Workshop on Multiparameter Persistent Homology
Casa Matemática Oaxaca, August 2018

Decomposition of exact 2-d persistence modules

Steve Oudot

Inria

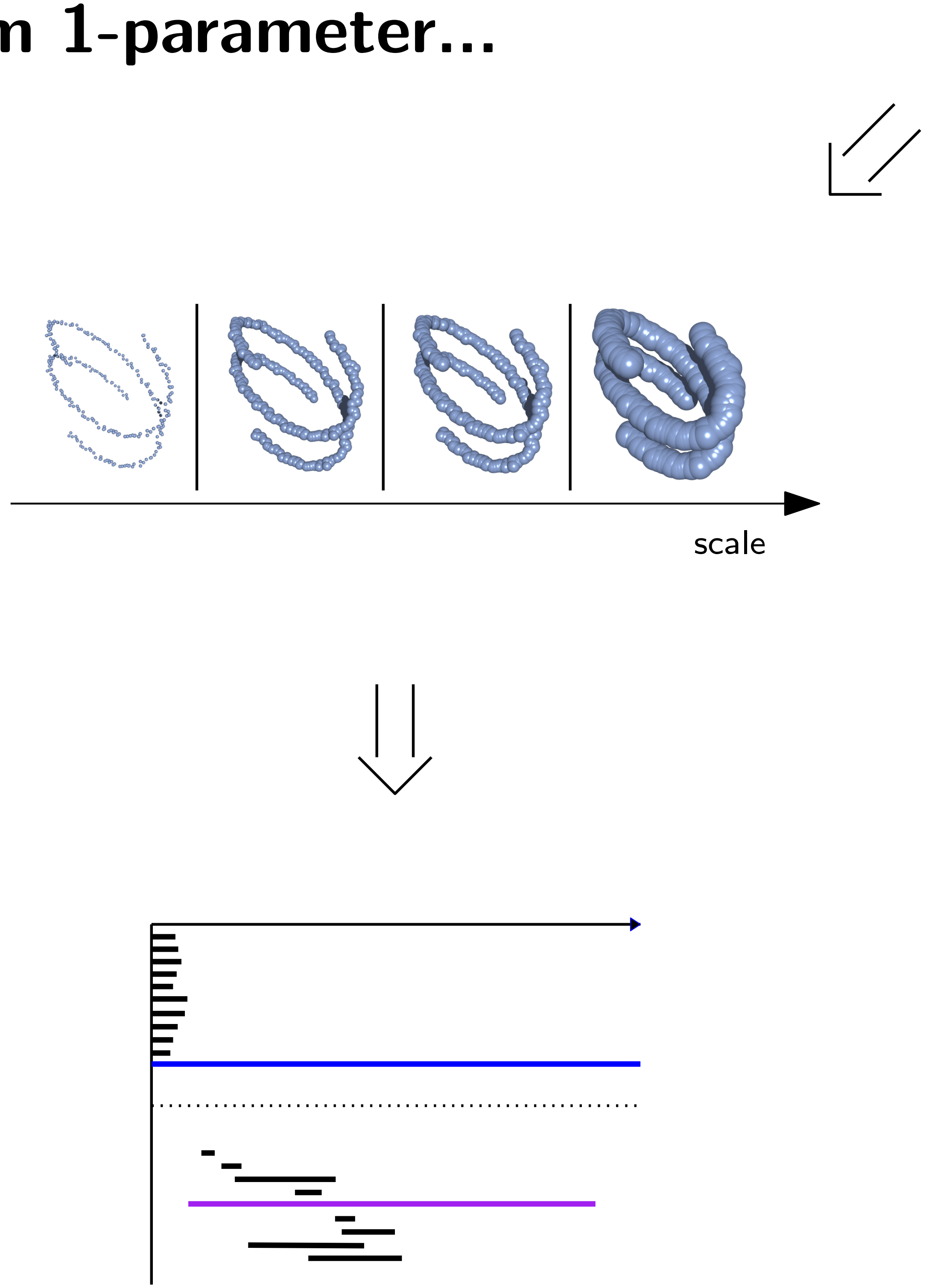
— joint work with J. Cochoy

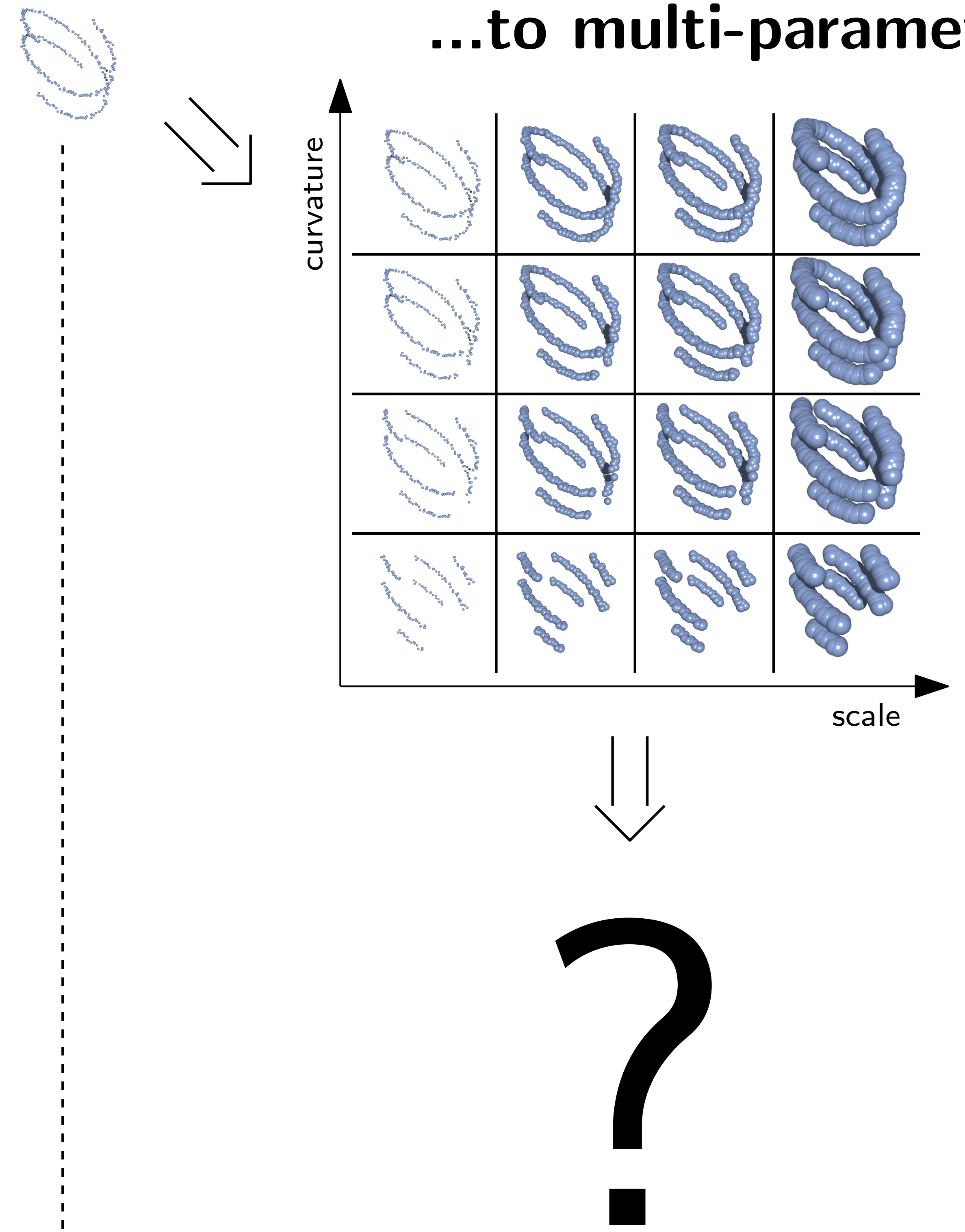
— arXiv 1605.09726 (math.RT)

Context: richer descriptors for data

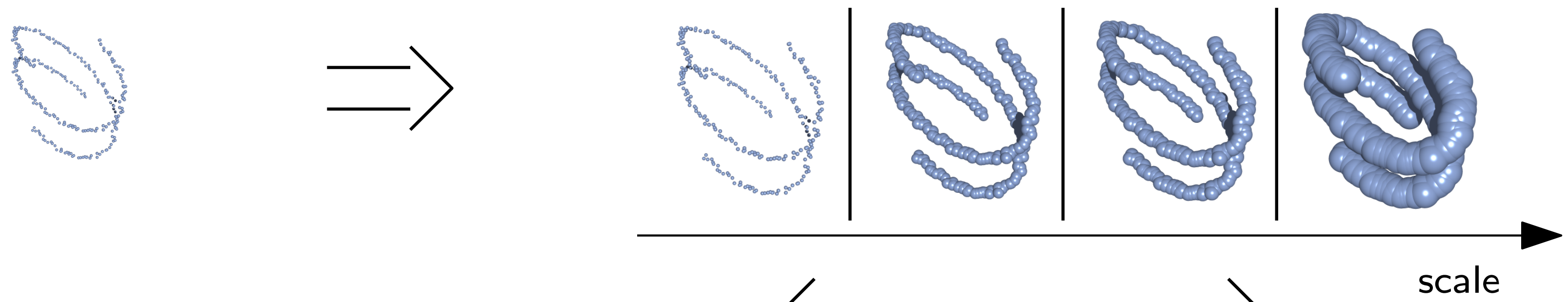
from 1-parameter...

