A Murnaghan-Nakayama Rule for k-Schur Functions

Anne Schilling (joint work with Jason Bandlow, Mike Zabrocki)

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Outline

History

The Murnaghan-Nakayama rule

The affine Murnaghan-Nakayama rule

Non-commutative symmetric functions

The dual formulation

Early history - Representation theory

Theorem (Frobenius, 1900)

The map from class functions on S_n to symmetric functions given by

$$f \mapsto \frac{1}{n!} \sum_{w \in S_n} f(w) p_{\lambda(w)}$$

sends

$$(\text{ trace function on } \lambda\text{-irrep of } S_n \text{ }) \mapsto s_\lambda$$

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Corollary

$$s_\lambda = \sum_\mu \frac{1}{z_\mu} \chi_\lambda(\mu) p_\mu \qquad p_\mu = \sum_\lambda \chi_\lambda(\mu) s_\lambda$$

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Early History - Combinatorics

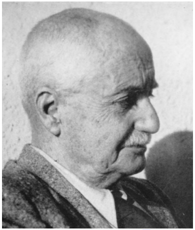
Theorem (Littlewood-Richardson, 1934)

$$p_r s_\mu = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda$$

where the summation is over all λ *such that* λ/μ *is a border strip of size* r*.*

Dudley Littlewood

Archibald Richardson



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Corollary

Iteration gives

$$\chi_\lambda(\mu) = \sum_T (-1)^{\mathrm{ht}(T)}$$

where the sum is over all border strip tableaux of shape λ and type μ.

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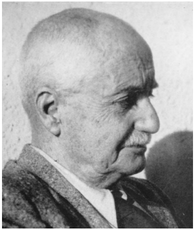

Early History - Further work

- Francis Murnaghan (1937) *On representations of the symmetric group*

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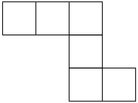
- Tadasi Nakayama (1941) *On some modular properties of irreducible representations of a symmetric group*

Border Strips

A *border strip* of size r is a connected skew partition consisting of r boxes and containing no 2×2 squares.

Example

$(4,3,3)/(2,2)$ is a border strip of size 6:



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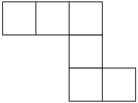

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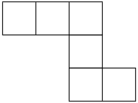

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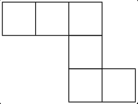

$$\left. \vphantom{\rule{0pt}{3em}} \right) = 2$$

The Murnaghan-Nakayama rule

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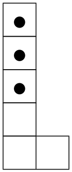

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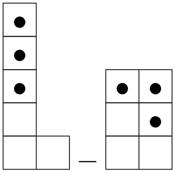

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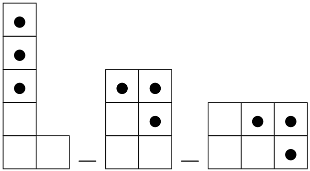

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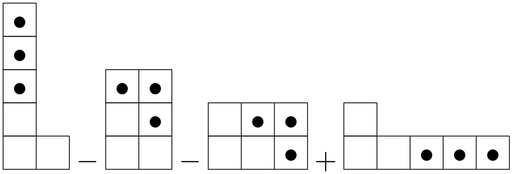
Example

$$p_3 s_{2,1} = s_{2,1,1,1,1} - s_{2,2,2} - s_{3,3} + s_{5,1}$$



Border strip tableaux

Definition

A *border strip tableau* of shape λ is a filling of λ satisfying:

- Restriction to any single entry is a border strip
- Restriction to first k entries is partition shape for every k

Type of a border strip tableau: $(\#$ of boxes labelled $i)_i$
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$$T = \begin{array}{|c|c|c|c}\cline{1-3} 1 & 3 & 3 \\ \cline{1-3} 1 & 2 & 3 \\ \hline 1 & 1 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$\text{type}(T) = (4, 1, 5)$$

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The affine Murnaghan-Nakayama rule

Theorem (Bandlow-S-Zabrocki, 2010)

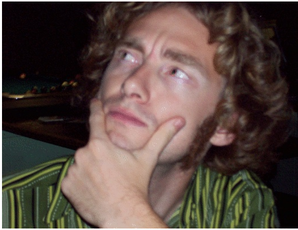
For $r \leq k$,

$$p_r s_\mu^{(k)} = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda^{(k)}$$

where the summation is over all λ such that λ/μ is a k-border strip of size r.

