Facets of Entropy

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Preliminaries

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- In information theory, entropy is the measure of the uncertainty contained in a discrete random variable, justified by fundamental coding theorems.

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- E.g., $n = 3$, the $2^3 - 1 = 7$ joint entropies are

$$H(X_1), H(X_2), H(X_3), H(X_1, X_2), H(X_2, X_3),$$

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- $H_\Omega : 2^{\mathcal{N}_n} \rightarrow \mathbb{R}$ is set function with $H_\Omega(\phi) = 0$.
- H_Ω is called the entropy function of Ω.

The Entropy Function as a Polymatroid

- It is well-known that for any Ω, H_Ω satisfies the following *polymatroidal axioms*. For any $\alpha, \beta \subset \mathcal{N}_n$,

  (P1) $H_\Omega(\phi) = 0$;

  (P2) $H_\Omega(\alpha) \leq H_\Omega(\beta)$ if $\alpha \subset \beta$;

  (P3) $H_\Omega(\alpha) + H_\Omega(\beta) \geq H_\Omega(\alpha \cap \beta) + H_\Omega(\alpha \cup \beta)$.

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- These are called Shannon's information measures.

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- That is,

$$\begin{aligned} \text{entropy} &\geq 0 \\ \text{mutual info} &\geq 0 \\ \text{conditional entropy} &\geq 0 \\ \text{conditional mutual info} &\geq 0 \end{aligned}$$

Laws of Information Theory

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- Pippenger's Question (1986): Are there any constraints on entropies other than the basic inequalities?

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- Define the region in $\mathcal{H}_n$:

$$\Gamma_n^* = \{\mathbf{h} \in \mathcal{H}_n : \mathbf{h} \text{ is entropic}\}$$

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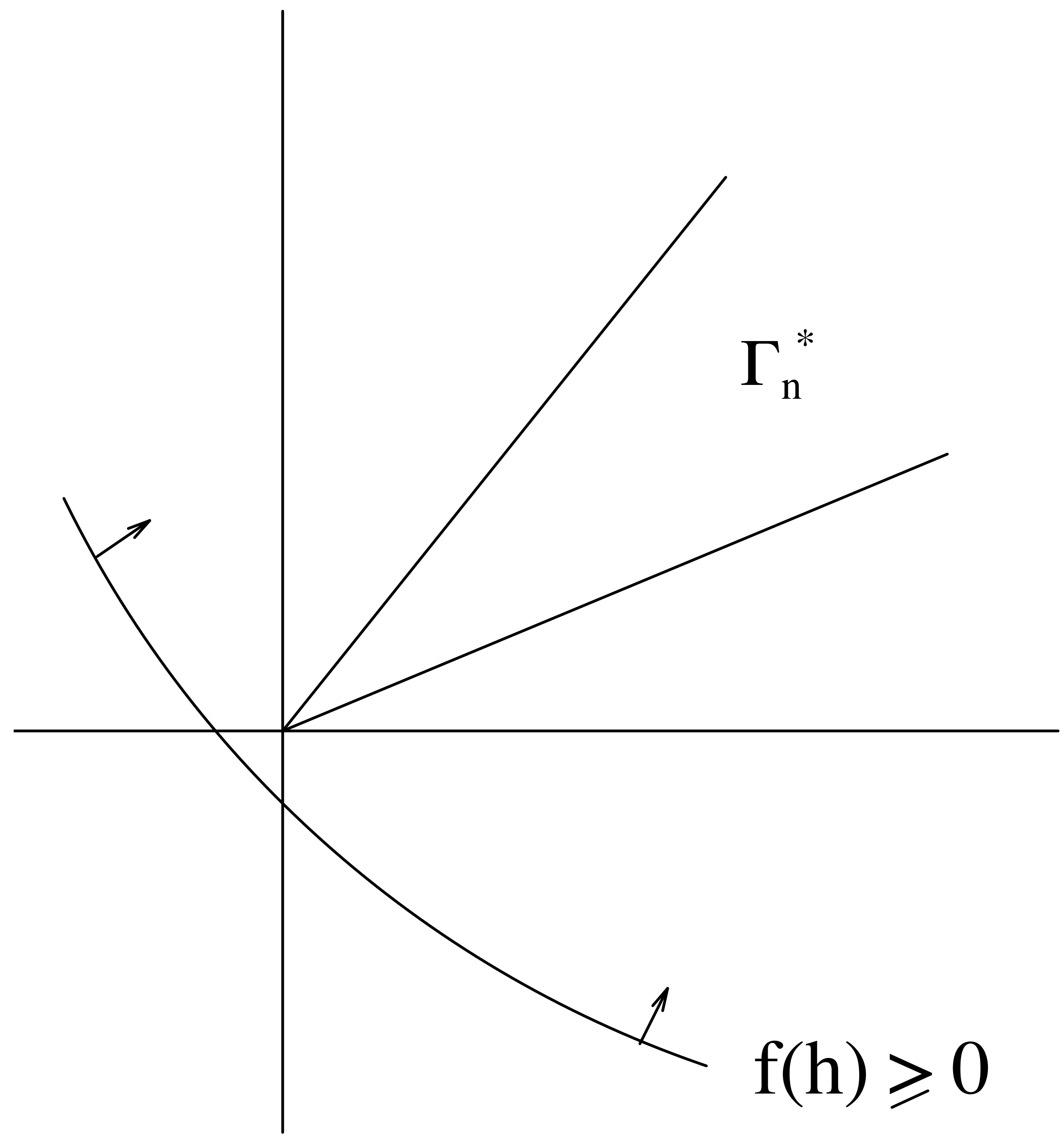
- In fact, $f(\mathbf{h}) \geq 0$ always holds if and only if