...to multi-parameter





Barcodes from decompositions (1-d)



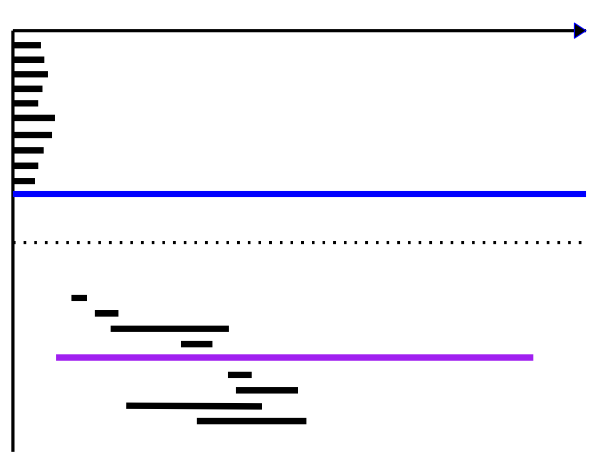
discrete setting: $M : \mathbb{Z} \to \mathrm{vect}_{\boldsymbol{k}}$

$\to$ fg graded module over $\boldsymbol{k}[t]$

$$M \simeq \bigoplus_{i\in I} t^{b_i}\,\boldsymbol{k}[t] \quad \oplus \quad \bigoplus_{j\in J} t^{b_j}\,\frac{\boldsymbol{k}[t]}{t^{s_j}\,\boldsymbol{k}[t]}$$

$\Downarrow$

dgm M:



continuous setting: $M : \mathbb{R} \to \mathrm{vect}_{\boldsymbol{k}}$

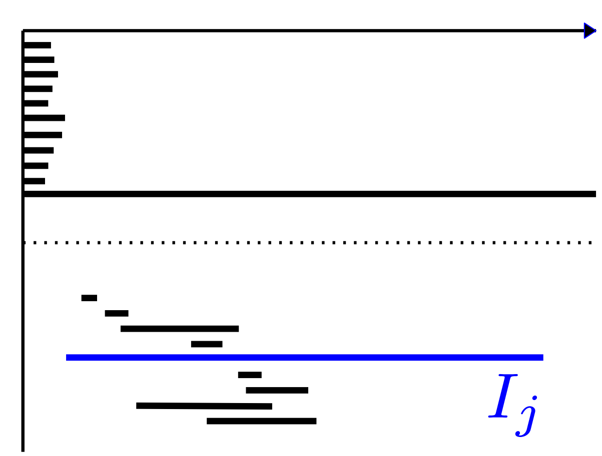
$\to$ pfd representation of poset $(\mathbb{R}, \leq)$

$$M \simeq \bigoplus_{j\in J} \boldsymbol{k}_{I_j}$$ [Crawley-Boevey]

indicator module on interval I_j

$\Downarrow$

dgm M:



Existence of decompositions (multi-d)

discrete setting: $M : \mathbb{Z}^d \to \mathsf{vect}_{\boldsymbol{k}}$

$$M \simeq \bigoplus_{j \in J} M_j \qquad \text{(indecomposables)}$$

- bounded support: by recurrence
- unbounded support: [Ringel]

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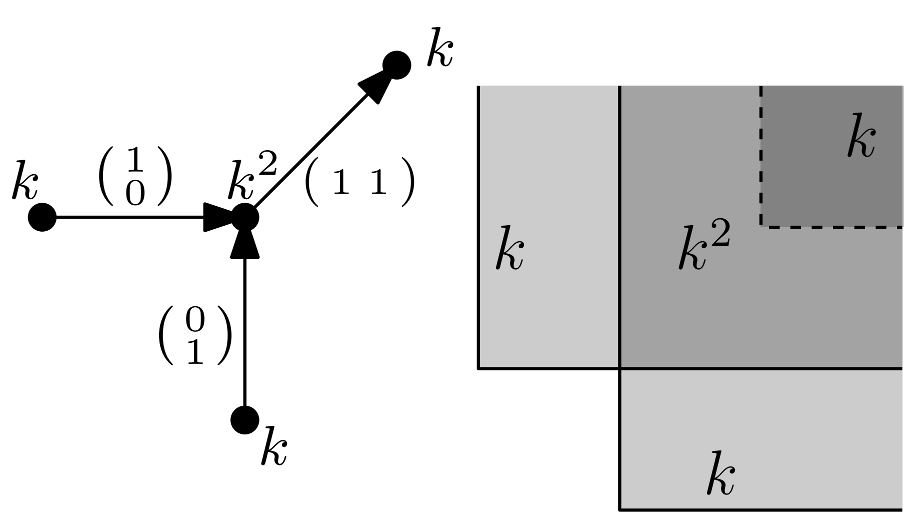
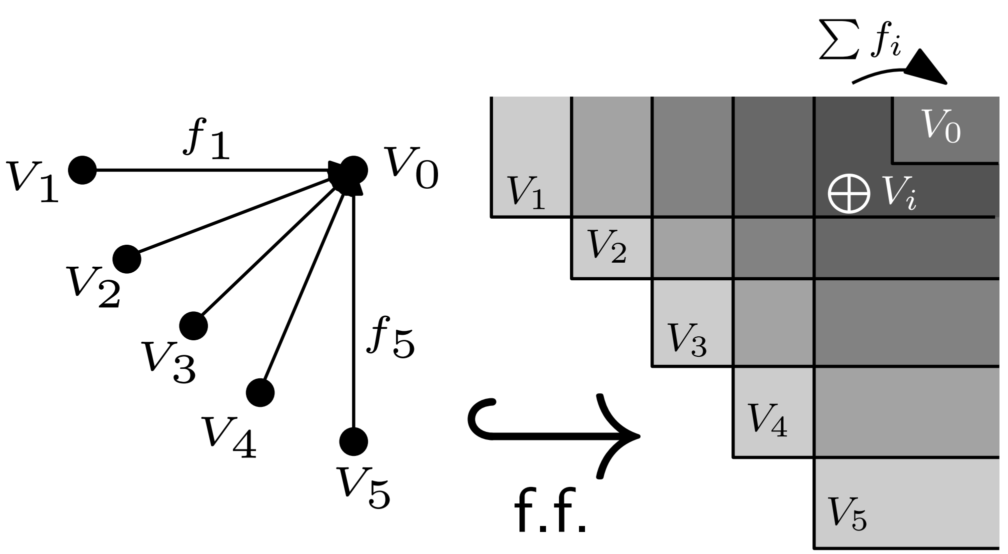
Q: shape of indecomposables?



non-thin summands



wild-type

Existence of decompositions (multi-d)

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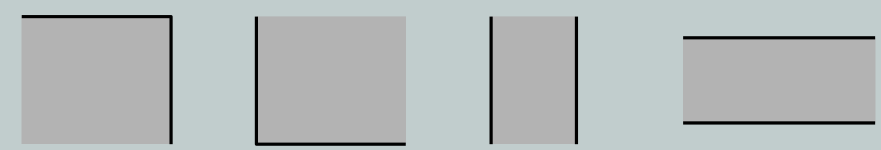
continuous setting: $M : \mathbb{R}^d \to \mathrm{vect}_{\boldsymbol{k}}$

Thm: [Cochoy, O.]

$M : \mathbb{R}^2 \to \mathrm{vect}_{\boldsymbol{k}}$ **exact**

$$\Longleftrightarrow$$

$$M \simeq \bigoplus_{j \in J} \boldsymbol{k}_{B_j}$$

B_j: block 

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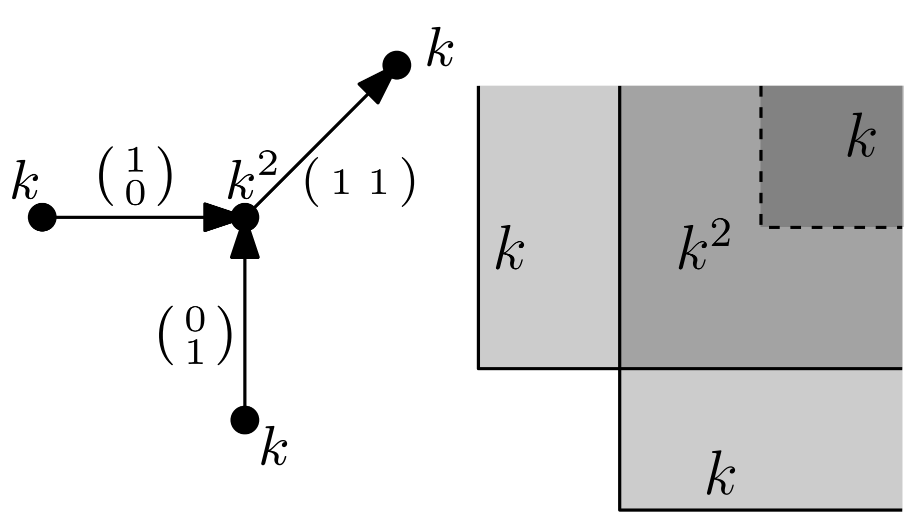

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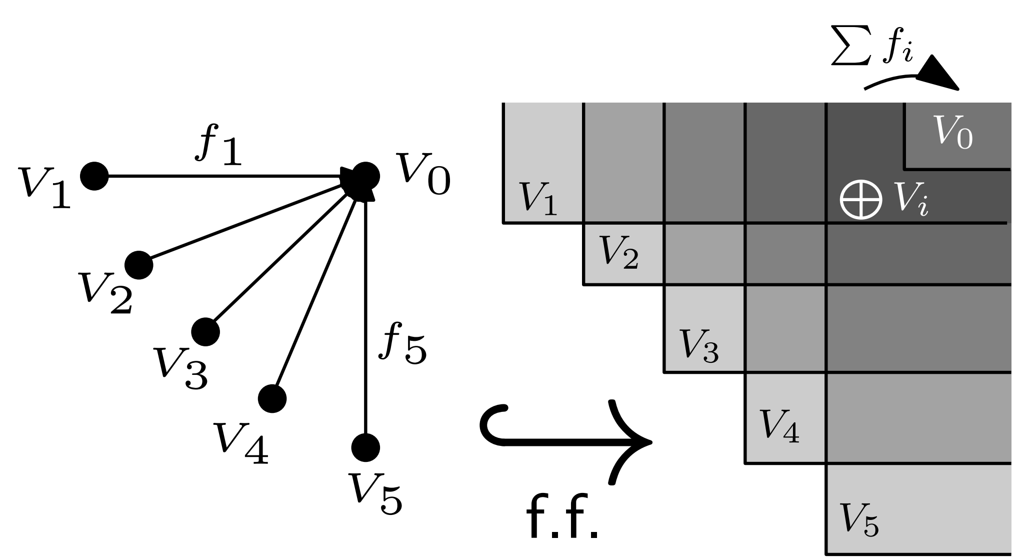
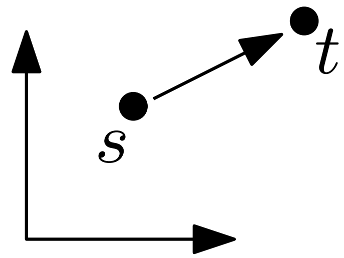


wild-type

Existence of decompositions (multi-d)