The affine Murnaghan-Nakayama rule

Jason Bandlow



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k-Schur functions

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$$s_\lambda^{(k)}(x;t) = \sum_{T \in A_\lambda^{(k)}} t^{ch(T)} s_{sh(T)}$$

k-Schur functions

Here we use the definition due to Lapointe and Morse in 2004:

$$h_r s_\lambda^{(k)}(x) = \sum_\mu s_\mu^{(k)}(x) \qquad \text{Pieri rule}$$

where the sum is over those μ such that $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ is a horizontal strip.

Partitions and cores

k-bounded partitions: First part $\leq k$

$k+1$-cores: No hook length $= k+1$

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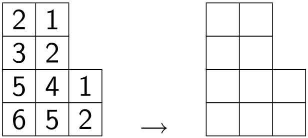
Example

$k = 3$



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$$\begin{array}{|c|c|c|}\hline 2 & 1 \\ \cline{1-2} 3 & 2 \\ \hline 5 & 4 & 1 \\ \hline 6 & 5 & 2 \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|c|}\hline 2 & 1 \\ \cline{1-2} 3 & 2 \\ \hline & 5 & 2 & 1 \\ \hline & 6 & 3 & 2 \\ \hline \end{array}$$

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\hline
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$\longrightarrow$

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3	2				
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			4	3	1

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\cline{1-2}
3 & 2 \\
\cline{1-5}
 & & 3 & 2 & 1 \\
\hline
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\hline
\end{array}
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\begin{array}{ccc}
2 & 1 & \\
3 & 2 & \\
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6 & 5 & 2
\end{array}
\quad \longrightarrow \quad
\begin{array}{cccccccc}
2 & 1 & & & & & & \\
3 & 2 & & & & & & \\
\color{red}{7} & \color{red}{6} & 3 & 2 & 1 & & & \\
\color{red}{11} & \color{red}{10} & \color{red}{7} & \color{red}{6} & \color{red}{5} & 3 & 2 & 1
\end{array}
$$

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The k-conjugate of a k-bounded partition λ is found by

$$\lambda \to \mathfrak{c}(\lambda) \to \mathfrak{c}(\lambda)' \to \lambda^{(k)}$$

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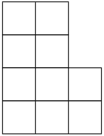

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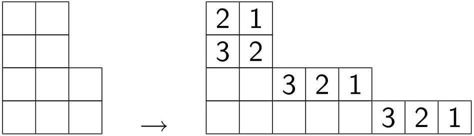

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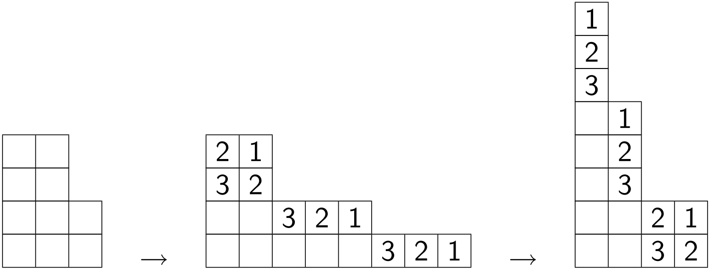

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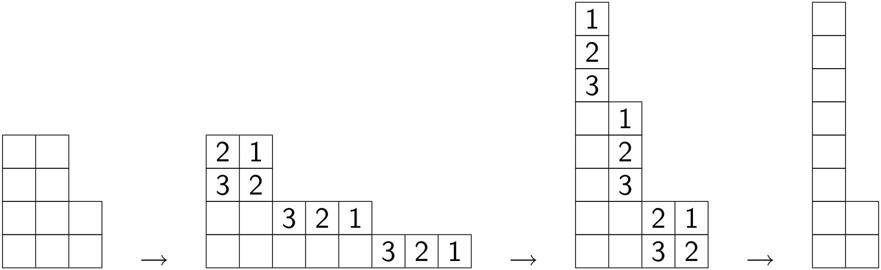

content

When $k = \infty$, the *content* of a cell in a diagram is

$$(\text{column index}) - (\text{row index})$$

Example

-3	-2		
-2	-1		
-1	0	1	2
0	1	2	3

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Example

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For $k<\infty$ we use the *residue* mod $k+1$ of the associated core

Example

1	2						
2	3						
3	0	1	2	3			
0	1	2	3	0	1	2	3

k-connected

A skew $k+1$ core is *k-connected* if the residues form a proper subinterval of the numbers $\{0, \ldots, k\}$, considered on a circle.