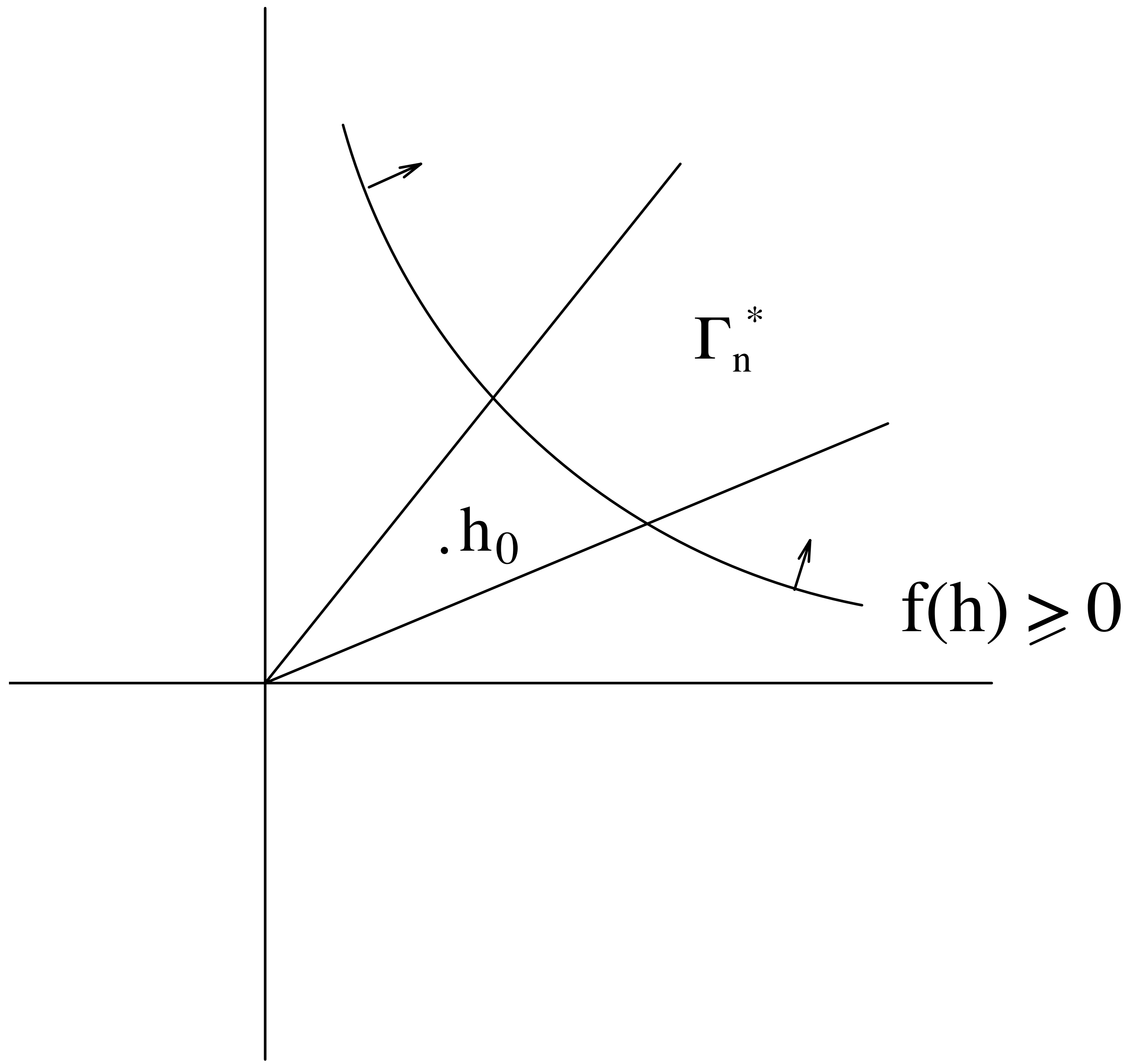
$$\overline{\Gamma}_n^* \subset \{\mathbf{h} \in \mathcal{H}_n : f(\mathbf{h}) \geq 0\}$$

because $\{\mathbf{h} \in \mathcal{H}_n : f(\mathbf{h}) \geq 0\}$ is closed.

$f(\mathbf{h}) \geq 0$ Always holds



$f(\mathbf{h}) \geq 0$ Does Not Always holds



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- $\Gamma_n^* \subset \Gamma_n$ since the basic inequalities are satisfied by any $X_1, \ldots, X_n$.

- An entropy inequality $f(\mathbf{h}) \geq 0$ is called a Shannon-type inequality if it is implied by the basic inequalities, or

$$\Gamma_n \subset \{\mathbf{h} \in \mathcal{H}_n : f(\mathbf{h}) \geq 0\}.$$

Machine-Proving of Entropy Inequalities

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ITIP and Xitip are linear programming based, while ITTP is axiom based.

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Pippenger's Problem Rephrased

- Is $\overline{\Gamma}^*_n = \Gamma_n$?
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- It is known that
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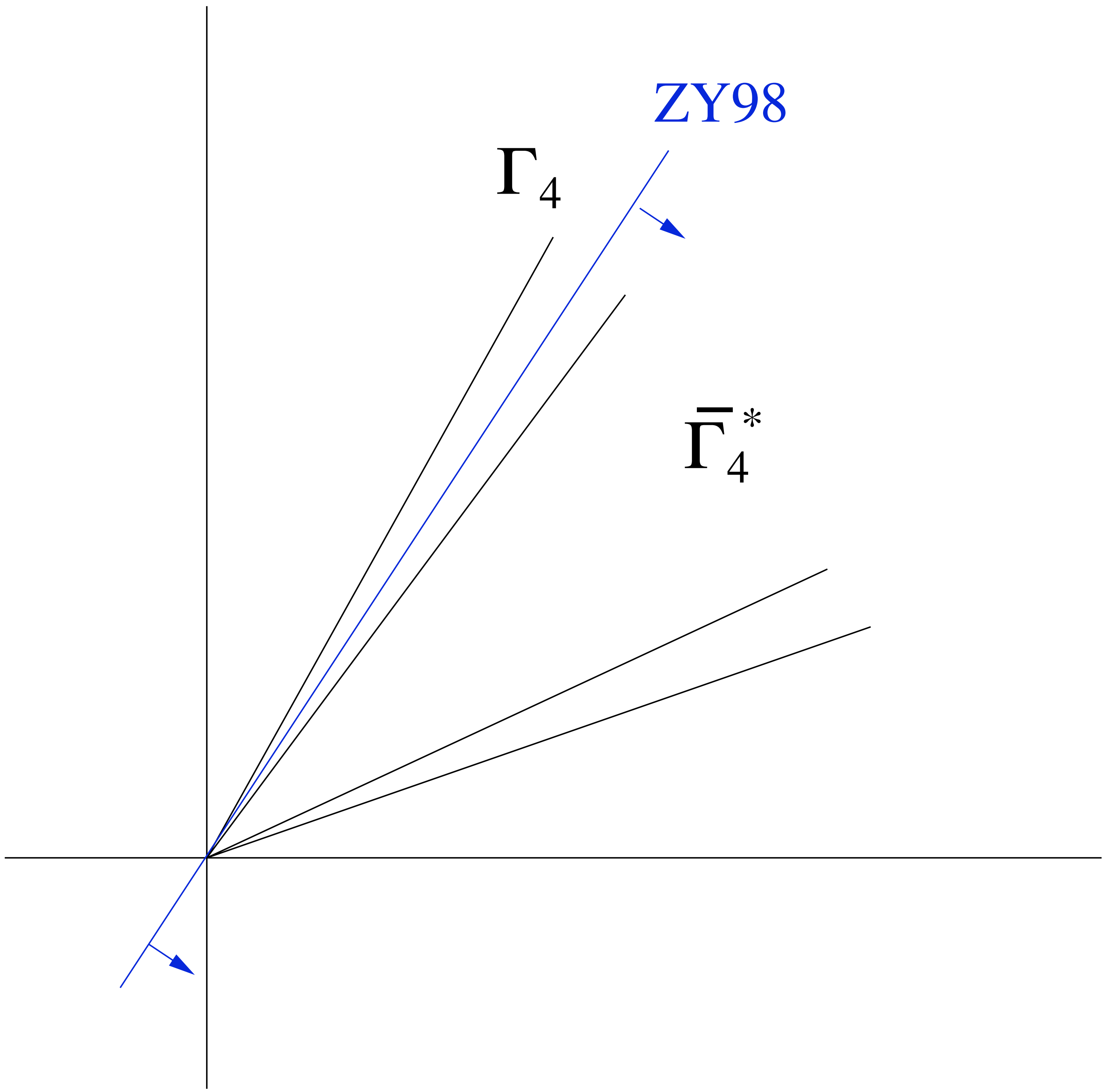
  2. $\Gamma^*_3 \neq \Gamma_3$, but $\overline{\Gamma}^*_3 = \Gamma_3$
- Therefore, unconstrained non-Shannon-type inequalities can exist only for 4 or more random variables.
- In general,
  - Γ^*_n is neither closed nor convex, but $\overline{\Gamma}^*_n$ is a convex cone.

