Mutations, dilogarithm, and pentagon relation

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to celebrate 60th birthday of Bernard Leclerc

This talk is mainly based on the review article:

[N21] T. Nakanishi, *Cluster algebras and scattering diagrams, Part III. Cluster scattering diagrams*, preliminary draft for a monograph, arXiv:2111.00800, v4, 108 pp.

This slide will be put on my website soon.

Cluster algebra vs cluster scattering diagram

Cluster scattering diagrams (CSDs) were introduced by [GHKK18].

[GHKK18] M. Gross, P. Hacking, S. Keel, M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. 31 (2018), 497–608, arXiv:1411.1394 [math.AG]

The following work is also fundamental for more general scattering diagrams.

[KS14] M. Kontsevich, Y. Soibelman, *Wall-crossing structures in Donaldson-Thomas invariants, integrable systems and mirror symmetry*, in Homological mirror symmetry and tropical geometry, Lect. Notes Unione Ital., vol. 15, Springer, 2014, pp. 197–308; arXiv:1303.3253 [math.AG]

	cluster algebra / cluster pattern	CSD
initial data	B: $r \times r$ skew-symmetrizable integer matrix	B: the same as left
(+ auxiliary data)	$\mathbf{x} = (x_1, \dots, x_r)$: r-tuple of variables $\mathbf{y} = (y_1, \dots, y_r)$: r-tuple of variables	N: lattice of rank r $e_1, \dots, e_r$: basis of N
principle of construction	mutation	consistency
structure behind	–	structure group G

The most basic result in [GHKK18]

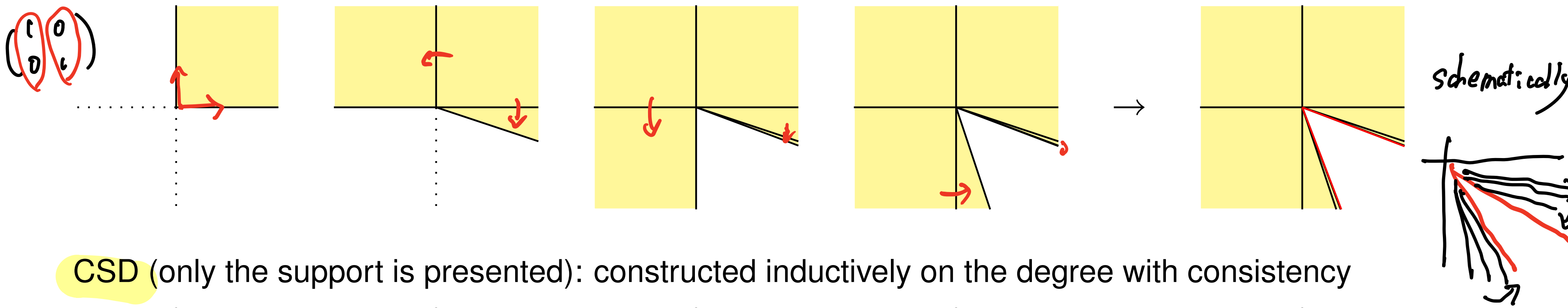
- The G-fan for a cluster pattern is embedded in the corresponding CSD. Thus, the CSD knows everything about the cluster pattern.
- In addition, the CSD contains a highly complex structure outside the G-fan (the Badlands).

Example: G-fan vs CSD

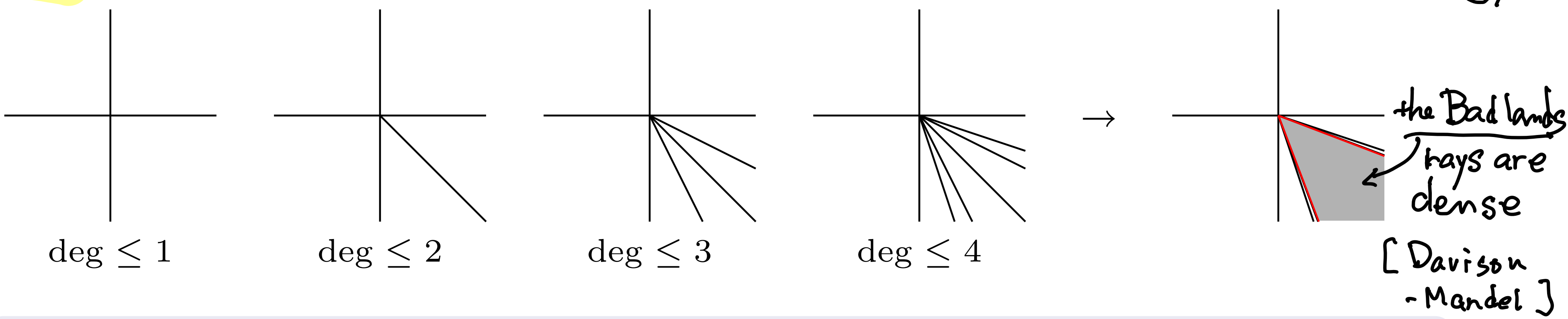
Example: Initial data

$$r = 2, \quad B = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$$

G-fan $=$ the geometrical presentation of G-matrices $=$ tropicalization of a cluster pattern (detropicalization: The cluster pattern can be reconstructed from it.)



CSD (only the support is presented): constructed inductively on the degree with consistency



The force is balanced between "the Light side" and "the Dark side".
So, it is natural to regard the cluster pattern and the CSD as "one inseparable object".

Badlands (the Dark side)

Badlands National Park, South Dakota, USA

Goal of Talk

Goal

I explain the roles of the dilogarithm (dilogarithm elements) and the pentagon relation in cluster algebras and CSDs.

The conclusion is very simple.

Summary (Message)

- The dilogarithm interpolates the two principles, *mutation* and *consistency*.
- The dilogarithm elements and the pentagon relation are *everything* for CSDs.

This point of view was implicit in [GHKK18] and clarified explicitly in [N21].