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Example

A 3-connected skew core:

0								
1	2							
2	3	0						
3	0	1	2	3	0			
0	1	2	3	0	1	2	3	0

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\begin{ytableau}
0 \\
{} & 2 \\
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{} & {} & {} & 2 & 3 & 0 \\
{} & {} & {} & {} & {} & {} & 2 & 3 & 0
\end{ytableau}
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0								
	2							
		0						
			2	3	0			
						2	3	0

A skew core which is not 3-connected:

0								
1	2							
2	3	0						
3	0	1	2	3	0			
0	1	2	3	0	1	2	3	0

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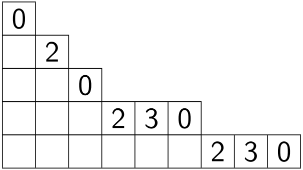

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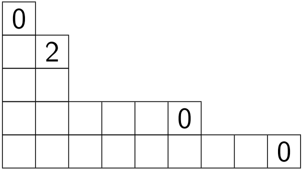

k-border strips

The skew of two k-bounded partitions λ/μ is a k-border strip of size r if it satisfies the following conditions:

- (size) $|\lambda/\mu| = r$
- (containment) $\mu \subset \lambda$ and $\mu^{(k)} \subset \lambda^{(k)}$
- (connectedness) $\mathfrak{c}(\lambda)/\mathfrak{c}(\mu)$ is k-connected
- (first ribbon condition) $\mathrm{ht}(\lambda/\mu) + \mathrm{ht}\left(\lambda^{(k)}/\mu^{(k)}\right) = r - 1$
- (second ribbon condition) $\mathfrak{c}(\lambda)/\mathfrak{c}(\mu)$ contains no 2×2 squares

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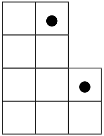
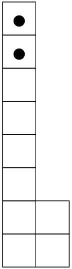
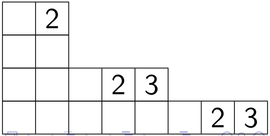
Example

$k = 3, r = 2$

$\lambda/\mu =$ 

$\lambda^{(3)}/\mu^{(3)} =$ 

$\mathfrak{c}(\lambda)/\mathfrak{c}(\mu) =$ 

k-ribbons at ∞

At $k = \infty$ the conditions

- (size) $|\lambda/\mu| = r$
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- (connectedness) $\mathfrak{c}(\lambda)/\mathfrak{c}(\mu)$ is k-connected
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Proposition

At $k = \infty$ the first four conditions imply the fifth.

The ribbon statistic at $k = \infty$

Let λ/μ be connected of size r, and

$\mathrm{ht}\,(\lambda/\mu) + \mathrm{ht}\,(\lambda'/\mu') = \#\text{vert. dominos} + \#\text{horiz. dominos} = r - 1$

Then λ/μ is a ribbon

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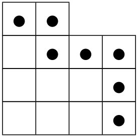
Example



$$3 + 3 = 6$$

The ribbon statistic at $k = \infty$

Let λ/μ be connected of size r, and

$$\mathrm{ht}\,(\lambda/\mu) + \mathrm{ht}\,(\lambda'/\mu') = \#\text{vert. dominos} + \#\text{horiz. dominos} = r - 1$$

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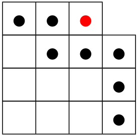

$$(3 + 1) + (3 + 1) = 8 \neq 7$$

Recapap for general k

Theorem (Bandlow-S-Zabrocki, 2010)

For $r \leq k$,

$$p_r s_\mu^{(k)} = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda^{(k)}$$

where the summation is over all λ such that λ/μ satifies

- *(size)* $|\lambda/\mu| = r$
- *(containment)* $\mu \subset \lambda$ *and* $\mu^{(k)} \subset \lambda^{(k)}$
- *(connectedness)* $\mathfrak{c}(\lambda)/\mathfrak{c}(\mu)$ *is k-connected*
- *(first ribbon condition)* $\mathrm{ht}(\lambda/\mu) + \mathrm{ht}\left(\lambda^{(k)}/\mu^{(k)}\right) = r - 1$
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Conjecture

The first four conditions imply the fifth.

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Conjecture

The first four conditions imply the fifth.

This has been verified for all $k, r \leq 11$, all μ of size ≤ 12 and all λ of size $|\mu| + r$.

The non-commutative setting

Sergey Fomin

Theorem (Fomin-Greene, 1998)

Any algebra with a linearly ordered set of generators $u_1, \dots, u_n$ satisfying certain relations contains a homomorphic image of Λ.