A Non-Shanon-Type Inequality

- The following unconstrained non-Shannon-type inequality was discovered by Zhang and Y (1998) for any 4 random variables:

$$\begin{aligned} & I(Z;U) - I(Z;U|X) - I(Z;U|Y) \\ \leq \quad & \frac{1}{2} I(X;Y) + \frac{1}{4}[I(X;Z,U) + I(Y;Z,U)] \end{aligned}$$

An Illustration of ZY98



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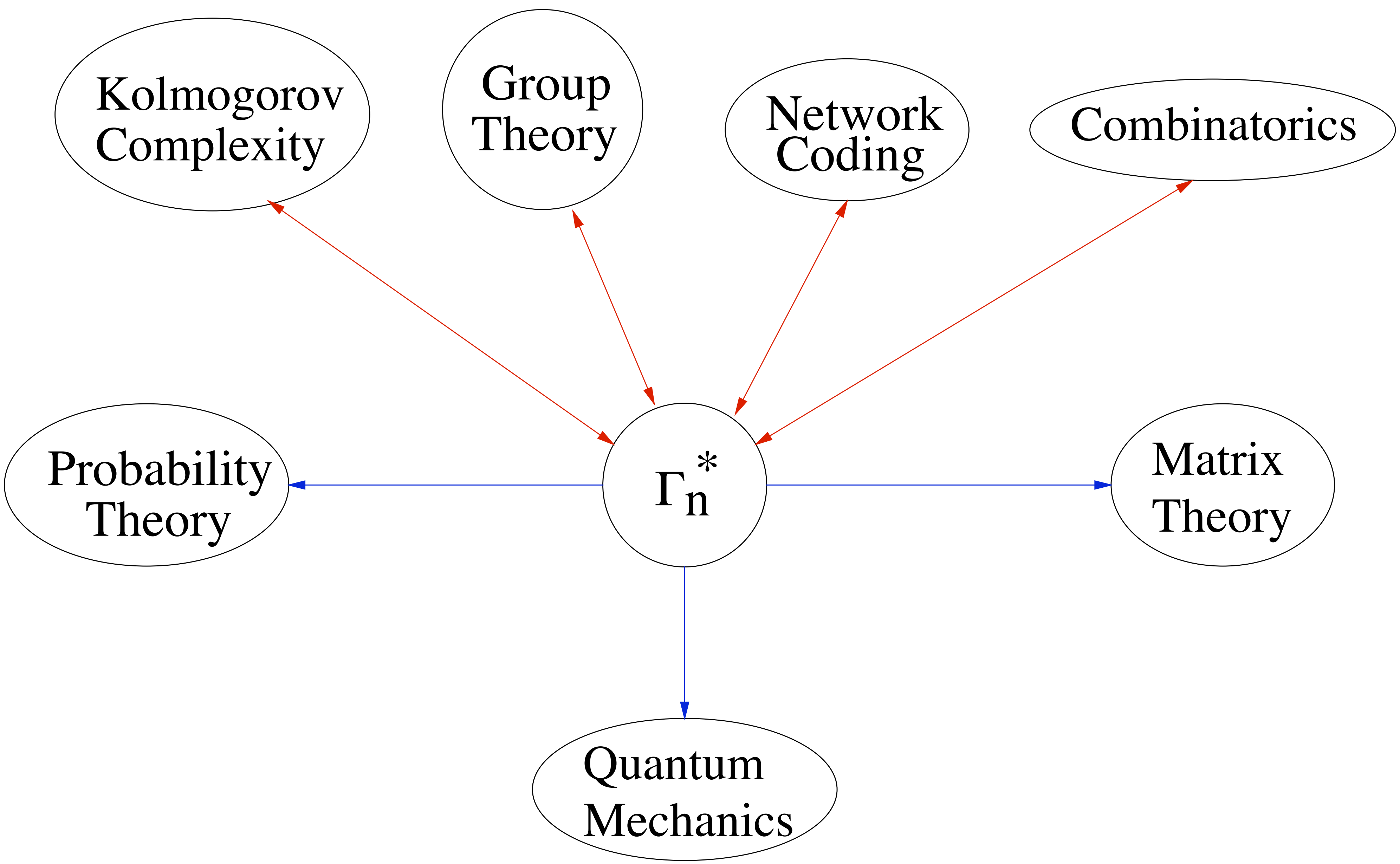
- In particular, Matúš showed that $\overline{\Gamma}_n^*$ is not a polytope, and hence there exist an infinitely number of linear non-Shannon-type inequalities!
- Dougherty, Freiling and Zeger (2006) have discovered several tens of non-Shannon-type inequalities by a search on the supercomputer at UCSD.