Fock-Goncharov decomposition

$(B, \mathbf{y})$: a given initial Y-seed

Consider a sequence of mutations

$$(B, \mathbf{y}) = (B(0), \mathbf{y}(0)) \overset{\mu_{k_0}}{\longrightarrow} (B(1), \mathbf{y}(1)) \overset{\mu_{k_1}}{\longrightarrow} \cdots \overset{\mu_{k_{P-1}}}{\longrightarrow} (B(P), \mathbf{y}(P)).$$

We regard each mutation μ_{k_s} as a field isomorphism

$$\begin{aligned} \mu(s): \quad \mathbb{Q}(\mathbf{y}(s+1)) &\longrightarrow \mathbb{Q}(\mathbf{y}(s)) \\ y_i(s+1) &\mapsto \begin{cases} y_{k_s}^{-1}(s) & i = k_s, \\ y_i(s) y_{k_s}(s)^{[\varepsilon b_{k_s i}(s)]_+} (1 + y_{k_s}(s)^{\varepsilon})^{-b_{k_s i}(s)} & i \neq k_s. \end{cases} \end{aligned}$$

Here, the RHS is independent of $\varepsilon \in \{1, -1\}$.

For ε, we especially choose the sign *(tropical sign)* ε_s of the corresponding c-vector $\mathbf{c}_{k_s}(s)$.

Then, we consider the decomposition

$$\mu(s) = \rho(s) \circ \tau(s),$$

$$\begin{aligned} \tau(s): \quad \mathbb{Q}(\mathbf{y}(s+1)) &\longrightarrow \mathbb{Q}(\mathbf{y}(s)) \\ y_i(s+1) &\mapsto \begin{cases} y_{k_s}^{-1}(s) & i = k_s, \\ y_i(s) y_{k_s}(s)^{[\varepsilon_s b_{k_s i}(s)]_+} & i \neq k_s, \end{cases} \\ \rho(s): \quad \mathbb{Q}(\mathbf{y}(s)) &\longrightarrow \mathbb{Q}(\mathbf{y}(s)) \\ y_i(s) &\mapsto y_i(s)(1 + y_{k_s}(s)^{\varepsilon_s})^{-b_{k_s i}(s)}. \end{aligned}$$

We call it the *Fock-Goncharov decomposition*.

The map $\tau(s)$ is the *tropical part*, while the map $\rho(s)$ is the *automorphism part* of $\mu(s)$.

Composition of the Fock-Goncharov decompositions

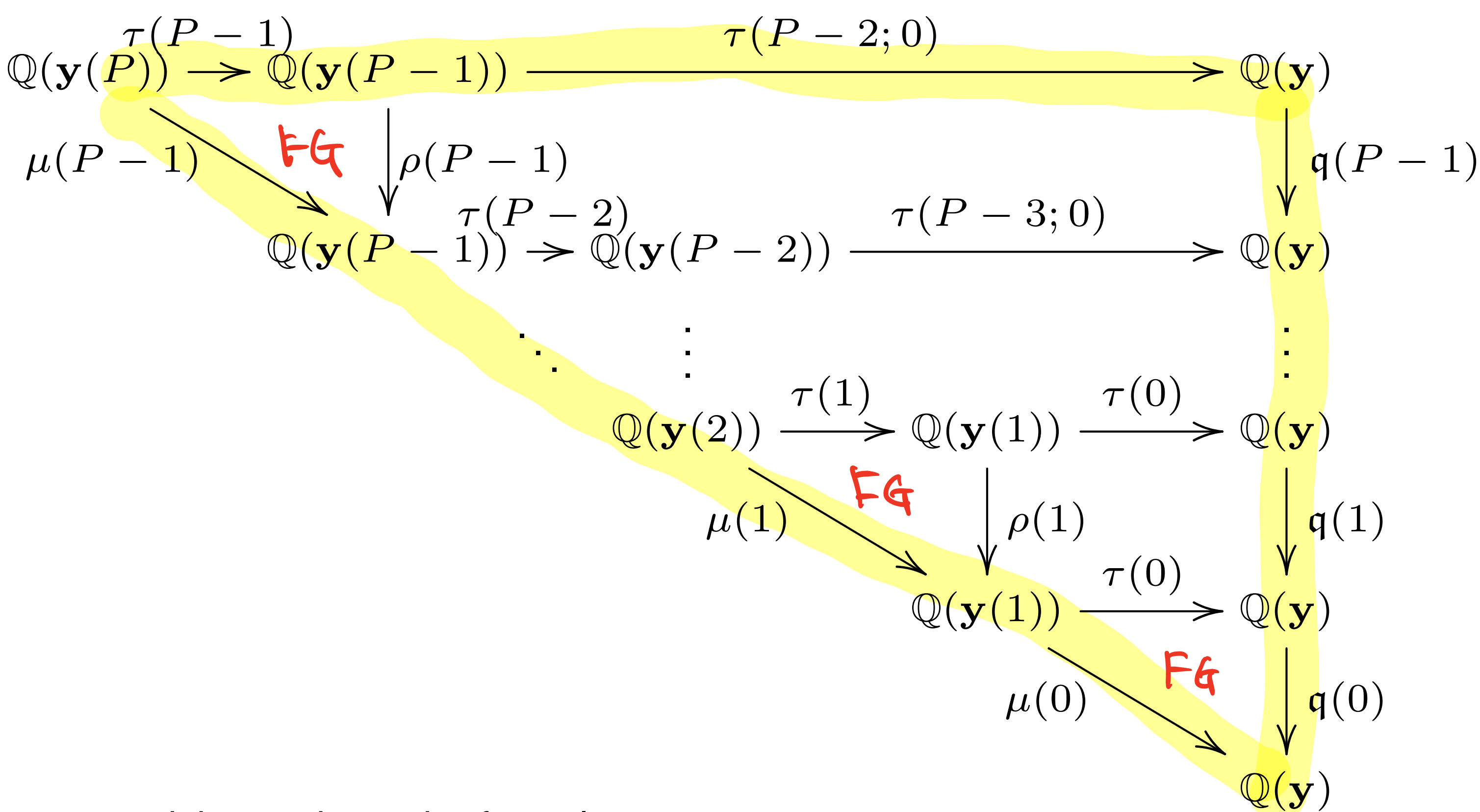
Next, we introduce compositions of the tropical parts ($s = 0, \dots, P-1$)

$$\tau(s;0) := \tau(0) \circ \tau(1) \circ \cdots \circ \tau(s) : \mathbb{Q}(\mathbf{y}(s+1)) \longrightarrow \mathbb{Q}(\mathbf{y}).$$