Example

The type A nilCoxeter algebra. Generators $s_1, \dots, s_{n-1}$. Relations

- $s_i^2 = 0$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- $s_i s_j = s_j s_i$ for $|i-j| > 2$.

Curtis Greene

The affine nilCoxeter algebra

The affine nilCoxeter algebra A_k is the $\mathbb{Z}$-algebra generated by $u_0, \ldots, u_k$ with relations

- $u_i^2 = 0$ for all $i \in [0, k]$
- $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for all $i \in [0, k]$
- $u_i u_j = u_j u_i$ for all i, j with $|i - j| > 1$

All indices are taken modulo $k + 1$ in this definition.

A word in the affine nilCoxeter algebra is called *cyclically decreasing* if

- its length is $\leq k$
- each generator appears at most once
- if u_i and u_{i-1} appear, then u_i occurs first (as usual, the indices should be taken mod k).

As elements of the nilCoxeter algebra, cyclically decreasing words are completely determined by their support.

Example

$k = 6$

$$(u_0 u_6)(u_4 u_3 u_2) = (u_4 u_3 u_2)(u_0 u_6) = u_4 u_0 u_3 u_6 u_2 = \cdots$$

Noncommutative $\mathbf{h}$ functions

For a subset $S \subset [0, k]$, we write u_S for the unique cyclically decreasing nilCoxeter element with support S.
For $r \leq k$ we define

$$\mathbf{h}_r = \sum_{|S|=r} u_S$$

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This immediately implies that the algebra $\mathbb{Q}[\mathbf{h}_1, \ldots, \mathbf{h}_k] \cong \mathbb{Q}[h_1, \ldots, h_k]$ where the latter functions are the usual homogeneous symmetric functions.

Noncommutative symmetric functions

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$\mathbf{s}_\lambda^{(k)}$ by the k-Pieri rule

k-Pieri rule

The k-Pieri rule is

$$\mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{\mu} \mathbf{s}_\mu^{(k)}$$

where the sum is over all k-bounded partitions μ such that μ/λ is a horizontal strip of length r and $\mu^{(k)}/\lambda^{(k)}$ is a vertical strip of length r. This can be re-written as

$$\mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{|S|=r} \mathbf{s}_{u_S \cdot \lambda}^{(k)}$$

The action on cores

There is an action of A_k on $k+1$-cores given by

$$u_i \cdot c = \begin{cases} 0 & \text{no addable } i\text{-residue} \\ c \cup \text{ all addable } i\text{-residues} & \text{otherwise} \end{cases}$$

Example

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	1	2						
	2	3						
	3	0	1	2	3			
$u_2 u_0 \cdot$	0	1	2	3	0	1	2	3

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$$u_2 u_0 \cdot \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 0 \\ \cline{1-2} 1 & 2 \\ \cline{1-3} 2 & 3 & 0 \\ \cline{1-6} 3 & 0 & 1 & 2 & 3 & 0 \\ \hline 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 \\ \hline \end{array}$$

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$$u_2 u_0 \cdot \begin{array}{|c|c|c|c|c|c|c|c|c|} \cline{1-1} \color{red}{0} \\ \cline{1-2} 1 & 2 \\ \cline{1-3} 2 & 3 & \color{red}{0} \\ \cline{1-6} 3 & 0 & 1 & 2 & 3 & \color{red}{0} \\ \hline 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & \color{red}{0} \\ \hline \end{array} = u_2 \cdot \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \cline{1-1} \color{red}{3} \\ \cline{1-2} 0 & \color{red}{1} \\ \cline{1-3} 1 & 2 & \color{red}{3} \\ \cline{1-4} 2 & 3 & 0 & \color{red}{1} \\ \cline{1-7} 3 & 0 & 1 & 2 & 3 & 0 & \color{red}{1} \\ \hline 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & \color{red}{1} \\ \hline \end{array} = 0$$

Multiplication rule

A corollary of the k-Pieri rule is that if $\mathbf{f}$ is any non-commutative symmetric function of the form

$$\mathbf{f} = \sum_u c_u u$$

then

$$\mathbf{f}\mathbf{s}_\lambda^{(k)} = \sum_u c_u \mathbf{s}_{u\cdot\lambda}^{(k)}$$

Hook words

Fomin and Greene define a *hook word* in the context of an algebra with a totally ordered set of generators to be a word of the form

$$u_{a_1} \cdots u_{a_r} u_{b_1} \cdots u_{b_s}$$

where

$$a_1 > a_2 > \cdots > a_r > b_1 \leq b_2 \leq \cdots \leq b_s$$

To extend this notion to A_k which has a *cyclically* ordered set of generators, we only consider words whose support is a proper subset of $[0, \cdots, k]$.