Subjects Related to Γ_n^*



COMBINATORICS

2-D Quasi-Uniform Array

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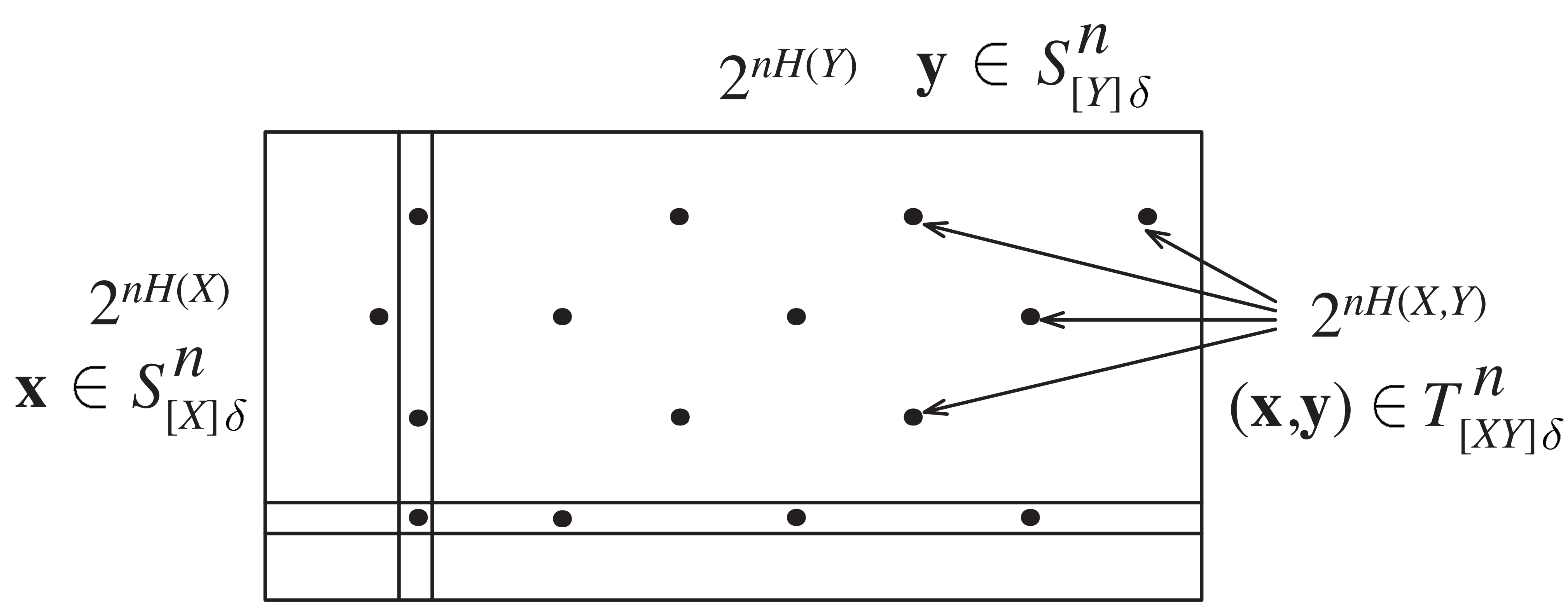
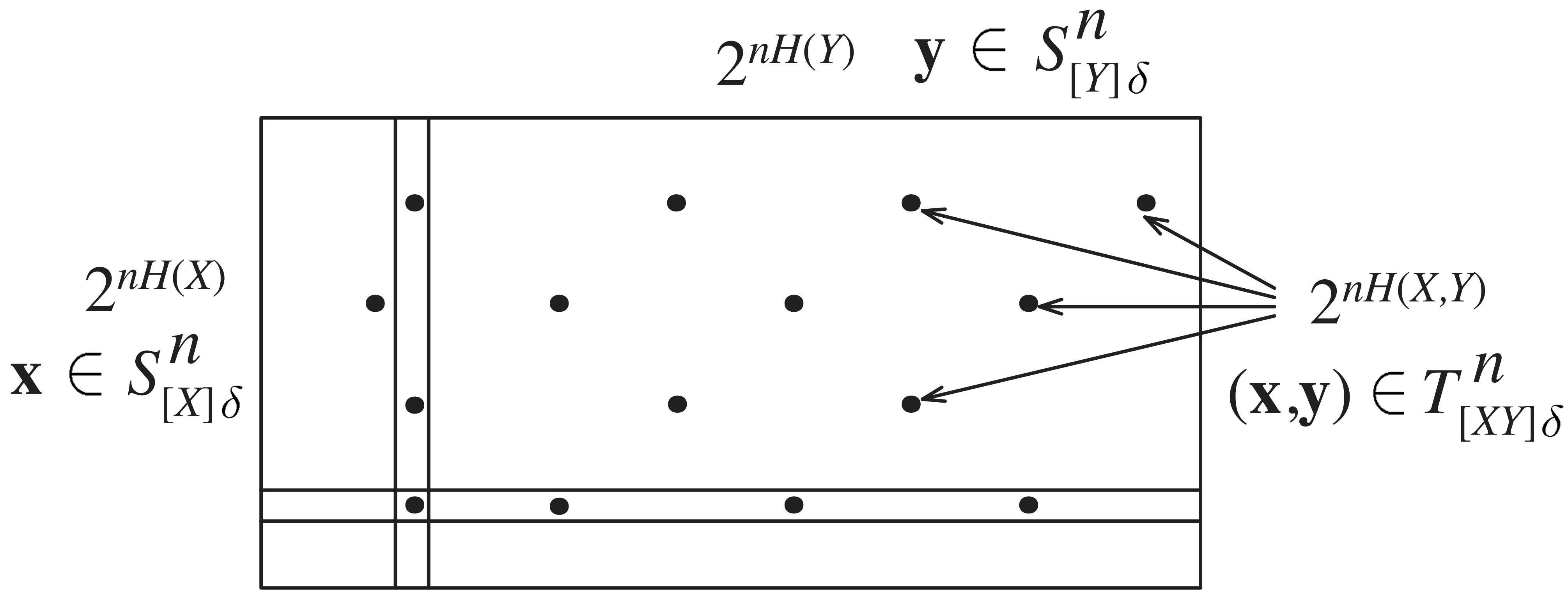
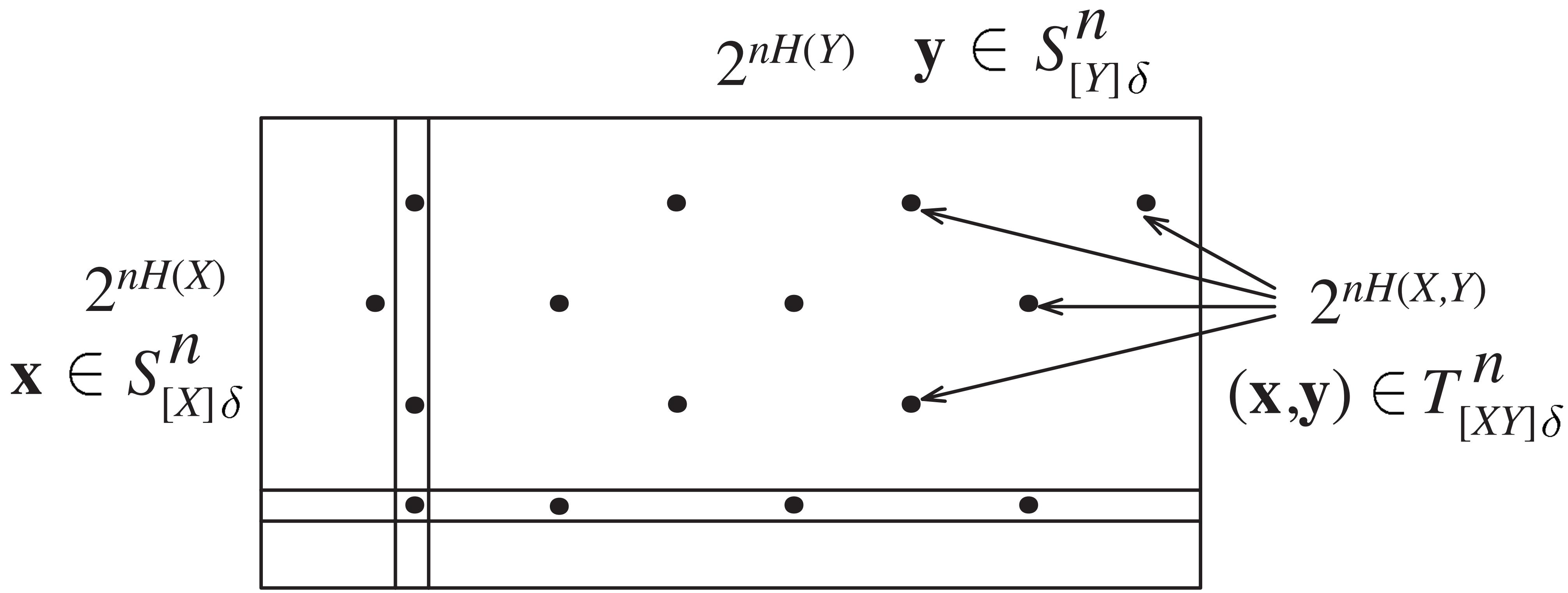
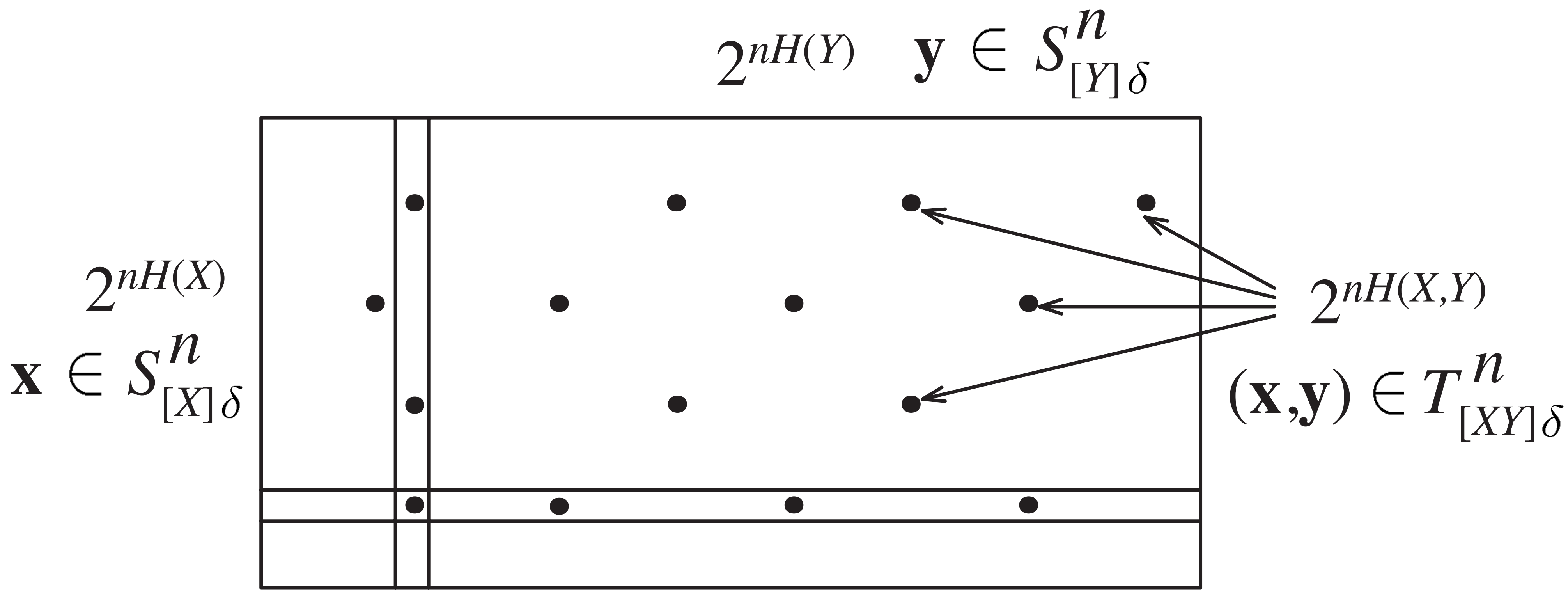



