes N}^{\langle,\rangle_\boxtimes}$ and an intertwining operator $\mathcal{Y}_\boxtimes$ such that $\mathcal{Y}_\boxtimes(a, z)b$ agrees with the 'identity' $M \otimes N \to \overline{M \otimes N}^{\langle,\rangle_\boxtimes}$.

For the appropriate category of modules/choice of intertwiners, this should be a tensor product.

Example: vacuum sector

If $M = N = V$:

$$\begin{aligned}\langle a \otimes b, a \otimes b \rangle_{\boxtimes,z} &= \langle Y(Y(\tilde{a}, \overline{z}^{-1} - z)a, z)b, b\rangle_V \\ &= \langle Y(\tilde{a}, \overline{z}^{-1})Y(a, z)b, b\rangle_V \\ &= \langle Y(a, z)b, Y(a, z)b\rangle_V \\ &= \|Y(a, z)b\|^2\end{aligned}$$

We have $\overline{V}^{\langle,\rangle_{\boxtimes,z}} \cong \mathcal{H}_V$, and the map corresponding to the 'identity' is $a \otimes b \mapsto Y(a, z)b$.

Rational evidence

Recent work of Bin Gui shows that:

- the weak conjecture implies the strong conjecture for rational VOAs
- if positivity holds, Mod(V) is naturally a unitary modular tensor category
- the positivity conjecture holds for certain WZW models of type A and D

Irrational evidence?

- This construction should produce unitary tensor categories outside of the rational setting.
- The strongest evidence comes from conformal nets, a different framework for studying 2d chiral CFTs.
- For conformal nets, unitary tensor categories have been constructed.
- $\langle \cdot , \cdot \rangle_{\boxtimes, z}$ is a translation of the inner product used for conformal nets, and work in progress makes this rigorous.

Part 2: From VOAs to local observables

Conformal nets

- A conformal net consists of a Hilbert space $\mathcal{H}_0$, along with a family of von Neumann algebras $\mathcal{A}(I) \subset B(\mathcal{H}_0)$ indexed by intervals $I \subset S^1$.

  'von Neumann algebra' means that it is closed under adjoints and pointwise limits

- Several axioms, including:

  $I \subset J \implies \mathcal{A}(I) \subset \mathcal{A}(J)$

  if I and J are disjoint, then $\mathcal{A}(I)$ and $\mathcal{A}(J)$ commute

- A representation of a conformal net is a Hilbert space $\mathcal{H}_\pi$ along with compatible representations $\pi_I : \mathcal{A}(I) \to B(\mathcal{H}_\pi)$.

Bridging the gap

- Conformal nets and unitary VOAs are supposed to encode the same physical ideas

- Theorems relating the mathematical structures have been hard to come by, and difficulty of theorems can be quite different in different settings (e.g. rigidity of $\otimes$-cat's)

Three-part project to relate these structures:

- Part 1: VOAs and conformal nets
- Part 2: Modules and representations
- Part 3: Tensor products

Goal is to gain new insight into VOAs and conformal nets

Tensor product of representations

Wassermann's approach:

The tensor product $\pi \boxtimes \lambda$ of reps π and λ is defined by:

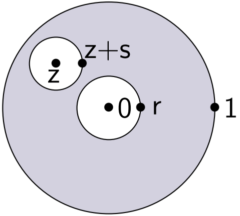
- consider certain dense subspaces $X \subset \mathcal{H}_\pi$ and $Y \subset \mathcal{H}_\lambda$
- complete $X \otimes Y$ with respect to an inner product:

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_2^* y_1 x_2^* x_1 \Omega, \Omega \rangle_{\mathcal{H}_0}.$$

Goal: build a dictionary between VOAs and conformal nets, identifying this inner product with $\langle \cdot , \cdot \rangle_\boxtimes$.

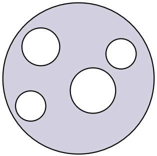
Motivation: Geometric VOAs

The geometric description of VOAs is given by identifying:



$$\longleftrightarrow \quad \left(v \otimes u \mapsto Y(s^{L_0} v, z) r^{L_0} u \right)$$

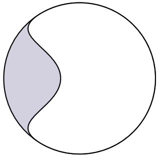
Combined with scale invariance and $\Omega =$ , this uniquely assigns a map to any:



in a way that is compatible with gluing/composition.