Exactness:



$$\begin{tikzcd}
M(s_x, t_y) \arrow[rr, "\delta"] & & M(t) \\
M(s) \arrow[u, "\beta"] \arrow[rr, "\alpha"] & & M(t_x, s_y) \arrow[u, "\gamma"]
\end{tikzcd}$$

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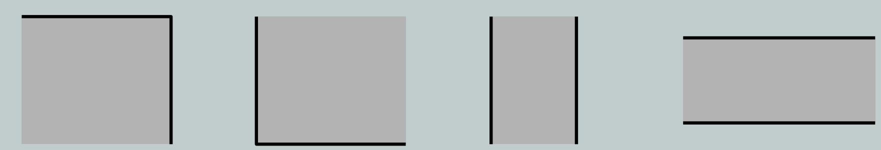
B_j: block 

$$M(s) \xrightarrow{\phi = (\alpha, \beta)} M(t_x, s_y) \oplus M(s_x, t_y) \xrightarrow{\psi = \gamma - \delta} M(t)$$

$\mathbf{Im}\ \boldsymbol{\phi} = \mathbf{Ker}\, \boldsymbol{\psi}$

$\mathbf{Im}\ \boldsymbol{\phi} \subseteq \mathbf{Ker}\, \boldsymbol{\psi}$: commutativity

$\mathbf{Im}\ \boldsymbol{\phi} \supseteq \mathbf{Ker}\, \boldsymbol{\psi}$: $\exists$ preimages in $M(t_x, s_y)$ and $M(s_x, t_y)$ $\Rightarrow$ $\exists$ common preimage in $M(s)$

Existence of decompositions (multi-d)

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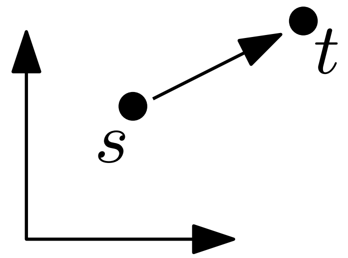

$$
\begin{tikzcd}
M(s_x, t_y) \arrow[rr, "\delta"] & & M(t) \\
M(s) \arrow[u, "\beta"] \arrow[rru, dotted, "\rho"] \arrow[rr, "\alpha"] & & M(t_x, s_y) \arrow[u, "\gamma"']
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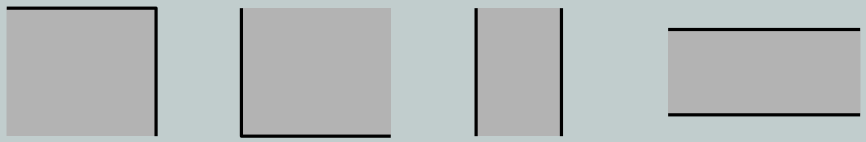

$$M(s) \xrightarrow{\phi = (\alpha, \beta)} M(t_x, s_y) \oplus M(s_x, t_y) \xrightarrow{\psi = \gamma - \delta} M(t) \qquad \mathbf{Im}\ \boldsymbol{\phi} = \mathbf{Ker}\, \boldsymbol{\psi}$$

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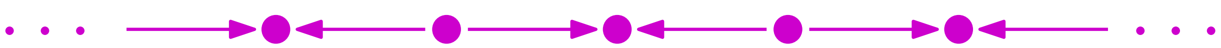
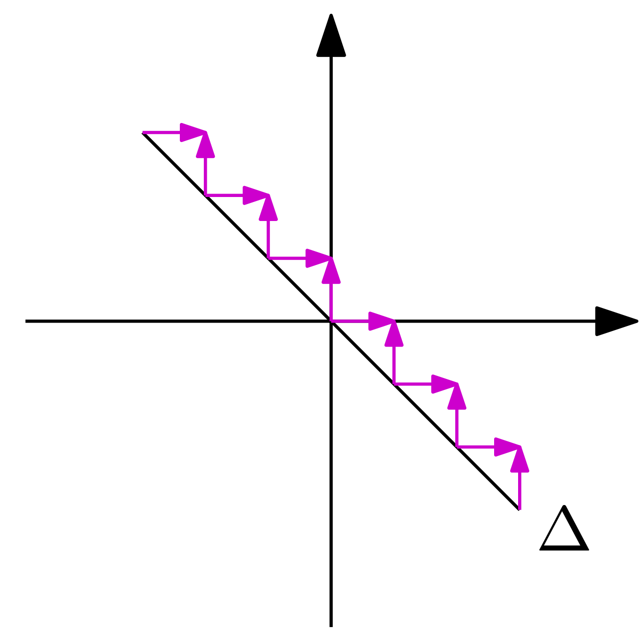
$$\Rightarrow \begin{cases} \mathrm{Im}\ \rho = \mathrm{Im}\ \gamma \cap \mathrm{Im}\ \delta \\ \mathrm{Ker}\, \rho = \mathrm{Ker}\, \alpha + \mathrm{Ker}\, \beta \end{cases}$$

Consequences

Stability of pfd zigzag modules:

$F, G : \mathbb{ZZ} \to$ vect$_{\boldsymbol{k}}$ $\rightsquigarrow$ (Kan ext.) $M, N : \mathbb{R}^2 \to$ vect$_{\boldsymbol{k}}$ **B-dec**





Thm: [Botnan, Lesnick] [Bjerkevik]

$$\mathrm{d_b}(F, G) := \mathrm{d_b}(\mathrm{dgm}\, M, \mathrm{dgm}\, N) = \mathrm{d}_i(M, N)$$

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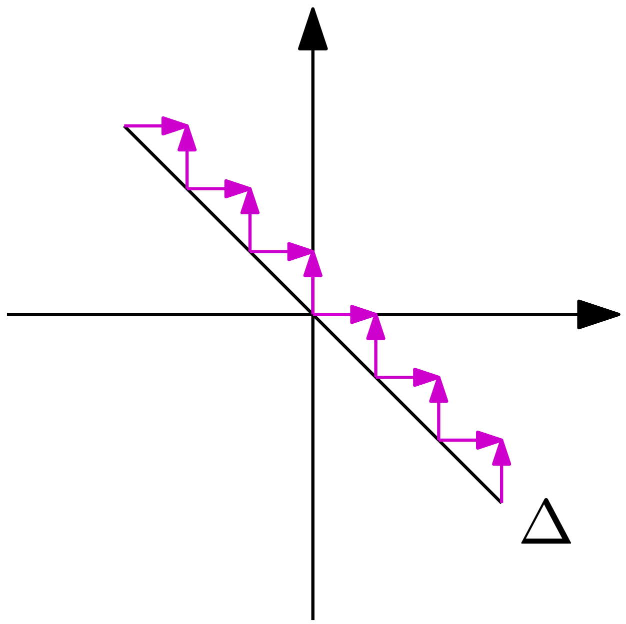
$$\cdots \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \cdots$$



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Application to interlevel-sets persistence:

$f, g : X \to \mathbb{R}$ **Morse** $\overset{}{\underset{\mathrm{H}_0(f^{-1}(\cdot);\, \boldsymbol{k})}{\rightsquigarrow}}$ $F, G : \mathrm{Int} \to \mathrm{vect}_{\boldsymbol{k}}$ $\underset{(a,b) \mapsto (-a,b)}{\rightsquigarrow}$ $M, N : \mathbb{R}^2_{>\Delta} \to \mathrm{vect}_{\boldsymbol{k}}$ **B-dec**

$\rightsquigarrow$ (right Kan ext.)

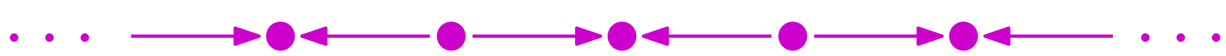
$M, N : \mathbb{R}^2 \to \mathrm{vect}_{\boldsymbol{k}}$ **B-dec**

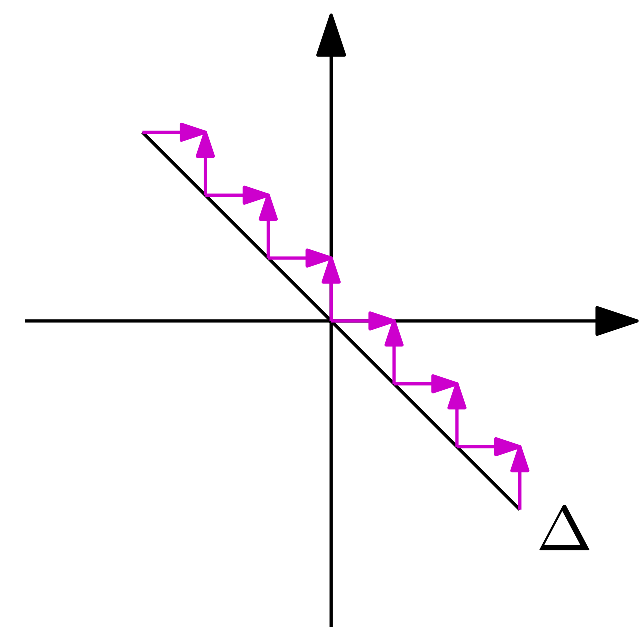
$\boxed{\textbf{thm} \Rightarrow \mathrm{d_b}(\mathrm{dgm}\, M, \mathrm{dgm}\, N) = \mathrm{d}_i(M, N) \leq \|f - g\|_\infty}$

Consequences

Stability of pfd zigzag modules:

$F, G : \mathbb{ZZ} \to \mathrm{vect}_{\boldsymbol{k}}$ $\underset{\text{(Kan ext.)}}{\rightsquigarrow}$ $M, N : \mathbb{R}^2 \to \mathrm{vect}_{\boldsymbol{k}}$ **B-dec**





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Application to interlevel-sets persistence:

$f, g : X \to \mathbb{R}$ **pfd** $\underset{\mathrm{H}_r(f^{-1}(\cdot);\, \boldsymbol{k})}{\rightsquigarrow}$ $F, G : \mathrm{Int} \to \mathrm{vect}_{\boldsymbol{k}}$ $\underset{(a,\, b) \mapsto (-a,\, b)}{\rightsquigarrow}$ $M, N : \mathbb{R}^2_{>\Delta} \to \mathrm{vect}_{\boldsymbol{k}}$ **exact**

$\big\downarrow$ (right Kan ext.)