$2^{nH(Y)}$ $\mathbf{y} \in S^n_{[Y]\delta}$
$2^{nH(X)}$
$\mathbf{x} \in S^n_{[X]\delta}$
$2^{nH(X,Y)}$
$(\mathbf{x},\mathbf{y}) \in T^n_{[XY]\delta}$



- Each row has approximately the same number of dots ($\sim 2^{nH(Y|X)}$) and each column has approximately the same number of dots ($\sim 2^{nH(X|Y)}$).



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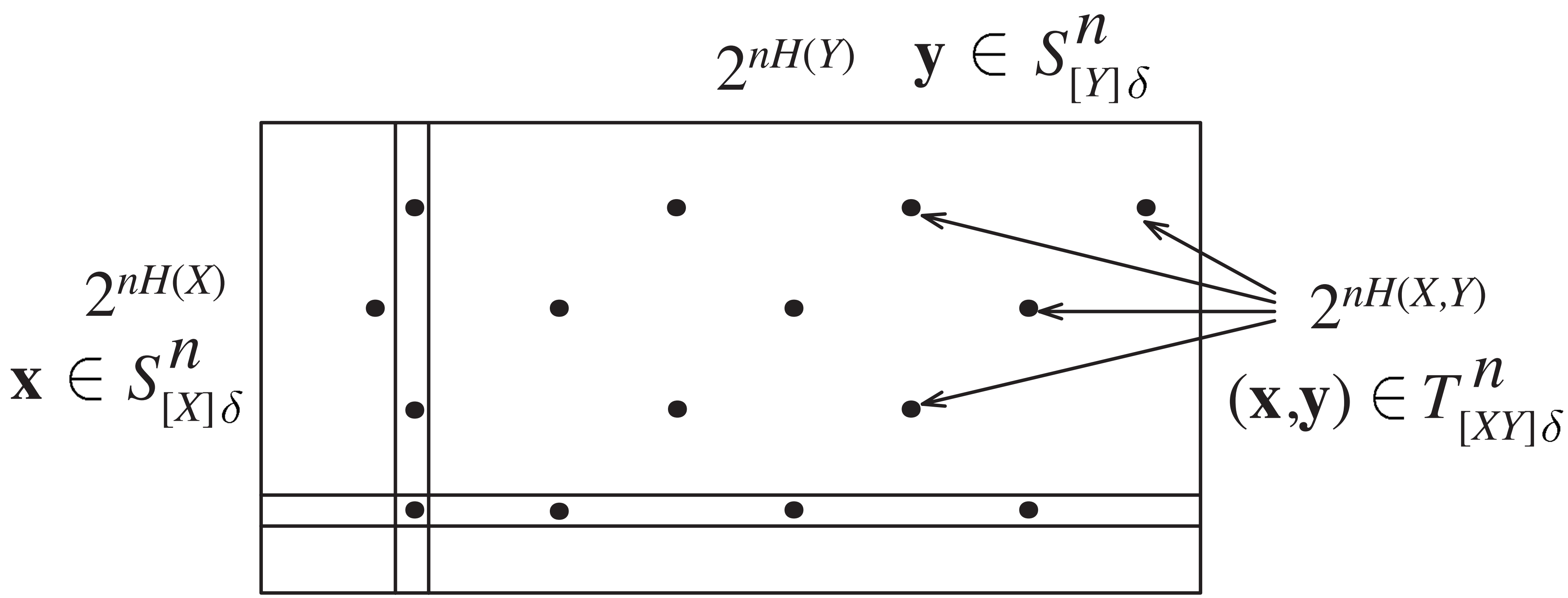
- Thus