Thanks to the choice of the sign ε_s, the following formula holds:

$$\tau(s;0)(y_i(s+1)) = y^{\mathbf{c}_i(s+1)} \quad (\mathbf{c}_i(s+1)\text{: } c\text{-vector})$$

We have a commutative diagram



By the commutativity, we have the formula

$$\mathfrak{q}(s)(y^{\mathbf{c}_i(s)}) = y^{\mathbf{c}_i(s)}(1 + y^{\mathbf{c}^+_{k_s}(s)})^{-b_{k_s i}(s)}, \quad \mathbf{c}^+_{k_s}(s) := \varepsilon_s \mathbf{c}_{k_s}(s).$$

Poisson bracket and dilogarithm

Following [Fock-Goncharov09, Gekhtman-N-Rupel16], we reformulate the above automorphism

$$\mathfrak{q}(s)(y^{\mathbf{c}_i(s)}) = y^{\mathbf{c}_i(s)}(1 + y^{\mathbf{c}^+_{k_s}(s)})^{-b_{k_s i}(s)}.$$

by the dilogarithm.

- Following [Gekhtamn-Shapiro-Weinstein02], consider a Poisson bracket on $\mathbb{Q}(\mathbf{y})$ as

$$\{y_i, y_i\} = d_i b_{ij} y_i y_i,$$

where $D = \mathrm{diag}(d_1, \dots, d_r)$ is any rational skew-symmetrizer of B.

- Recall the Euler dilogarithm:

$$\mathrm{Li}_2(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} x^j, \quad x\frac{d}{dx}(-\mathrm{Li}_2(-x)) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j = \log(1+x).$$

- Also recall the following fact [Nakanishi-Zelevinsky12]:

$$DB(s) = C(s)^T (DB) C(s).$$

- Using the above formulas, one can derive

$$\{\varepsilon_s d_{k_s}^{-1} \mathrm{Li}_2(-y^{\mathbf{c}^+_{k_s}(s)}), y^{\mathbf{c}_i(s)}\} = y^{\mathbf{c}_i(s)} \log(1 + y^{\mathbf{c}^+_{k_s}(s)})^{-b_{k_s i}(s)}.$$

Thus, the automorphism $\mathfrak{q}(s)$ is described as the time-one flow by the Hamiltonian

$$\mathcal{H}(s) := \frac{\varepsilon_s}{d_{k_s}} \mathrm{Li}_2(-y^{\mathbf{c}^+_{k_s}(s)}).$$

This gives an intrinsic connection between mutations and dilogarithm.
([FG09] used this observation to quantize mutations with the *quantum dilogarithm*.)

Lie algebra $\mathfrak{g}_\Omega$

Temporally, forget about the initial skew-symmetrizable matrix B.

- initial data

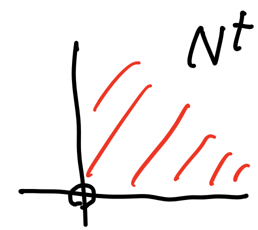
$\Omega = (\omega_{ij})$: skew-symmetric $r \times r$ rational matrix
$N \simeq \mathbb{Z}^r$: lattice of rank r
$e_1, \dots, e_r$: basis of N
The data determines the followings:

- (a). skew-symmetric bilinear form $\{\cdot,\cdot\} : N \times N \to \mathbb{Q}$:

$$\{e_i, e_j\} := \omega_{ij}.$$

- (b). semi-group of positive elements

$$N^+ := \left\{ n = \sum_{i=1}^{r} a_i e_i \mid a_i \in \mathbb{Z}_{\geq 0},\ n \neq 0 \right\}.$$



- Lie algebra $\mathfrak{g}_\Omega$

Define an N^+-graded Lie algebra $\mathfrak{g}_\Omega$ associated with the above data as follows:

- basis (generator): X_n $(n \in N^+)$

$$\mathfrak{g}_\Omega = \bigoplus_{n \in N^+} \mathbb{Q} X_n.$$

- Lie bracket

$$[X_n, X_{n'}] = \{n, n'\} X_{n+n'}.$$

It is easy to check the Jacobi identity.

Structure group G_Ω

- completion $\widehat{\mathfrak{g}}_\Omega$

For $n = \sum_i a_i e_i \in N^+$, define $\deg(n)$ as

$$\deg(n) = \sum_{i=1}^{r} a_i.$$

Then, we have the completion $\widehat{\mathfrak{g}}_\Omega$ of $\mathfrak{g}$ with respect to $\deg$. An element of $\widehat{\mathfrak{g}}_\Omega$ has the form

$$\sum_{n \in N^+} c_n X_n \quad \text{(possibly infinite sum)}$$

- Group G_Ω

We define a group G_Ω as follows:

$$G_\Omega = \{\exp(X) \mid X \in \widehat{\mathfrak{g}}_\Omega\},$$

where

$$\exp : \widehat{\mathfrak{g}} \to G_\Omega$$

is a formal bijection, and the product is defined by the Baker-Campbell-Hausdorff (BCH) formula

$$\begin{aligned} &\exp(X)\exp(Y) \\ &= \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots\right). \end{aligned}$$

- This is the relation of the formal sum $\exp x = \sum_{k=1}^{\infty} x^k/k!$ of an element x of a Lie algebra.
- Since $\widehat{\mathfrak{g}}$ is N^+-graded, the infinite sum of the RHS is well-defined.

This construction of G_Ω is due to [Kontsevich-Soibelman14]. There is no specific name of G_Ω. We call it the *structure group* of the forthcoming scattering diagrams.