$M, N : \mathbb{R}^2 \to \mathrm{vect}_{\boldsymbol{k}}$ **exact**

thm + our result $\Rightarrow \mathrm{d_b}(\mathrm{dgm}\, M, \mathrm{dgm}\, N) = \mathrm{d}_i(M, N) \leq \|f - g\|_\infty$

see also [Carlsson, de Silva, Kališnik, Morozov]

Proof of the theorem (1-d case) [Crawley-Boevey]

Overview:

1. Define a *counting functor* for each interval I:

$$C_I : \left| \begin{array}{l} \mathsf{vect}_{\boldsymbol{k}}^{\mathbb{R}} \to \mathsf{vect}_{\boldsymbol{k}} \\ M \mapsto \boldsymbol{k}^{\mathrm{mult}(\boldsymbol{k}_I; M)} \end{array} \right. \quad (\mathrm{mult}(\boldsymbol{k}_I; M) := \max\{n \mid M \simeq \boldsymbol{k}_I^n \oplus N\})$$

2. Define an *embedding operator* (non-functorial) for each interval I:

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3. Show that $M = \bigoplus_I M_I$

   - show that the M_I's are in direct sum
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Counting functor (1-d case)

For $I = (a, b)$:

- $\operatorname{Im}_I^+(t) := \bigcap_{a<s\leq t} \operatorname{Im} M(s \to t)$ (elements alive at least since a and still at t)
- $\operatorname{Im}_I^-(t) := \sum_{s\leq a} \operatorname{Im} M(s \to t)$ (elements born before a and still alive at t)

$\operatorname{Im}_I^+(t)/\operatorname{Im}_I^-(t)$ (elements alive at t that were born at a)

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- $\mathrm{Ker}_I^+(t) := \bigcap_{s \geq b} \mathrm{Ker}\, M(t \to s)$ (elements alive at t but not after b)

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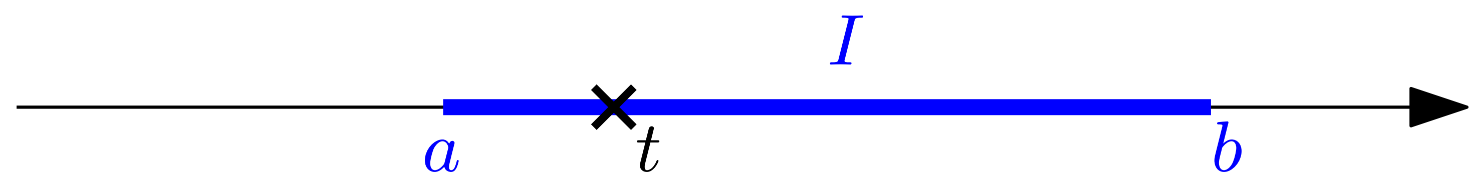

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- $\mathrm{Im}\,{}^{-}_{I}(t) := \sum_{t\leq s<b} \mathrm{Ker}\, M(s \to t)$ (elements alive at t and dead before b)

$$C_I(t) := \underbrace{\left(\mathrm{Im}\,{}^{+}_{I}(t) \cap \mathrm{Ker}\,{}^{+}_{I}(t)\right)}_{\text{(alive at least since } a \text{ but not after } b\text{)}} \Big/ \underbrace{\left(\left(\mathrm{Im}\,{}^{+}_{I}(t) \cap \mathrm{Ker}\,{}^{-}_{I}(t)\right) + \left(\mathrm{Im}\,{}^{-}_{I}(t) \cap \mathrm{Ker}\,{}^{+}_{I}(t)\right)\right)}_{\text{(alive since } a \text{ but dead before } b\text{)} + \text{(alive until } b \text{ but born before } a\text{)}}$$

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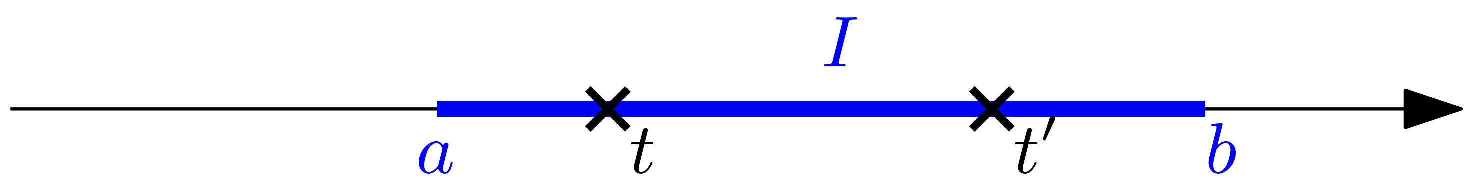

Prop: For $t \leq t' \in (a, b)$, $M(t \longrightarrow t')$ induces $C_I(t) \xrightarrow{\simeq} C_I(t')$

$$C_I(M) := \varprojlim_{t \in I} C_I(t)$$

functorial construction

Prop: $\dim C_I(M) = \mathrm{mult}(\boldsymbol{k}_I; M)$

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Proof of the theorem (1-d case) [Crawley-Boevey]

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Embedding of summands (1-d case)

$$C_I(t) := \underbrace{\left(\mathrm{Im}_I^+(t) \cap \mathrm{Ker}_I^+(t)\right)}_{C_I^+(t)} / \underbrace{\left(\left(\mathrm{Im}_I^+(t) \cap \mathrm{Ker}_I^-(t)\right) + \left(\mathrm{Im}_I^-(t) \cap \mathrm{Ker}_I^+(t)\right)\right)}_{C_I^-(t)}$$

$$C_I(M) := \varprojlim_{t \in I} C_I(t) \qquad\qquad C_I^\pm(M) := \varprojlim_{t \in I} C_I^\pm(t)$$

$0 \longrightarrow C_I^-(t) \longrightarrow C_I^+(t) \longrightarrow C_I(t) \longrightarrow 0$ is exact for all $t \in I$

$\Downarrow$ (Mittag-Leffler)

$0 \longrightarrow C_I^-(M) \longrightarrow C_I^+(M) \longrightarrow C_I(M) \longrightarrow 0$ is exact

$W :=$ vector space complement of $C_I^-(M)$ in $C_I^+(M)$ $\rightsquigarrow$ $W \simeq C_I(M)$

$M_I(t) := \pi_t(M_I)$ where the π_t are the (injective) cone maps for $C_I^+(M)$

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Direct sum (1-d case)

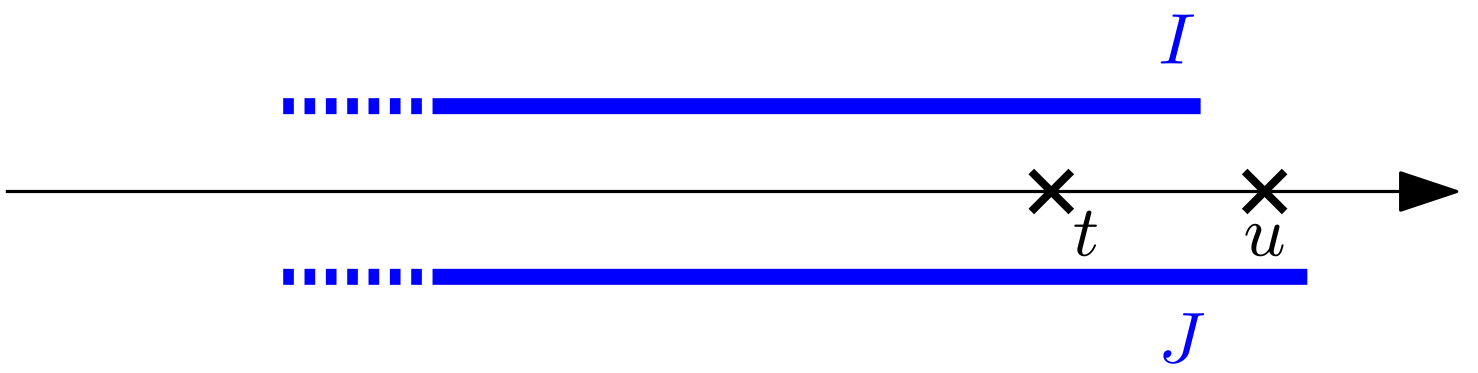
Base case: M_I vs. M_J with $\sup I \neq \sup J$

$<$

$$M_I(t) \cap M_J(t) \neq 0$$

$$\Rightarrow M_I(u) \cap M_J(u) \neq 0$$

$\Rightarrow M_I(u) \neq 0$ (contradiction)