Motivation: Thin annuli

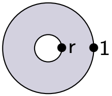
Given an inward pointing holomorphic vector field ρ on the disk and positive number t, we obtain an annulus by flowing along ρ for time t:



We associate to this annulus the operator $e^{tT(\rho)}$, where

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \qquad T(\rho) = \frac{1}{2\pi i} \oint T(z)\rho(z)dz.$$

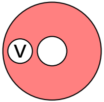
The special case $\rho = -z$ corresponds to



$$\longleftrightarrow \qquad r^{L_0}$$

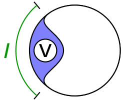
Motivation: Insertion operators

Just like we had



$$\longleftrightarrow \quad Y(s^{L_0}v, z)r^{L_0}$$

we have



$$\longleftrightarrow \quad Y(s^{L_0}v, z)e^{tT(\rho)}.$$

- We say $\text{supp}(\rho, t) \subset I$ if the complement of I is thin.
- Let $\text{int}(\rho, t)$ be the shaded interior of the 'annulus.'

Bounded localized vertex operators

Given a VOA V, we try to construct a conformal net $\mathcal{A}_V$ on $\mathcal{H}_V$:

$$\mathcal{A}_V(I) = \mathrm{vNA}\Big(\{Y(a,z)e^{tT(\rho)}e^{-itT(\rho_\perp)} \ : \ \mathrm{supp}(\rho,t) \subset I,\ z \in \mathrm{int}(\rho,t),\ a \in V\}\Big)$$

We say V has **bounded localized** vertex operators if:

1) The generators are bounded
2) $\mathcal{A}_V(I)$ and $\mathcal{A}_V(J)$ commute when I and J are disjoint

(So you get a conformal net)

Existence of BLVO

Theorem ('16, '18)

The class of VOAs with bounded localized vertex operators...

1) ...is closed under taking tensor products and unitary subalgebras

2) ...includes WZW models and the free fermion

The proof of (2) goes via delicate calculations for the free fermion Segal CFT.

Conjecture

Every unitary VOA has bounded localized vertex operators.

BLVO modules

If M is a V-module, the representation π^M of $\mathcal{A}_V$ is given by

$$\pi_I^M(Y(a,z)e^{tT(\rho)}) = Y^M(a,z)e^{tT(\rho)}$$

if such a representation exists.

Theorem ('18)

If V has bounded localized vertex operators, $W \subset V$ is a unitary subalgebra, and M is a W-submodule of V, then π^M exists.

Conjecture

There is an equivalence: V-modules $\longleftrightarrow$ reps. of $\mathcal{A}_V$

Verified, for example, for WZW models with $\mathfrak{g}$ simply laced, W-algebras, and some more.

Tensor products

For V a unitary VOA, M and N unitary modules, we must guess:

Conjecture

$\pi^M \boxtimes \pi^N \cong \pi^{M \boxtimes N}$

Theorem ('19?)

If V has bounded localized vertex operators, W a unitary subalgebra, M and N W-submodules of V, then

- *the positivity conjecture holds for M and N:*

$$\left\langle Y^N(\mathcal{Y}(\tilde{a}_2, \overline{z}^{-1} - z)a_1, z)b_1, b_2 \right\rangle_N \geq 0$$

- *there is a natural unitary $\pi^M \boxtimes \pi^N \cong \overline{M \otimes N}^{\langle \, , \, \rangle_\boxtimes}$*

Remember that when W is rational, the right-hand side is the Hilbert space of $M \boxtimes N$.

Applications for conformal nets

- The original goal of the project was to compute fusion rules for conformal nets via VOAs, which is done via $\pi^M \boxtimes \pi^N \cong \mathcal{H}_{M \boxtimes N}$.

- Fusion rules for conformal nets are much harder to compute, but also more powerful; e.g. rigidity of representation category follows from fusion rules (via subfactor theory).

- Ideal outcome is an equivalence $\mathrm{Mod}(V) \cong \mathrm{Rep}(\mathcal{A}_V)$.

Thank you!