Normal subgroup $G^{>\ell}$ and quotient $G^{\leq\ell}$

We write G_Ω as G, when there is no confusion.

- Normal subgroup $G^{>\ell}$

For any positive integer ℓ, we define

$$(N^+)^{>\ell} := \{n \in N^+ \mid \deg(n) > \ell\}.$$

Let $G^{>\ell}$ be the set of all elements of G having the form

$$\exp\left(\sum_{n\in(N^+)^{>\ell}} c_n X_n\right) \quad \text{(possibly infinite sum)}.$$

Then, $G^{>\ell}$ is a normal subgroup of G.

- Quotient group $G^{\leq\ell}$

For the above $G^{>\ell}$, we define

$$G^{\leq\ell} := G/G^{>\ell}.$$

By the construction, we have

$$G = \varprojlim G^{\leq\ell}.$$

Infinite product in G

The infinite product in G is given by the limit of the finite product in $G^{\leq\ell}$ compatible with the canonical projections $\pi_\ell : G \to G_\ell$.

Dilogarithm elements (Algebraic formulation of dilogarithm)

- Recall the Euler dilogarithm

$$\mathrm{Li}_2(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} x^j, \quad x\frac{d}{dx}(-\mathrm{Li}_2(-x)) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j = \log(1+x).$$

- Dilogarithm element: For each $n \in N^+$, we define the *dilogarithm element* for n

$$\Psi[n] := \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} X_{jn} \right) \in G.$$

- y-representation of G: We define the action of X_n on the formal power series ring $\mathbb{Q}[[\mathbf{y}]]$ of variables $\mathbf{y} = (y_1, \dots, y_r)$ as

$$\tilde{X}_n(y^{n'}) := \{n, n'\} y^{n+n'}.$$

This is a derivation, and it induces the group homomorphism

$$\rho_y : G \mapsto \mathrm{Aut}(\mathbb{Q}[[\mathbf{y}]]), \quad \exp(X) \mapsto \sum_{k=0}^{\infty} \tilde{X}^k / k!$$

We call it the *y-representation* of G. Under this action, we have

$$\Psi[n](y^{n'}) = y^{n'}(1+y^n)^{\{n,n'\}}.$$

In particular, we recover the automorphism part $\mathfrak{q}(s)$ of the Fock-Goncharov decomposition as

$$\Psi[\mathbf{c}^+_{k_s}(s)]^{-\varepsilon_s/d_{k_s}}(y^{\mathbf{c}_i(s)}) = y^{\mathbf{c}_i(s)}(1+y^{\mathbf{c}^+_{k_s}(s)})^{-b_{k_s i}(s)} = \mathfrak{q}(s)(y^{\mathbf{c}_i(s)})$$

- A dilogarithm element $\Psi[n]^{-1}$ corresponds to (the time-one flow of) the Hamiltonian $\mathrm{Li}_2(-y^n)$.
- The Poisson bracket is replaced with the group G and its action.

Pentagon relation (Algebraic formulation of pentagon identity)

Advantages of working with the group G

- We can study the relations among $\Psi[n]$'s in G.
- Infinite products are available in the group G.
- All relevant representations (y-rep, x-rep, principal x-rep, etc) and their mutations are treated in a unified and more intrinsic way.

The dilogarithm elements $\Psi[n]$ ($n \in N^+$) satisfy a remarkable relation in G.

Theorem [GHKK18, N21]

For any $n', n \in N^+$ and $c, c' \in \mathbb{Q}$, the following relations hold:
(a). If $\{n', n\} = 0$,

$$\text{(commutative relation)} \quad \Psi[n']^{c'} \Psi[n]^{c} = \Psi[n]^{c} \Psi[n']^{c'}.$$

(b). If $\{n', n\} = c \; (\neq 0)$,

$$\text{(pentagon relation)} \quad \Psi[n']^{1/c} \Psi[n]^{1/c} = \Psi[n]^{1/c} \Psi[n+n']^{1/c} \Psi[n']^{1/c}.$$

Proof. (a) $[X_n, X_{n'}] = \{n, n'\} X_{n+n'} = 0$. (b) Use y-representation. □
This is an algebraic formulation of the *pentagon identity (Abel's identity)* for the Euler dilogarithm

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\frac{1-x}{1-xy}\right) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\left(\frac{1-y}{1-xy}\right)$$
$$= \frac{\pi^2}{2} - \log x \log(1-x) - \log y \log(1-y) - \log\left(\frac{1-x}{1-xy}\right) \log\left(\frac{1-y}{1-xy}\right).$$

Walls

We continue to use the initial data:

$\Omega = (\omega_{ij})$: skew-symmetric $r \times r$ rational matrix,

$N \simeq \mathbb{Z}^r$: lattice of rank r; $e_1, \dots, e_r$: basis of N,

$G = G_\Omega$: the group determined by the above data

- Additional definitions
  - $M := \mathrm{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^r$, $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^r$
    (A scattering diagram is defined in the space $M_\mathbb{R}$.)
  - $\langle \cdot, \cdot \rangle : N \times M_\mathbb{R} \to \mathbb{R}$: the canonical paring and its linear extension.
  - For $n \in N^+$, we define the hypersurface $n^\perp$ in $M_\mathbb{R}$ as
    $$n^\perp := \{ z \in M_\mathbb{R} \mid \langle n, z \rangle = 0 \}.$$
  - For $n \in N^+$, we say it is *primitive* if it is not divisible by $t \in \mathbb{Z}_{>1}$ in N^+.
    Let N^+_{pr} denote the set of all primitive elements in N^+.
  - For $n \in N^+_{\mathrm{pr}}$, let $G_n^\parallel$ be the abelian subgroup of G consisting of all elements $\exp(\sum_{j=1}^\infty c_j X_{jn})$ (possibly infinite sum). We call it the *parallel subgroup* for n.