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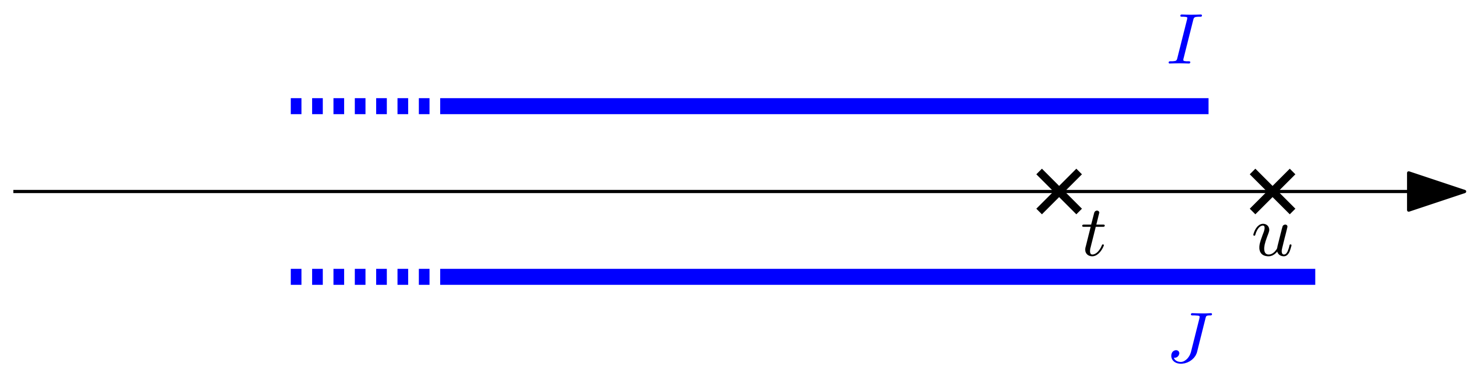


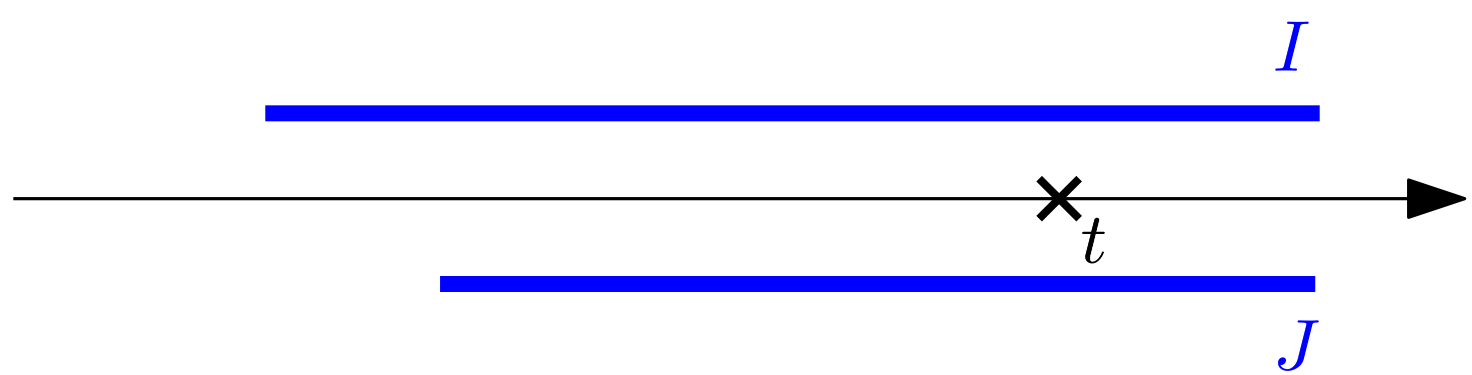
$$\Rightarrow M_I(u) \cap M_J(u) \neq 0$$

$\Rightarrow M_I(u) \neq 0$ (contradiction)

Variant case: M_I vs. M_J with $\sup I = \sup J$ and $\inf I \neq \inf J$
$<$

$$\mathrm{Ker}\,^{\pm}_{I}(t) = \mathrm{Ker}\,^{\pm}_{J}(t)$$

$$\mathrm{Im}\,^{+}_{I}(t) \subseteq \mathrm{Im}\,^{-}_{J}(t)$$



$$C^{+}_{I}(t) \subseteq C^{-}_{J}(t) \Rightarrow M_I(t) \cap M_J(t) = 0$$

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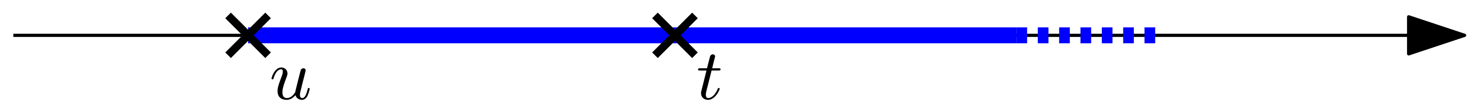
   - show that the sum of the M_I's covers M

Covering M (1-d case)

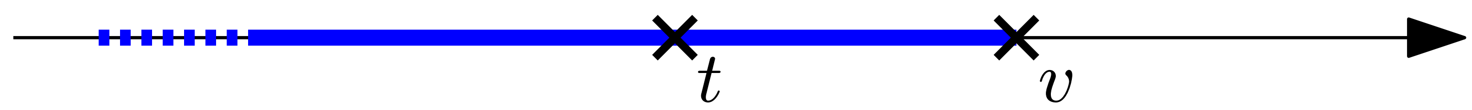
Approach: show that $\sum_I M_I(t) = M(t)$ for every $t \in \mathbb{R}$

Suppose $X := \sum_I M_I(t) \subsetneq M(t)$:

$$u := \inf\{s \leq t \mid X \subsetneq \operatorname{Im} M(s \to t)\}$$



$$v := \sup\{s \geq t \mid \operatorname{Ker} M(t \to s) \subsetneq X\}$$



Then:

$$\left.\begin{array}{l} \operatorname{Im}^-_{(u,v)}(t) \subseteq X \not\supseteq \operatorname{Im}^+_{(u,v)}(t) \\ \operatorname{Ker}^-_{(u,v)}(t) \subseteq X \not\supseteq \operatorname{Ker}^+_{(u,v)}(t) \end{array}\right] \Rightarrow C^-_{(u,v)}(t) \subseteq X \not\supseteq C^+_{(u,v)}(t)$$

$$\Downarrow$$

$$M_{(u,v)}(t) \not\subseteq X := \sum_I M_I(t)$$

(contradiction)

Proof of the theorem (1-d case) [Crawley-Boevey]

Overview:

1. Define a *counting functor* for each interval I:

$$C_I : \left| \begin{array}{l} \mathsf{vect}_{\boldsymbol{k}}^{\mathbb{R}} \to \mathsf{vect}_{\boldsymbol{k}} \\ M \mapsto \boldsymbol{k}^{\mathrm{mult}(\boldsymbol{k}_I; M)} \end{array} \right. \qquad (\mathrm{mult}(\boldsymbol{k}_I; M) := \max\{n \mid M \simeq \boldsymbol{k}_I^n \oplus N\})$$

2. Define an *embedding operator* (non-functorial) for each interval I:

$$M \mapsto M_I \leq M \text{ such that } M_I \simeq \boldsymbol{k}_I^{\mathrm{mult}(\boldsymbol{k}_I; M)}$$

3. Show that $M = \bigoplus_I M_I$

   - show that the M_I's are in direct sum
   - show that the sum of the M_I's covers M

Proof of the theorem (exact 2-d case) [Cochoy, O.]

Overview:

1. Define a *counting functor* for each block B:

$$C_B : \left| \begin{array}{l} \text{Exact } \mathrm{vect}_{\boldsymbol{k}}^{\mathbb{R}^2} \to \mathrm{vect}_{\boldsymbol{k}} \\ M \mapsto \boldsymbol{k}^{\mathrm{mult}(\boldsymbol{k}_B; M)} \qquad (\mathrm{mult}(\boldsymbol{k}_B; M) := \max\{n \mid M \simeq \boldsymbol{k}_B^n \oplus N\}) \end{array} \right.$$

2. Define an *embedding operator* (non-functorial) for each block B:

$$M \mapsto M_B \leq M \text{ such that } M_B \simeq \boldsymbol{k}_B^{\mathrm{mult}(\boldsymbol{k}_B; M)}$$