$$2^{nH(X,Y)} \leq 2^{nH(X)} 2^{nH(Y)} \Rightarrow H(X,Y) \leq H(X) + H(Y)$$



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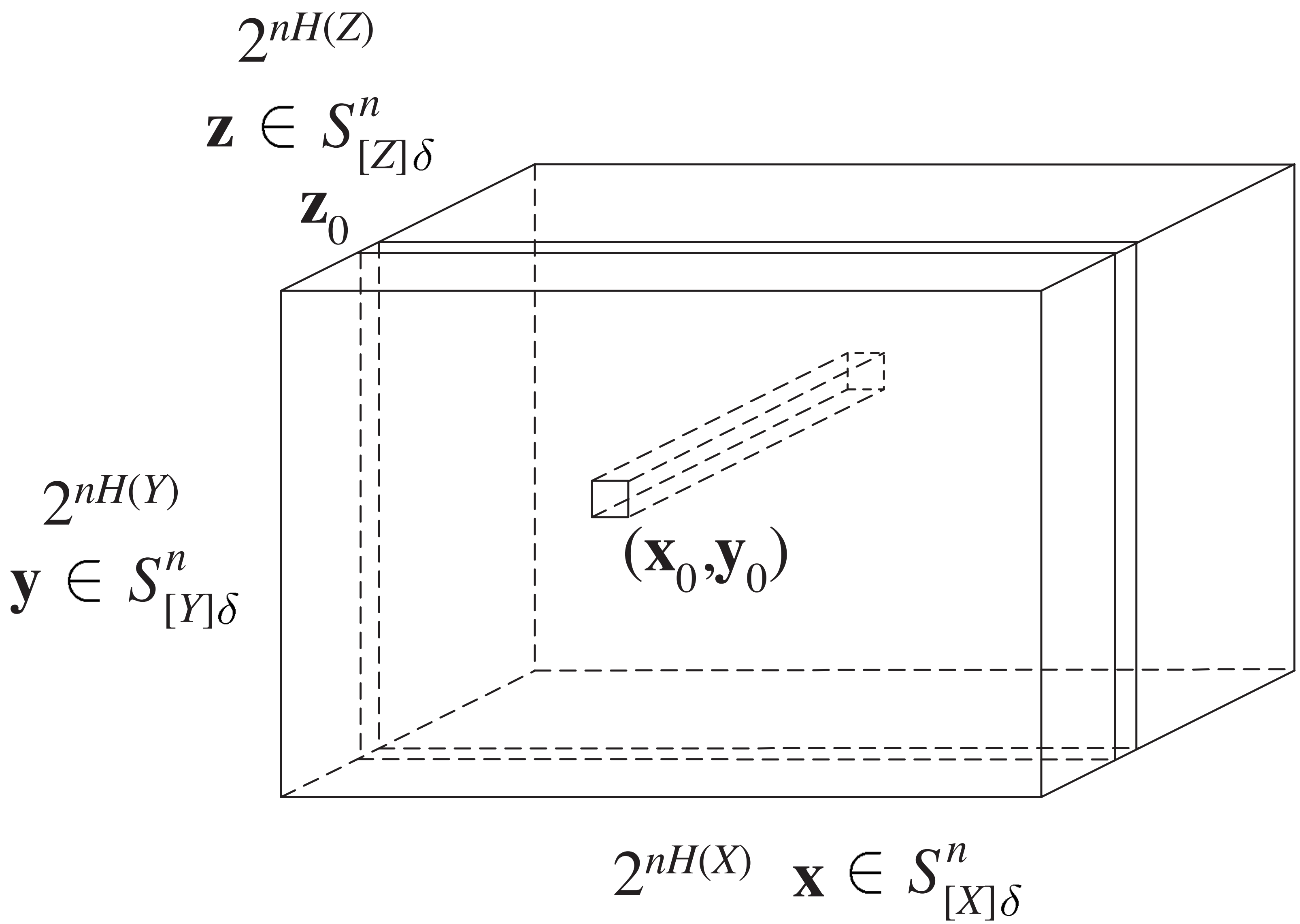
- Then the basic inequality $I(X;Y) \geq 0$ is about the unfilled entries in the array.

3-D Quasi-Uniform Array



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- For an n-dimensional quasi-uniform array, if all the "dots" are assigned equal probabilities, then the projection on every lower dimensional plane has a uniform distribution over its support.
- Do quasi-uniform arrays fully capture all constraints on the entropy function?
- YES. T. Chan (2001) showed that all constraints on the entropy function can be obtained through quasi-uniform arrays, and vice versa.

GROUP THEORY

Entropy and Groups
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Entropy and Groups (Chan-Y 99)

- Let G be a finite group and $G_1, G_2, \ldots, G_n$ be subgroups of G.
- Let $G_\alpha = \cap_{i\in\alpha} G_i$, also a subgroup.
- A probability distribution for n random variables $X_1, X_2, \ldots, X_n$ can be constructed from any finite group G and subgroups $G_1, G_2, \ldots, G_n$, with

$$H(X_\alpha) = \log \frac{|G|}{|G_\alpha|}$$

which depends only on the orders of G and $G_1, G_2, \ldots, G_n$.

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- For example, for any X_1, X_2,

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corresponds to for any finite group G and subgroups G_1, G_2,

$$\log \frac{|G|}{|G_1|} + \log \frac{|G|}{|G_2|} \geq \log \frac{|G|}{|G_1 \cap G_2|}$$

or

$$|G||G_1 \cap G_2| \geq |G_1||G_2|$$

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- For example, ZY98 can be written as

$$
\begin{aligned}
H(X_1) + H(X_2) & & 6H(X_3, X_4) \\
+2H(X_1, X_2) & & +4H(X_1, X_3) \\
+4H(X_3) + 4H(X_4) & \leq & +4H(X_1, X_4) \\
+5H(X_1, X_3, X_4) & & +4H(X_2, X_3) \\
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\end{aligned}
$$

Non-shannon-Type Group Inequalities

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- This corresponds to

$$\begin{aligned} |G_3 \cap G_4|^6 |G_1 \cap G_3|^4 & & |G_1||G_2||G_3|^4|G_4|^4 \\ |G_1 \cap G_4|^4 |G_2 \cap G_3|^4 & \leq & |G_1 \cap G_2|^2 |G_1 \cap G_3 \cap G_4|^5 \\ |G_2 \cap G_4|^4 & & |G_2 \cap G_3 \cap G_4|^5 \end{aligned}$$

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- It can be proved that the correspondence between entropy inequalities and group inequalities is one-to-one.