- Wall

We call a triplet $\mathbf{w} = (\mathfrak{d}, g)_n$ a *wall*, where

  - normal vector: $n \in N^+_{\mathrm{pr}}$
  - support: $\mathfrak{d} \subset n^\perp$, a cone in $M_\mathbb{R}$ of dimension $r-1$ (not necessarily strongly convex)
  - wall element: $g \in G_n^\parallel$

Ex: For $n \in N^+_{\mathrm{pr}}$, $\mathbf{w} = (n^\perp, \Psi[n])_n$ is a wall.

Scattering diagrams

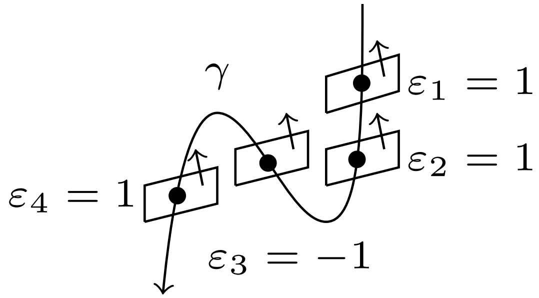
- Scattering diagram
  - A collection of walls $\mathfrak{D} = \{\mathbf{w}_\lambda = (\mathfrak{d}_\lambda, g_\lambda)_{n_\lambda}\}_{\lambda\in\Lambda}$ is a *scattering diagram* if it satisfies the following *finiteness condition*:
    For any positive integer ℓ, there are only finitely many $\mathbf{w}_\lambda$ such that $\pi_\ell(g_\lambda) \neq \mathrm{id}$.
    (Here, $\pi_\ell : G \to G^{\leq \ell}$ is the canonical projection.)
  - For each positive integer ℓ, the following (finite) subset $\mathfrak{D}_\ell$ of $\mathfrak{D}$ is called the *reduction* of $\mathfrak{D}$ at degree ℓ:
    $$\mathfrak{D}_\ell = \{\mathbf{w}_\lambda \in \mathfrak{D} \mid \pi_\ell(g_\lambda) \neq \mathrm{id}\}.$$
  - The union of the supports of walls $\mathrm{Supp}(\mathfrak{D}) := \bigcup_{\lambda\in\Lambda} \mathfrak{d}_\lambda$ is called the *support* of $\mathfrak{D}$.
- Path-ordered product

For a scattering diagram $\mathfrak{D}$ and a smooth curve γ in $M_\mathbb{R}$ satisfying a certain generic condition (an admissible curve), the *path-ordered product* $\mathfrak{p}_{\mathfrak{D},\gamma} \in G$ is defined as follows:
For each positive integer ℓ, when γ crosses the walls $\mathbf{w}_1, \dots, \mathbf{w}_k$ of $\mathfrak{D}_\ell$ in this order, we set

$$\mathfrak{p}_{\mathfrak{D}_\ell,\gamma} = g_k^{\varepsilon_k} \cdots g_1^{\varepsilon_1},$$

$$\mathfrak{p}_{\mathfrak{D},\gamma} = \lim_{\ell\to\infty} \mathfrak{p}_{\mathfrak{D}_\ell,\gamma}$$ (well-defined thanks to the finiteness condition.)

Here, the *intersection sign* ε_i is defined as below.



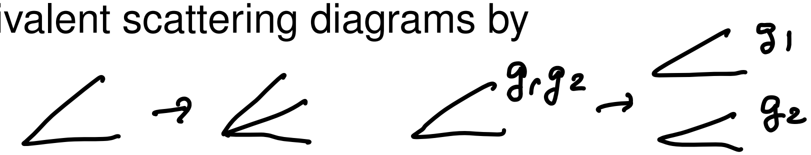
Consistent scattering diagrams

- <u>equivalence</u>

Scattering diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ are *equivalent*.

$\overset{\text{def}}{\iff}$ For any admissible curve γ, $\mathfrak{p}_{\mathfrak{D},\gamma} = \mathfrak{p}_{\mathfrak{D}',\gamma}$ holds.

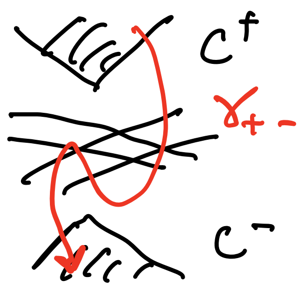
For a given scattering diagram $\mathfrak{D}$, we have infinitely many equivalent scattering diagrams by splitting and unifying the supports of walls and wall elements.



- <u>consistency</u>

A scattering diagram $\mathfrak{D}$ is *consistent* if, for any admissible <u>closed</u> curve γ, $\mathfrak{p}_{\mathfrak{D},\gamma} = \mathrm{id}$ holds.

- <u>Existence theorem</u>



$$\mathcal{C}^+ := \{ z \in M_{\mathbb{R}} \mid \langle e_i, z \rangle \geq 0 \quad (i = 1, \dots, r) \},$$
$$\mathcal{C}^- := \{ z \in M_{\mathbb{R}} \mid \langle e_i, z \rangle \leq 0 \quad (i = 1, \dots, r) \}.$$

Let γ_{+-} be any admissible curve starting in $\mathrm{Int}(\mathcal{C}^+)$ and ending in $\mathrm{Int}(\mathcal{C}^-)$.
For any consistent scattering diagram $\mathfrak{D}$, any wall of $\mathfrak{D}$ does not intersect $\mathrm{Int}(\mathcal{C}^{\pm})$. So, an element $g(\mathfrak{D}) := \mathfrak{p}_{\mathfrak{D},\gamma_{+-}} \in G$ is uniquely determined, and it only depends on the equivalence class of $\mathfrak{D}$.

Theorem ([KS14, GHKK18])

The following map is bijective:

$$\begin{array}{ccc} \{\text{equivalence classes of consistent scattering diagrams}\} & \longrightarrow & G \\ [\mathfrak{D}] & \mapsto & g(\mathfrak{D}). \end{array}$$

The proof depends on some (abstract) decompositions of G.

Rank 2 example: CSD of type A_2

consistent scattering diagrams

Q. How can we construct ~~CSDs~~ more explicitly?

Some (special) consistent scattering diagrams can be constructed only from dilogarithm elements and the pentagon relation.