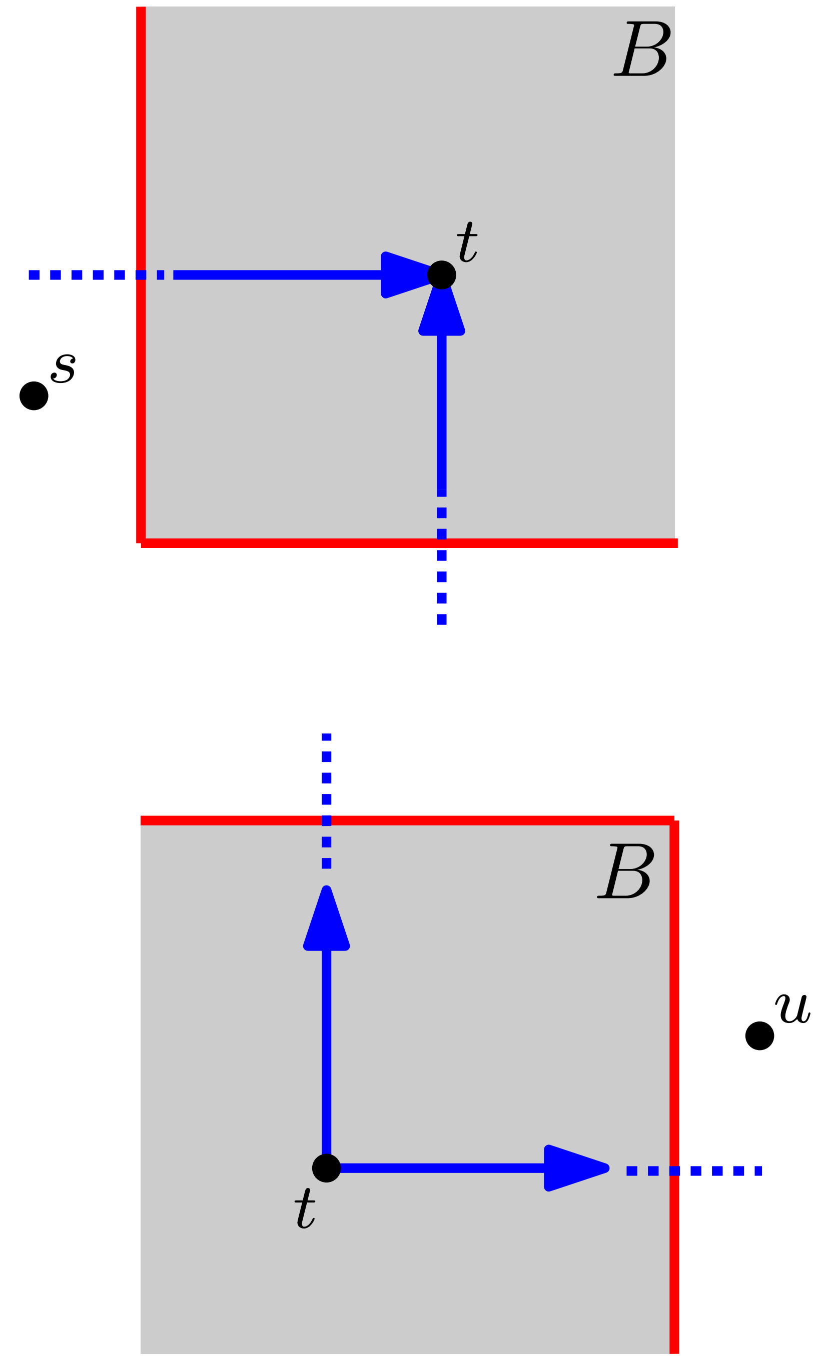
3. Show that $M = \bigoplus_B M_B$
   - show that the M_B's are in direct sum
   - show that the sum of the M_B's covers M

Specificity of the exact 2-d case

product order on $\mathbb{R}^2$ is not total

$$\sum_{\substack{s\notin B\\ s\leq t}} \mathrm{Im}\ M(s\to t) \nsubseteq \bigcap_{\substack{s\in B\\ s\leq t}} \mathrm{Im}\ M(s\to t)$$

$$\sum_{\substack{u\in B\\ u\geq t}} \mathrm{Ker}\, M(t\to u) \nsubseteq \bigcap_{\substack{u\notin B\\ u\geq t}} \mathrm{Ker}\, M(t\to u)$$



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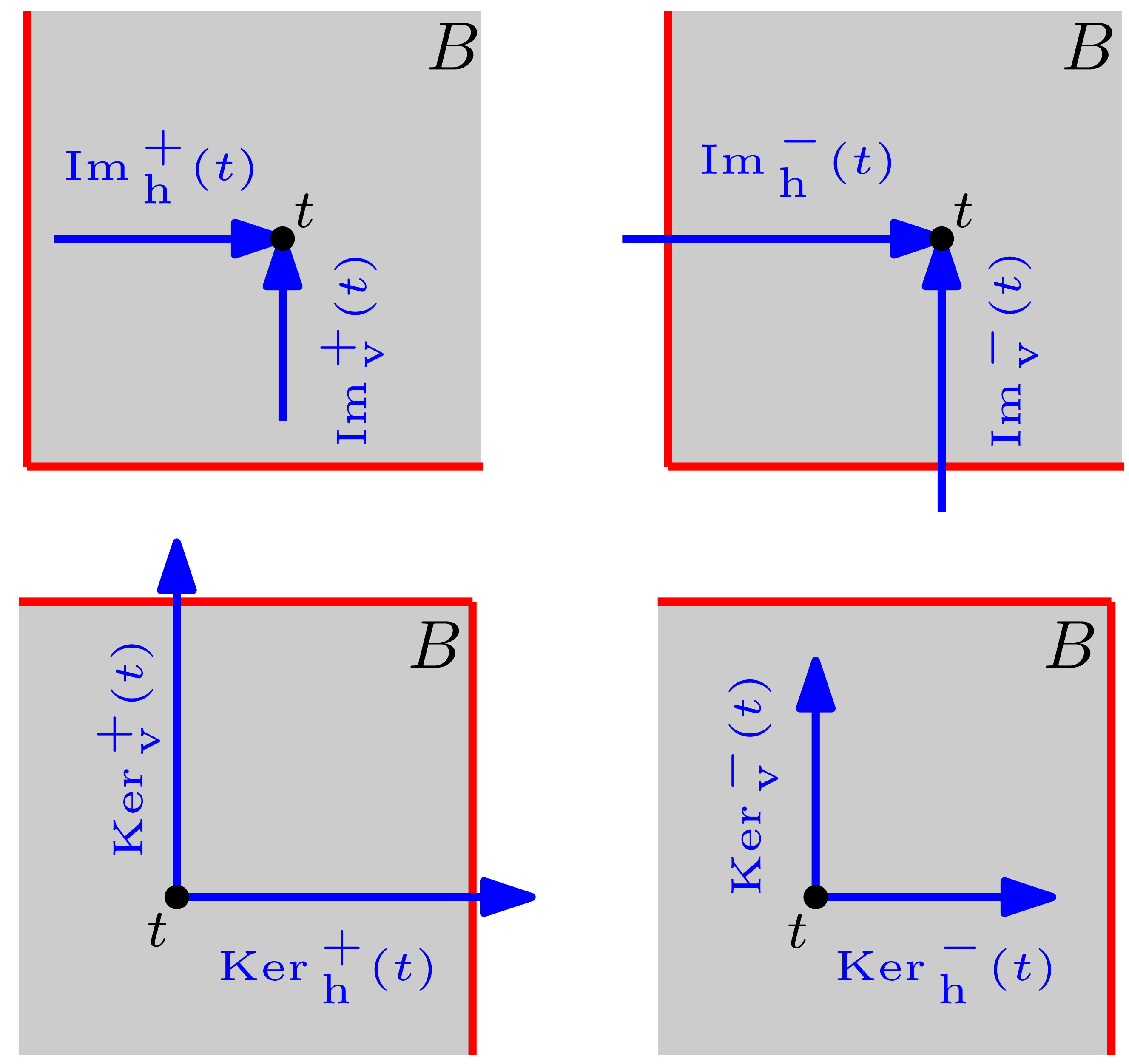
$$\sum_{\substack{u\in B\\ u\geq t}} \mathrm{Ker}\, M(t\to u) \not\subseteq \bigcap_{\substack{u\notin B\\ u\geq t}} \mathrm{Ker}\, M(t\to u)$$

exactness $\Rightarrow$ may restrict focus to horizontal and vertical lines

$$\bigcap_{\substack{s\in B\\ s\leq t}} \mathrm{Im}\, M(s\to t) = \mathrm{Im}\,{}^{+}_{h}(t)\cap \mathrm{Im}\,{}^{+}_{v}(t)$$

$$\sum_{\substack{u\in B\\ u\geq t}} \mathrm{Ker}\, M(t\to u) = \mathrm{Ker}\,{}^{-}_{h}(t) + \mathrm{Ker}\,{}^{-}_{v}(t)$$



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exactness $\Rightarrow$ may restrict focus to horizontal and vertical lines

$$\bigcap_{\substack{s \in B \\ s \leq t}} \mathrm{Im}\, M(s \to t) = \mathrm{Im}\,{}_h^+(t) \cap \mathrm{Im}\,{}_v^+(t) \;=:\; \mathrm{Im}\,{}_B^+(t)$$

$$\Big(\mathrm{Im}\,{}_h^-(t) + \mathrm{Im}\,{}_v^-(t)\Big) \cap\; \mathrm{Im}\,{}_B^+(t) \;=:\; \mathrm{Im}\,{}_B^-(t)$$

$$\sum_{\substack{u \in B \\ u \geq t}} \mathrm{Ker}\, M(t \to u) = \mathrm{Ker}\,{}_h^-(t) + \mathrm{Ker}\,{}_v^-(t) \;=:\; \mathrm{Ker}\,{}_B^-(t)$$

$$\mathrm{Ker}\,{}_B^-(t) + \Big(\mathrm{Ker}\,{}_h^+(t) \cap \mathrm{Ker}\,{}_v^+(t)\Big) \;=:\; \mathrm{Ker}\,{}_B^+(t)$$

Specificity of the exact 2-d case

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duality:

$$\mathrm{Im}\,^{\pm}_{M^*,B}(t) = \left(\mathrm{Ker}\,^{\mp}_{M,B}(t)\right)^{\perp}$$

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exactness $\Rightarrow$ may restrict focus to horizontal and vertical lines

$$\bigcap_{\substack{s\in B\\ s\leq t}} \mathrm{Im}\, M(s\to t) = \mathrm{Im}\,^{+}_{h}(t) \cap \mathrm{Im}\,^{+}_{v}(t)$$
$$=:\ \mathrm{Im}\,^{+}_{B}(t)$$

$$\left(\mathrm{Im}\,^{-}_{h}(t) + \mathrm{Im}\,^{-}_{v}(t)\right) \cap\ \mathrm{Im}\,^{+}_{B}(t)$$
$$=:\ \mathrm{Im}\,^{-}_{B}(t)$$

$$\sum_{\substack{u\in B\\ u\geq t}} \mathrm{Ker}\, M(t\to u) = \mathrm{Ker}\,^{-}_{h}(t) + \mathrm{Ker}\,^{-}_{v}(t)$$
$$=:\ \mathrm{Ker}\,^{-}_{B}(t)$$

$$\mathrm{Ker}\,^{-}_{B}(t) + \left(\mathrm{Ker}\,^{+}_{h}(t) \cap \mathrm{Ker}\,^{+}_{v}(t)\right)$$
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Specificity of the exact 2-d case

product order on $\mathbb{R}^2$ is not total

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definitions of counting functor and embedding operator go through

exactness $\Rightarrow$ may restrict focus to horizontal and vertical lines

$$\bigcap_{\substack{s \in B \\ s \leq t}} \mathrm{Im}\, M(s \to t) = \mathrm{Im}\,^{+}_{h}(t) \cap \mathrm{Im}\,^{+}_{v}(t) =: \mathrm{Im}\,^{+}_{B}(t)$$