Relation between Finite Group and Quasi-Uniform Array

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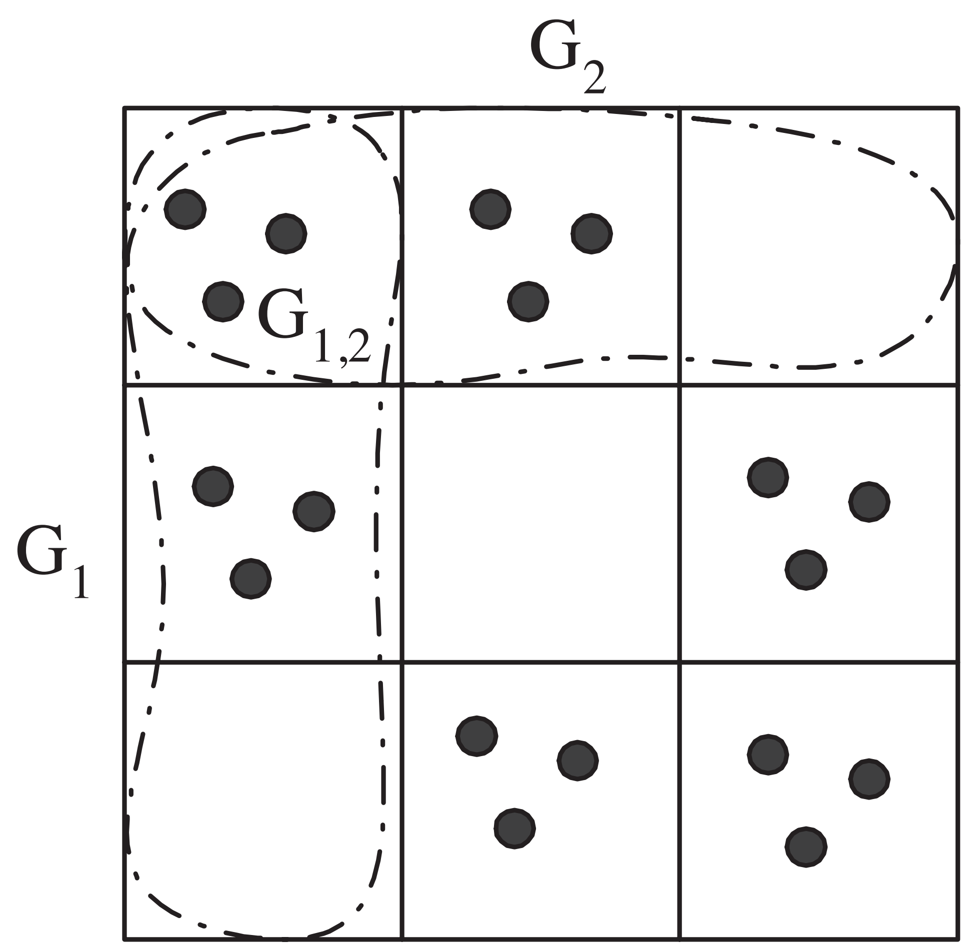

KOLMOGOROV COMPLEXITY

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- "Non-Shannon-type" Kolmogorov complexity inequalities can be obtained accordingly.

PROBABILITY THEORY

Compatibility of Conditional Independence

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- The Implication Problem Is a given conditional independency implied by a given set of conditional independencies?

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- Matroid Theory is a powerful tool for studying this problem.

MATRIX THEORY

Differential Entropy Inequalities

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- Chan (2006) showed that a differential entropy inequality is valid iff the coefficients of the random variables are *balanced* and its discrete counterpart is valid.

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  is valid.

- The coefficients in ZY98 are balanced, so it is also valid for differential entropy.

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- Then

$$h(\mathbf{X}) = \frac{1}{2} \log \left[(2\pi e)^n |K| \right]$$

and for any subset α of $\{1, 2, \ldots, n\}$,

$$h(\mathbf{X}_\alpha) = \frac{1}{2} \log \left[(2\pi e)^{|\alpha|} |K_\alpha| \right]$$

where K_α is the corresponding submatrix of K.

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- Substituting these joint differential entropies into the inequality

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$$\begin{aligned} & |K_1||K_2||K_{1,2}|^2|K_3|^4|K_4|^4|K_{1,3,4}|^5|K_{2,3,4}|^5 \\ \leq \quad & |K_{3,4}|^6|K_{1,3}|^4|K_{1,4}|^4|K_{2,3}|^4|K_{2,4}|^4 \end{aligned}$$

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- Many other "non-Shannon-type" determinant inequalities can be obtained this way.

NETWORK CODING

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- For multi-source network coding, the problem has been studied by Y and Zhang (1999), Song, Y, and Cai (2006). Yan, Y and Zhang (2007) finally obtained a complete characterization (implicit) of the network capacity in terms of Γ_n^*.

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Every constraint on the entropy function is useful in
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- The implications of non-Shannon-type inequalities in information theory is finally understood in the context of network coding.

Secret Sharing

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- Recently, Beimel et al. (2008) has applied ZY98 to obtain a lower bound in a secret sharing problem.
- Secret sharing can be regarded as a special case of secure network coding (Cai and Y, 2002).