Throughout all examples below, let

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \{e_2, e_1\} = 1.$$

Example 1. We have the pentagon relation

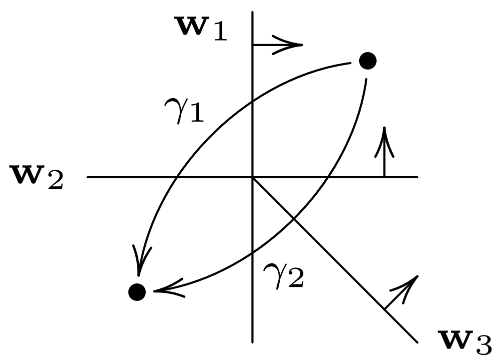
$$\Psi[e_2]\Psi[e_1] = \Psi[e_1]\Psi[e_1 + e_2]\Psi[e_2].$$

This is interpreted as a (unique) consistent relation

$$\mathfrak{p}_{\mathfrak{D},\gamma_1} = \mathfrak{p}_{\mathfrak{D},\gamma_2}$$

for the consistent scattering diagram $\mathfrak{D}$ with walls

$$\mathbf{w}_1 = (e_1^{\perp}, \Psi[e_1])_{e_1}, \quad \mathbf{w}_2 = (e_2^{\perp}, \Psi[e_2])_{e_2}, \quad \mathbf{w}_3 = (\mathbb{R}_{\geq 0}(e_1^* - e_2^*), \Psi[e_1 + e_2])_{e_1+e_2}.$$



This is indeed a CSD of type A_2. (The definition of a CSD will be given later.)
The support of $\mathfrak{D}$ also coincides with the G-fan of type A_2.

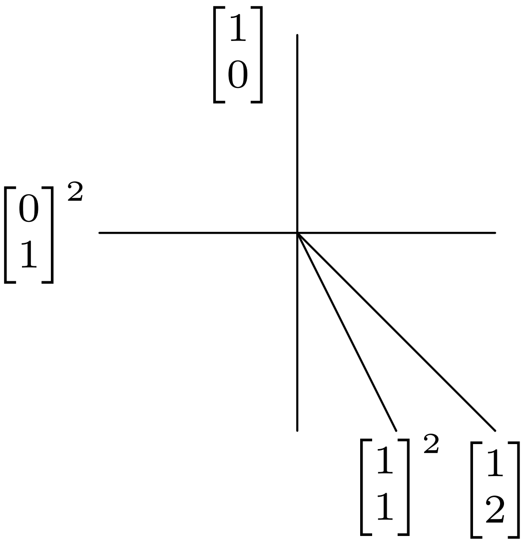
Rank 2 example: CSD of type B_2

Example 2. Below, for $n = n_1 e_1 + n_2 e_2$, we write $\Psi[n]$ as $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$.

Applying the pentagon relation repeatedly, we have

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2.$$

The LHS is anti-ordered, while the RHS is ordered. This gives a (unique) consistent relation for the following consistent scattering diagram $\mathfrak{D}$:



Here, the identification $M_{\mathbb{R}} \simeq \mathbb{R}^2$ is given by $e_1^* \mapsto \mathbf{e}_1$, $e_2^*/2 \mapsto \mathbf{e}_2$.

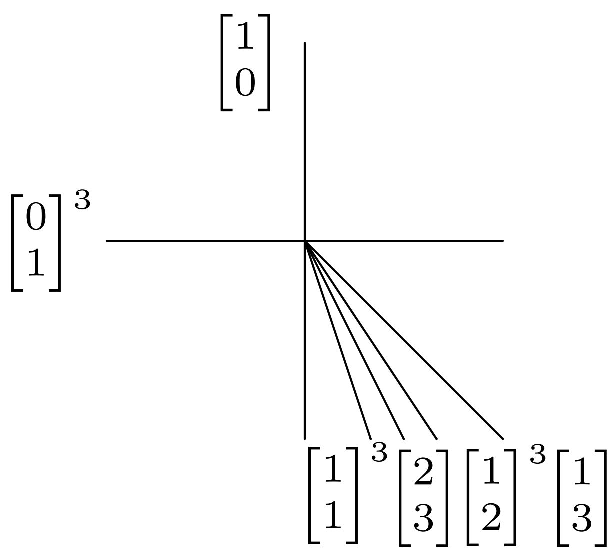
This is a CSD of type B_2. The support of $\mathfrak{D}$ also coincides with the G-fan of type B_2.

Rank 2 example: CSD of type G_2

Example 3. Applying the pentagon relation repeatedly, we have

$$\begin{aligned}
\begin{bmatrix}0\\1\end{bmatrix}^3 \begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}0\\1\end{bmatrix} \left(\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix}^2 \begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}^2 \right) \\
&= \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix}^2 \begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix}^2 \begin{bmatrix}1\\3\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}^3 \\
&= \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix}^3 \begin{bmatrix}2\\3\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix}^3 \begin{bmatrix}1\\3\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}^3 .
\end{aligned}$$

The LHS is anti-ordered, while the RHS is ordered. This gives a (unique) consistent relation for the following consistent scattering diagram $\mathfrak{D}$:



Here, the identification $M_{\mathbb{R}} \simeq \mathbb{R}^2$ is given by $e_1^* \mapsto \mathbf{e}_1$, $e_2^*/3 \mapsto \mathbf{e}_2$.
This is a CSD of type G_2. The support of $\mathfrak{D}$ also coincides with the G-fan of type G_2.

Initial data for CSD

• initial data for CSD:
skew-symmetrizable $r \times r$ integer matrix B
decomposition

$$B = \Delta\Omega$$

Δ: positive integer diagonal matrix, Ω: skew-symmetric rational matrix.

Thus, Δ^{-1} is a skew-symmetrizer of B.
(Such a decomposition is not unique, but we do not care at this moment.)