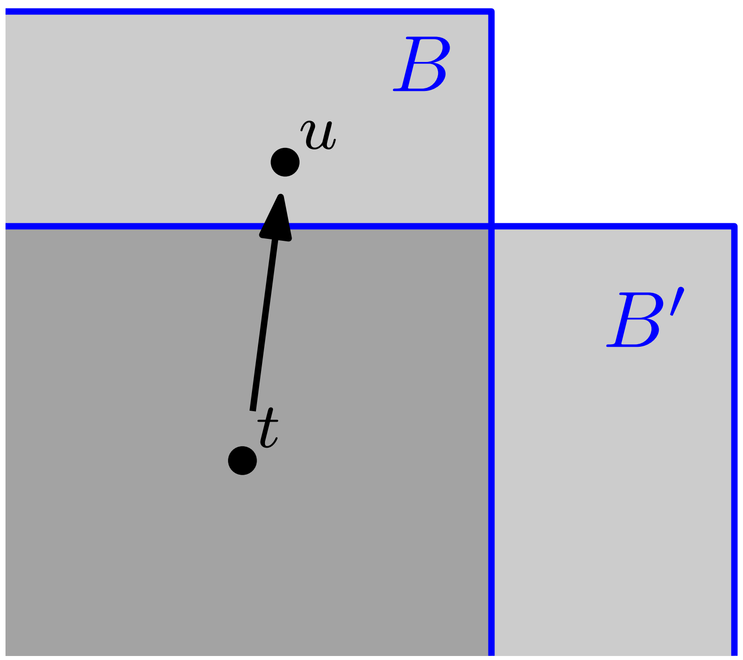
$$\sum_{\substack{u \in B \\ u \geq t}} \mathrm{Ker}\, M(t \to u) = \mathrm{Ker}\,^{-}_{h}(t) + \mathrm{Ker}\,^{-}_{v}(t) =: \mathrm{Ker}\,^{-}_{B}(t)$$

$$\left(\mathrm{Im}\,^{-}_{h}(t) + \mathrm{Im}\,^{-}_{v}(t)\right) \cap\ \mathrm{Im}\,^{+}_{B}(t) =: \mathrm{Im}\,^{-}_{B}(t)$$

$$\mathrm{Ker}\,^{-}_{B}(t) + \left(\mathrm{Ker}\,^{+}_{h}(t) \cap \mathrm{Ker}\,^{+}_{v}(t)\right) =: \mathrm{Ker}\,^{+}_{B}(t)$$

Direct sum (exact 2-d case)

Base case: M_B vs. $M_{B'}$ with $\sup B \neq \sup B'$



$$M_B(t) \cap M_{B'}(t) \neq 0 \Rightarrow M_B(u) \cap M_{B'}(u) \neq 0$$

$\Rightarrow M_{B'}(u) \neq 0$ (contradiction)

Direct sum (exact 2-d case)

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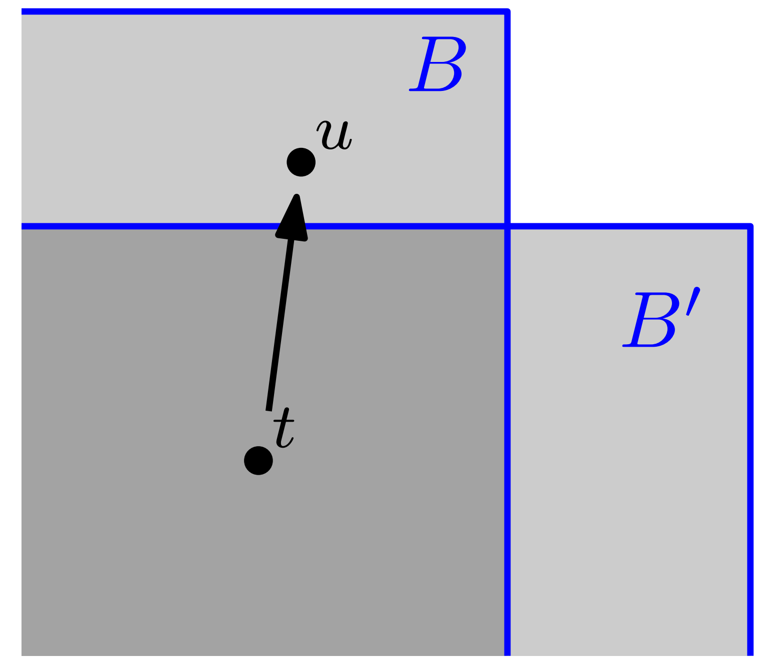
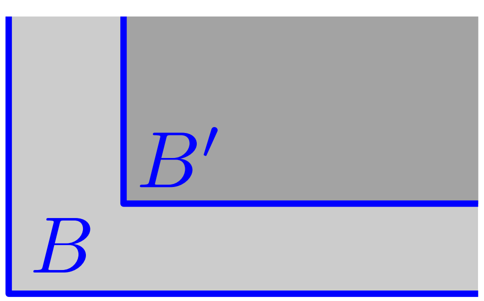


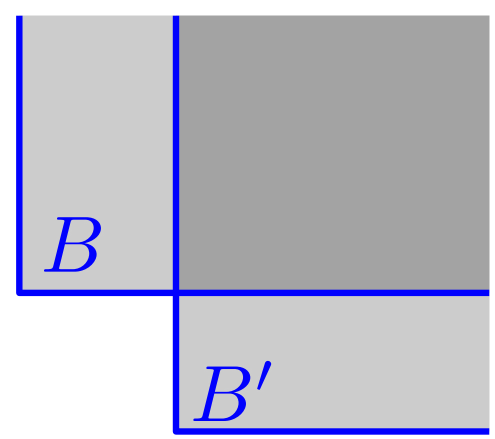
$$M_B(t) \cap M_{B'}(t) \neq 0 \Rightarrow M_B(u) \cap M_{B'}(u) \neq 0$$

$$\Rightarrow M_{B'}(u) \neq 0 \text{ (contradiction)}$$

Variant case: M_B vs. $M_{B'}$ with $\sup B = \sup B'$ and $\inf B \neq \inf B'$



$$\operatorname{Im}^{+}_{B}(t) \subseteq \operatorname{Im}^{-}_{B'}(t)$$



$$\operatorname{Ker}^{\pm}_{B}(t) = \operatorname{Ker}^{\pm}_{B'}(t)$$

$$\operatorname{Im}^{+}_{B}(t) \cap \operatorname{Im}^{+}_{B'}(t) \subseteq \operatorname{Im}^{-}_{B'}(t)$$

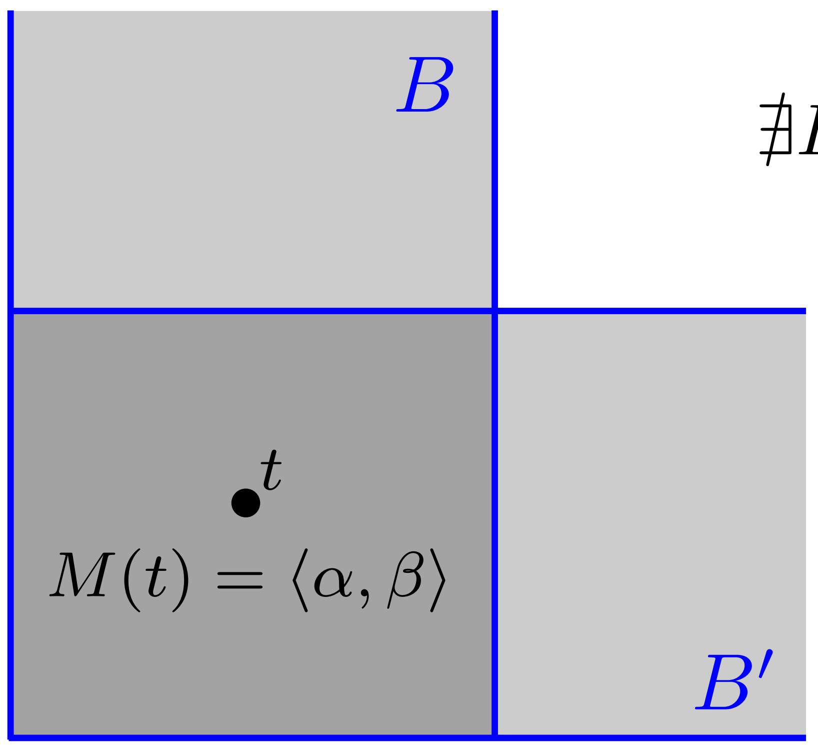
$$\Rightarrow M_B(t) \cap M_{B'}(t) = 0$$

Covering M (exact 2-d case)

Approach: show that $\sum_B M_B(t) = M(t)$ for every $t \in \mathbb{R}^2$

Suppose $X := \sum_B M_B(t) \subsetneq M(t)$:

Problem: $\left\{\mathrm{Im}\,{}^{\pm}_{B}(t)\right\}_B$ separates any $X \subsetneq M(t)$, but $\left\{\mathrm{Ker}\,{}^{\pm}_{B}(t)\right\}_B$ doesn't



$\nexists B"$ s.t. $\mathrm{Ker}\,{}^{-}_{B"}(t) \subseteq \langle \alpha + \beta \rangle \not\supseteq \mathrm{Ker}\,{}^{+}_{B"}(t)$