QUANTUM MECHANICS

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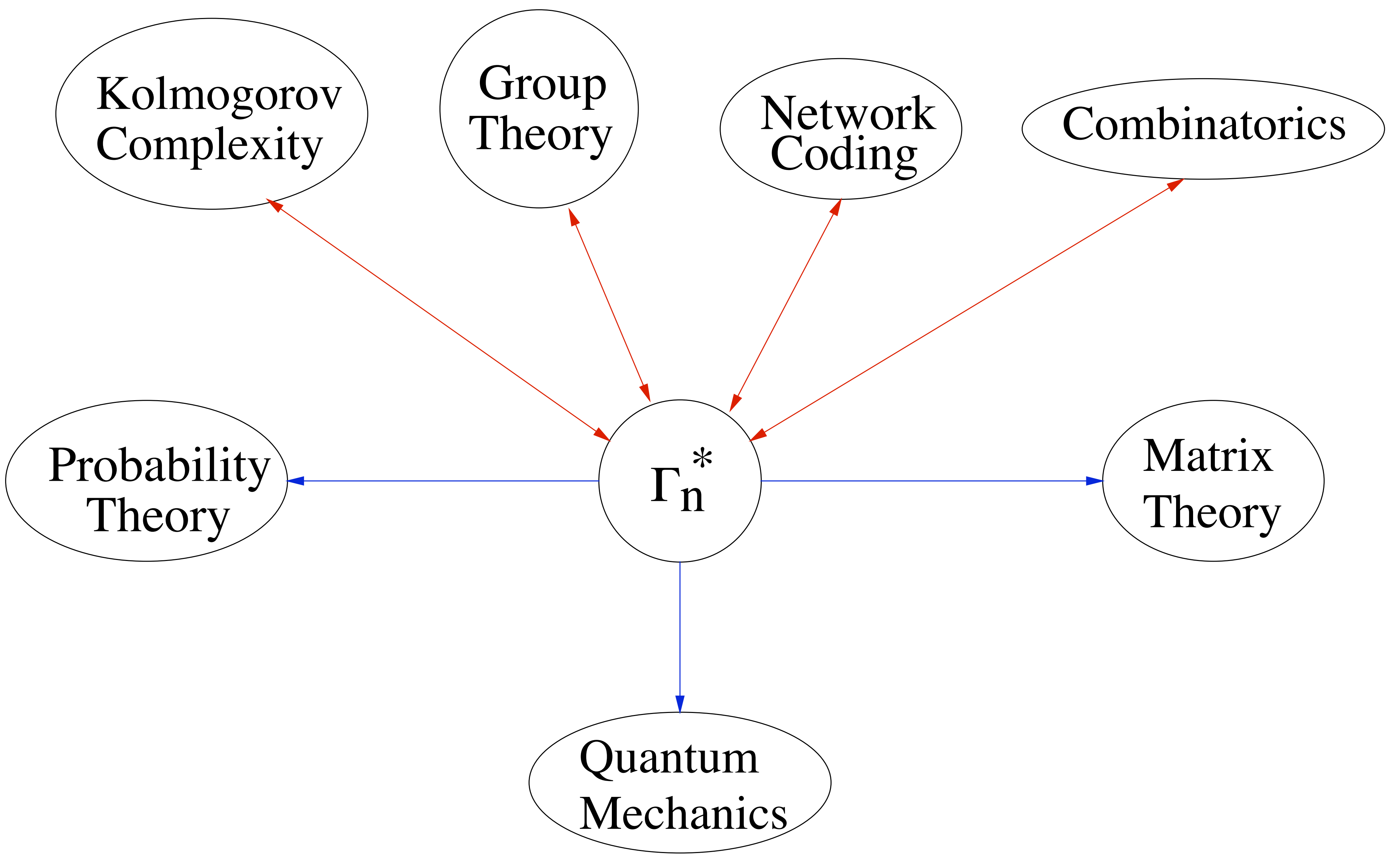
- Linden and Winter (2005) discovered for a 4-party system a constrained inequality for the von Neumann entropy which is independent of strong subadditivity.

Summary



Kolmogorov Complexity
Group Theory
Network Coding
Combinatorics
Probability Theory
Γ_n^*
Matrix Theory
Quantum Mechanics

Concluding Remarks

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- Matroid theory plays a role here and there, in particular in the study of conditional independence and network coding.
- The relations between these fields need a deeper understanding. The combinatorial structure to study is the quasi-uniform array.
- "Non-Shannon-type" inequalities in different fields need further understanding.

ARCHIVE

From nicholas@cs.ubc.ca Tue Jul 7 23:04:11 1998
X400-Received: by /PRMD=ca/ADMD=telecom.canada/C=ca/; Relayed; Tue, 7 Jul
1998
8:03:59 UTC-0700
Date: Tue, 7 Jul 1998 8:03:59 UTC-0700
X400-Originator: nicholas@cs.ubc.ca
X400-Recipients: non-disclosure:;
X400-Content-Type: P2-1984 (2)
X400-MTS-Identifier: [/PRMD=ca/ADMD=telecom.canada/C=ca/;980707080359]
Content-Identifier: 4429
X-UIDL: 900031892.048
From: Nicholas Pippenger <nicholas@cs.ubc.ca>
To: zzhang@milly.usc.edu, whyeung@ie.cuhk.edu.hk
MIME-Version: 1.0 (Generated by Ean X.400 to MIME gateway)

I have just seen your paper "On the Characterization of Entropy
Function via Information Inequalities" in the IEEE Transactions on
Information Theory. Please allow me to congratulate you on a most
beautiful result! I worked on the problem of whether \overbar{\Gamma}^*_n
= \Gamma_n during the 80s, without any success. I presented it as an open
problem at the SPOC (Specific Problems on Communication and Computation)
Conference in 1986--I believe there were proceedings published by Springer,
but they seem to be out of print now.

It was wonderful to see your paper.

 - Nick Pippenger

What Are the Laws of Information Theory?

Nicholas Pippenger
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650 Harry Road
San Jose, California 95120-6099

Shannon defined the *entropy* $H(X)$ of a random variable X assuming values in a finite set $\mathcal{X}$ to be $-\sum_{x\in\mathcal{X}} \Pr(X=x)\log\Pr(X=x)$. The entropy $H(X,Y,Z)$ of a finite set $\{X,Y,Z\}$ of random variables is defined by regarding the tuple (X,Y,Z) as a single random variable. In information theory, one also deals with *conditional entropies*, like $H(X \mid Y) = H(X,Y) - H(Y)$; *mutual informations*, like $I(X;Y) = H(X) + H(Y) - H(X,Y)$; and *conditional mutual informations*, like $I(X;Y \mid Z) = H(X,Y) + H(X,Z) - H(X,Y,Z) - H(Z)$. All identities and inequalities concerning these quantities, however, can be reduced to ones involving only "plain" entropies, like $H(X,Y,Z)$, by invoking these definitions. The identities are known (see [H] and [R]). The problem posed here is to determine the inequalities.

If $\{X_t\}_{t\in T}$ is a family of random variables, and if $S \subseteq T$, let H_S denote the entropy of the subfamily $\{X_s\}_{s\in S}$. The resulting map $H : 2^T \to \mathbf{R}$ satisfies the following conditions (known as the *polymatroid axioms*).

(1) $H_S \geq 0$ and $H_\emptyset = 0$.
(2) $H_R \leq H_S$ if $R \subseteq S$.
(3) $H_{R\cup S} + H_{R\cap S} \leq H_R + H_S$.

These conditions are immediate consequences of the fact that the logarithm vanishes at unity, is increasing and is concave. Are there any other conditions? If so, what are they? If not, show that any function satisfying (1), (2) and (3) can be approximated arbitrarily closely by the entropies of some family of random variables. (I say "approximated arbitrarily closely" to avoid the question of what happens on the boundary of the polytope defined by (1), (2) and (3).)

[H] K. T. Hu, "On the Amount of Information", *Theory of Prob. and Appl.*, 7 (1962) 439–447.

[R] F. M. Reza, *An Introduction to Information Theory*, McGraw-Hill, New York, 1961.

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ACKNOWLEDGMENTS

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Thank You