• As we did so far,
$\Omega = (\omega_{ij})$: the above skew-symmetric rational matrix
$N \simeq \mathbb{Z}^r$: lattice of rank r; $e_1, \dots, e_r$: basis of N
$\{e_i, e_j\} = \omega_{ij}$: skew-symmetric form on N
G_Ω: the group defined by the above data

• Meanwhile, from $\Delta = \mathrm{diag}(\delta_1, \dots, \delta_r)$, we have
$N^\circ := \bigoplus_{i=1}^r \mathbb{Z}\delta_i e_i$: sublattice of N
$M^\circ := \mathrm{Hom}(N^\circ, \mathbb{Z}) = \bigoplus_{i=1}^r \mathbb{Z}e_i^*/\delta_i$: $M \subset M^\circ \subset M_\mathbb{R}$
Also, for $n \in N^+$, let $\delta(n)$ be the smallest positive rational number such that $\delta(n)n \in N^\circ$. We call it the *normalization factor* of n (e.g., $\delta(e_i) = \delta_i$).

• We have a homomorphism of abelian groups

$$p^* : N \to M^\circ \subset M_\mathbb{R}, \quad n \mapsto \{\cdot, n\}.$$

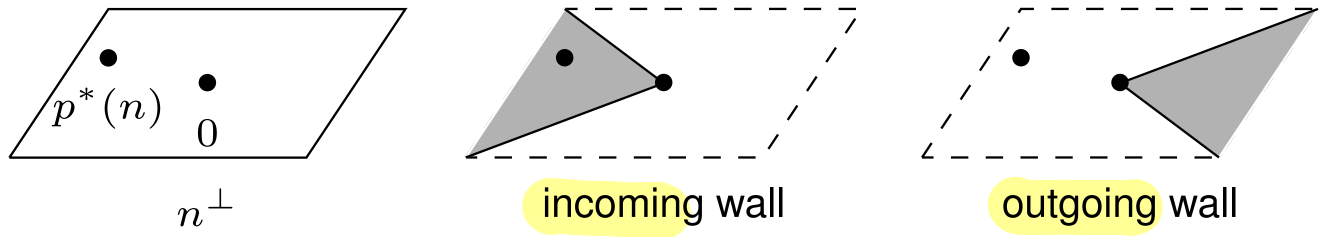
The representation matrix of p^* with respect to the above bases is B.

Cluster Scattering Diagrams (CSDs)

- incoming and outgoing walls

A wall $\mathbf{w} = (\mathfrak{d}, g)_n$ of a scattering diagram $\mathfrak{D}$ with the structure group G_Ω is *incoming* (resp. *outgoing*) if $p^*(n) \in \mathfrak{d}$ (reps. otherwise).

Since $\langle n, p^*(n)\rangle = \{n, n\} = 0$, we have $p^*(n) \in n^\perp$.



- Cluster scattering diagrams

We are ready to define cluster scattering diagrams.

Theorem-Definition [GHKK18]

For any skew-symmetrizable $r \times r$ integer matrix B and its decomposition $B = \Delta\Omega$, there is a unique (up to equivalence) consistent scattering diagram $\mathfrak{D}$ with the structure group G_Ω satisfying the following condition:

The set of all incoming walls in $\mathfrak{D}$ is given by $\{\mathbf{w}_{e_i} := (e_i^\perp, \Psi[e_i]^{\delta_i})_{e_i} \mid i = 1, \dots, r\}$.

A consistent scattering diagram satisfying the above condition is called a *cluster scattering diagram (CSD)* associated with B and denoted by $\mathfrak{D}(B)$.

For another decomposition $B = \Delta'\Omega'$, one can identify the corresponding CSD through the isomorphism of the structure groups $G_\Omega \simeq G_{\Omega'}$.

Ordering Lemma

Let us temporarily concentrate on the rank 2 case.
We say that a (possibly infinite) product of $\Psi[n]^c$ ($c \in \mathbb{Q}$) is *ordered* (resp. *anti-ordered*) if, for any adjacent factors $\Psi[n']^{c'}\Psi[n]^c$, $\{n', n\} \geq 0$ (resp. $\{n', n\} \leq 0$) holds.

Ordering Lemma [N21]

Any finite anti-ordered product of $\Psi[n]^{\delta(n)}$ is rewritten as a (possibly infinite) ordered product of $\Psi[n]^{\delta(n)}$ by applying the pentagon relation (possibly infinitely many times).

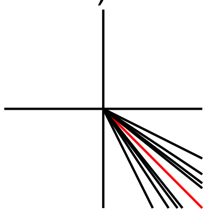
Proof. One can given an explicit algorithm. Also, there is a program for SageMath [N21]. □

Examples: Let

$$B = \begin{pmatrix} 0 & -\delta_1 \\ \delta_2 & 0 \end{pmatrix} = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\delta_1, \delta_2 \in \mathbb{Z}_{>0}).$$

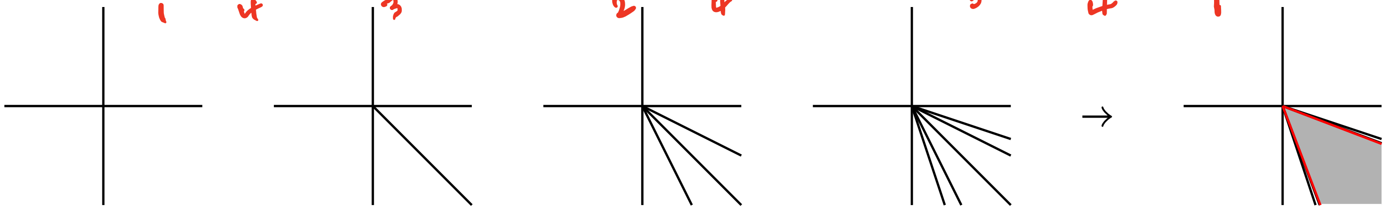
(1). type $A_1^{(1)}$: $(\delta_1, \delta_2) = (2, 2)$. ([Reineke11], [Matsushita21] by the pentagon relation)

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}^2 \cdots \prod_{j=0}^{\infty} \begin{bmatrix} 2^j \\ 2^j \end{bmatrix}^{2^{2-j}} \cdots \begin{bmatrix} 2 \\ 3 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2.$$