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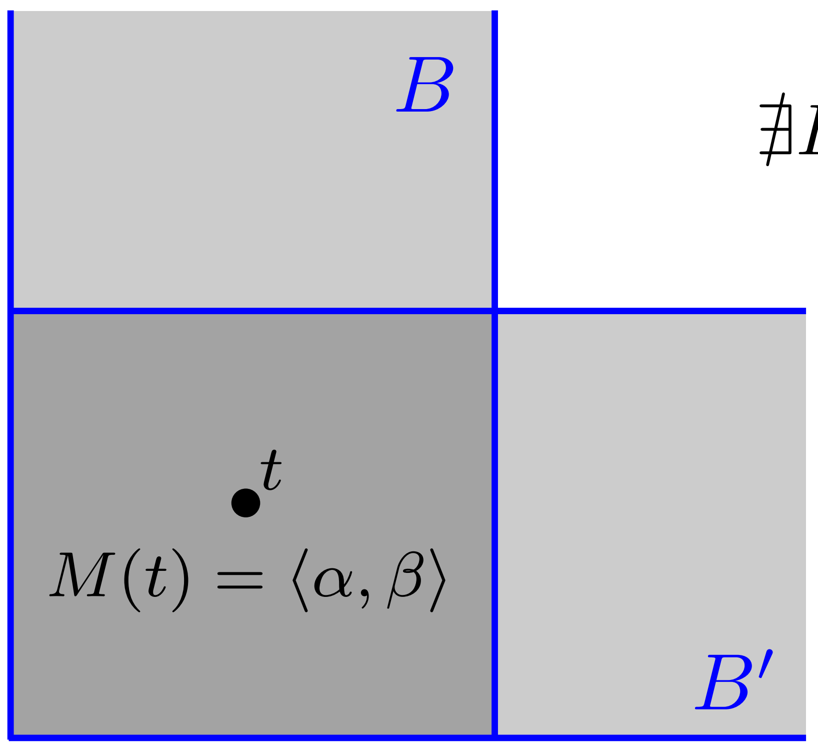

$\nexists B"$ s.t. $\mathrm{Ker}\,^{-}_{B"}(t) \subseteq \langle \alpha + \beta \rangle \nsupseteq \mathrm{Ker}\,^{+}_{B"}(t)$

Notes:

affects only the coverage by death quadrants

$\{\ker^{\mp}_{B}(t)^{\perp}\}_B$ separates any $Y \subsetneq M^*(t)$

Covering M (exact 2-d case)

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Problem: $\left\{\mathrm{Im}\,^{\pm}_{B}(t)\right\}_B$ separates any $X \subsetneq M(t)$, but $\left\{\mathrm{Ker}\,^{\pm}_{B}(t)\right\}_B$ doesn't

Fix: isolate the contribution of death quadrants to the coverage:

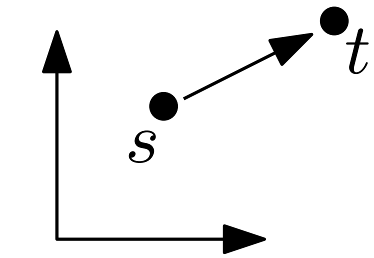
$N(t) := \mathrm{Im}\,^{+}_{\mathbb{R}^2}(t) \cap \mathrm{Ker}\,^{-}_{\mathbb{R}^2}(t)$ ⟵ contribution of death quadrants

$M = N \oplus \bigoplus_{B:\text{ band or birth quadrant}} M_B$ ⟵ coverage by other blocks

$N^* = \bigoplus_{B:\text{ birth quadrant in } (\mathbb{R}^2)^{\mathrm{op}}} N_B^*$ ⟵ coverage of N by death quadrants

A conjecture

Exactness:



$$\begin{tikzcd}
M(s_x, t_y) \arrow[rr, "\delta"] & & M(t) \\
M(s) \arrow[u, "\beta"] \arrow[rru, dotted, "\rho"] \arrow[rr, "\alpha"] & & M(t_x, s_y) \arrow[u, "\gamma"']
\end{tikzcd}$$

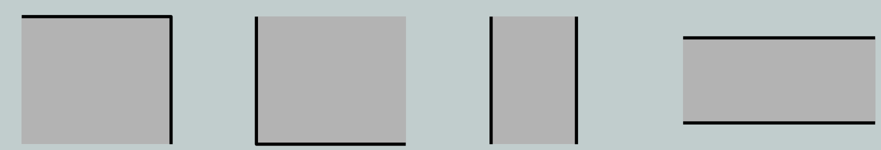
continuous setting: $M : \mathbb{R}^d \to \mathsf{vect}_{\boldsymbol{k}}$

Thm: [Cochoy, O.]

$M : \mathbb{R}^2 \to \mathsf{vect}_{\boldsymbol{k}}$ **exact**

$$\Longleftrightarrow$$

$$M \simeq \bigoplus_{j\in J} \boldsymbol{k}_{B_j}$$

B_j: block 

$$M(s) \xrightarrow{\phi = (\alpha, \beta)} M(t_x, s_y) \oplus M(s_x, t_y) \xrightarrow{\psi = \gamma - \delta} M(t)$$

$\mathbf{Im}\ \boldsymbol{\phi} = \mathbf{Ker}\, \boldsymbol{\psi}$

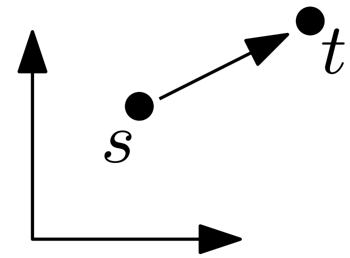
$\mathbf{Im}\ \boldsymbol{\phi} \subseteq \mathbf{Ker}\, \boldsymbol{\psi}$: commutativity

$\mathbf{Im}\ \boldsymbol{\phi} \supseteq \mathbf{Ker}\, \boldsymbol{\psi}$: $\exists$ preimages in $M(t_x, s_y)$ and $M(s_x, t_y)$ $\Rightarrow$ $\exists$ common preimage in $M(s)$

$$\Rightarrow \begin{cases} \mathrm{Im}\ \rho = \mathrm{Im}\ \gamma \cap \mathrm{Im}\ \delta \\ \mathrm{Ker}\, \rho = \mathrm{Ker}\, \alpha + \mathrm{Ker}\, \beta \end{cases}$$

A conjecture

Exactness:



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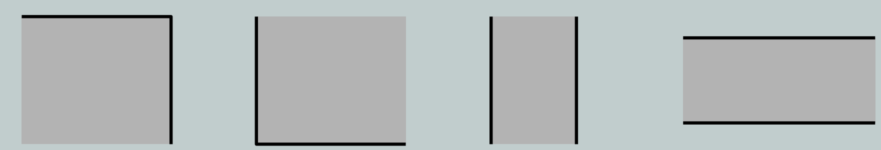


$$M(s) \xrightarrow{\phi=(\alpha,\beta)} M(t_x,s_y) \oplus M(s_x,t_y) \xrightarrow{\psi=\gamma-\delta} M(t) \qquad \mathbf{Im}\ \boldsymbol{\phi} = \mathbf{Ker}\,\boldsymbol{\psi}$$

Im $\boldsymbol{\phi} \subseteq$ **Ker** $\boldsymbol{\psi}$: commutativity

Im $\boldsymbol{\phi} \supseteq$ **Ker** $\boldsymbol{\psi}$: $\exists$ preimages in $M(t_x,s_y)$ and $M(s_x,t_y)$ $\Rightarrow$ $\exists$ common preimage in $M(s)$

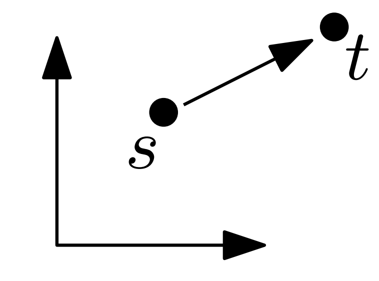
blocks

$$\left[\begin{array}{l} \mathrm{Im}\ \rho = \mathrm{Im}\ \gamma \cap \mathrm{Im}\ \delta \\ \mathrm{Ker}\,\rho = \mathrm{Ker}\,\alpha + \mathrm{Ker}\,\beta \end{array}\right]$$ weak exactness

rectangles

A conjecture

Exactness:



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continuous setting: $M : \mathbb{R}^d \to \mathrm{vect}_{\boldsymbol{k}}$

Conjecture:

$M : \mathbb{R}^2 \to \mathrm{vect}_{\boldsymbol{k}}$ **weakly exact**

$$\Longleftrightarrow$$

$$M \simeq \bigoplus_{j \in J} \boldsymbol{k}_{B_j}$$

B_j: rectangle

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blocks

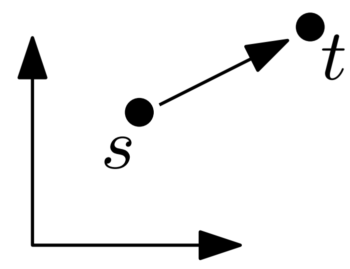
$$\Rightarrow \left[\begin{array}{l} \mathrm{Im}\ \rho = \mathrm{Im}\ \gamma \cap \mathrm{Im}\ \delta \\ \mathrm{Ker}\, \rho = \mathrm{Ker}\, \alpha + \mathrm{Ker}\, \beta \end{array} \right]$$

weak exactness

rectangles

A conjecture

Exactness:



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continuous setting: $M:\mathbb{R}^d\to\mathrm{vect}_{\boldsymbol{k}}$

Conjecture:

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$\mathbf{Im}\ \boldsymbol{\phi}\supseteq\mathbf{Ker}\,\boldsymbol{\psi}$: $\exists$ preimages in $M(t_x,s_y)$ and $M(s_x,t_y)$ $\Rightarrow$ $\exists$ common preimage in $M(s)$

$$\begin{cases}\mathrm{Im}\ \rho=\mathrm{Im}\ \gamma\cap\mathrm{Im}\ \delta\\ \mathrm{Ker}\,\rho=\mathrm{Ker}\,\alpha+\mathrm{Ker}\,\beta\end{cases}$$

counting functor & embedding operator ✓

direct sum ✓

coverage