(2). non-affine type: $(\delta_1, \delta_2) = (3, 3)$. Use my program! ← the Badlands

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}^3 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}^3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}^9 \begin{bmatrix} 3 \\ 2 \end{bmatrix}^{39} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^9 \begin{bmatrix} 2 \\ 2 \end{bmatrix}^{18} \begin{bmatrix} 2 \\ 3 \end{bmatrix}^{39} \begin{bmatrix} 1 \\ 2 \end{bmatrix}^9 \begin{bmatrix} 1 \\ 3 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}^3 \mod G^{>5}.$$



deg ≤ 1 deg ≤ 2 deg ≤ 3 deg ≤ 4

Theorems on CSDs

Theorem A. (Positive realization [GHKK18])

For any skew-symmetrizable matrix B, there is a CSD $\mathfrak{D}(B)$ such that any wall element have the form $\Psi[n]^{\delta(n)}$.

To prove it, an alternative construction of a CSD was introduced in [GHKK18].

Theorem B. ([GHKK18])

For a CSD $\mathfrak{D}(B)$ with minimal support, the corresponding G-fan is embedded in $\mathrm{Supp}(\mathfrak{D}(B))$ under the identification $M_{\mathbb{R}} \simeq \mathbb{R}^r$ with $e_i^* \mapsto \delta_i \mathbf{e}_i$.

the construction in Theorem A $\Longrightarrow$ the *mutation invariance* of $\mathfrak{D}(B)$ $\Longrightarrow$ Theorem B.

Theorem B $\Longrightarrow$ the *sign-coherence of C-matrices*.

Theorems A & B $\Longrightarrow$ the *Laurent positivity*.

Modifying the construction for Theorem A with Ordering Lemma, we obtain the following result.

Theorem C. ([N21])

Every consistency relation of a CSD $\mathfrak{D}(B)$ reduces to a trivial one $g = g$ by applying the commutative relation and the pentagon relation (possibly infinitely many times).

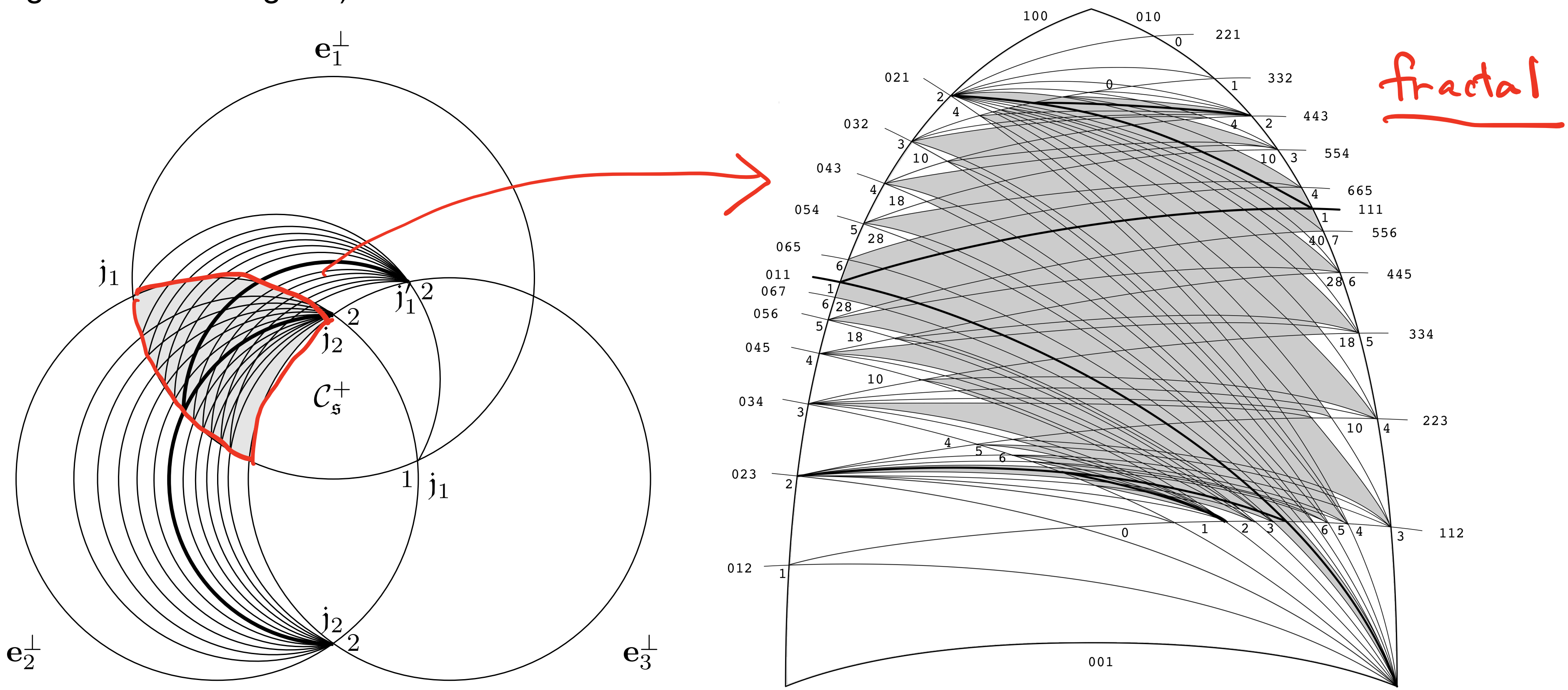
Summary (Message)

- The dilogarithm interporates the two principles (mutation and the consistency).
- The dilogarithm elements and the pentagon relation are *everything* for CSDs.

Example: the Badlands in a rank 3 CSD

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \Delta = I, \quad \Omega = B.$$

the stereo graphic projection of the support: (The right figure is the magnified one of the shaded region in the left figure.)



See

[N21] T. Nakanishi, *Cluster algebras and scattering diagrams, Part III. Cluster scattering diagrams*, preliminary draft for a monograph, arXiv:2111.00800, 106 pp.