Hook words

There is a *canonical order* on any proper subset of $[0, k]$ given by thinking of the smallest (in integer order) element which does not appear as the smallest element of the circle.

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Example

For $\{0, 1, 3, 4, 6\} \subset [0, 6]$, we have the order

$$2 < 3 < 4 < 5 < 6 < 0 < 1$$

Hook words in A_k have (support = proper subset) and form

$$u_{a_1} \cdots u_{a_r} u_{b_1} \cdots u_{b_s}$$

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Hook word representations are unique; therefore the number of *ascents* in a hook word is well-defined as $s-1$.

The non-commutative rule

Theorem (Bandlow-S-Zabrocki, 2010)

$$\mathbf{p}_r \mathbf{s}_\mu^{(k)} = \sum_w (-1)^{asc(w)} \mathbf{s}_{w\cdot\mu}^{(k)}$$

where the sum is over all words in the affine nilCoxeter algebra satisfying

- *(size)* $\mathrm{len}(w) = r$
- *(containment)* $w \cdot \mu \neq 0$
- *(connectedness)* w *is a* k*-connected word*
- *(ribbon condition)* w *is a hook word*

Comparison between characterizations

Characterize the image of the map ($w \to w \cdot \mu = \lambda$):

conditions on words:

- (size)
  $\mathrm{len}(w) = r$

conditions on shapes:

- (size)
  $|\lambda/\mu| = r$

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- (second ribbon condition)
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Iteration

Iterating the rule

$$p_r s_\lambda^{(k)} = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu^{(k)}$$

gives

$$p_\lambda = \sum_T (-1)^{\mathrm{ht}(T)} s_{sh(T)}^{(k)} = \sum_\mu \bar{\chi}_\lambda^{(k)}(\mu) s_\mu^{(k)}$$

where the sum is over all k-ribbon tableaux, defined analogously to the classical case.

Duality

In the classical case, the inner product immediately gives

$$p_\lambda = \sum_\mu \chi_\lambda(\mu) s_\mu \iff s_\mu = \sum_\lambda \frac{1}{z_\lambda} \chi_\lambda(\mu) p_\lambda$$

In the affine case we have

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We would like the inverse matrix

$$s_\lambda^{(k)} = \sum_\mu \frac{1}{z_\mu} \chi_\lambda^{(k)}(\mu) p_\mu$$

Conceptual reasons

Λ ring of symmetric functions
$\mathcal{P}^k$ set of partitions $\{\lambda \mid \lambda_1 \leq k\}$

$$\Lambda_{(k)} := \mathbb{C}\langle h_\lambda \mid \lambda \in \mathcal{P}^k\rangle = \mathbb{C}\langle e_\lambda \mid \lambda \in \mathcal{P}^k\rangle = \mathbb{C}\langle p_\lambda \mid \lambda \in \mathcal{P}^k\rangle$$
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Hall inner product $\langle \cdot, \cdot \rangle$:
for $f \in \Lambda_{(k)}$ and $g \in \Lambda^{(k)}$ define $\langle f, g \rangle$ as the usual Hall inner product in Λ
$\{h_\lambda\}$ and $\{m_\lambda\}$ with $\lambda \in \mathcal{P}^k$ form dual bases of $\Lambda_{(k)}$ and $\Lambda^{(k)}$

$\Lambda_{(k)}$	is a subalgebra
$\Lambda^{(k)}$	is **not** closed under multiplication, but comultiplication

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k-Schur functions $\{s_\lambda^{(k)} \mid \lambda \in \mathcal{P}^k\}$ form basis of $\Lambda_{(k)}$
(Schubert class of cohomology of affine Grassmannian $H_*(Gr)$)

dual k-Schur functions $\{\mathfrak{S}_\lambda^{(k)} \mid \lambda \in \mathcal{P}^k\}$ form basis of $\Lambda^{(k)}$
(Schubert class of homology of affine Grassmannian $H^*(Gr)$)

Back to Frobenius

For V any S_n representation, we can find the decomposition into irreducible submodules with

$$\sum_{\mu} \frac{1}{z_\mu} \chi_V(\mu) p_\mu = \sum_{\lambda} c_\lambda s_\lambda$$

So finding

$$s_\lambda^{(k)} = \sum_{\mu} \frac{1}{z_\mu} \chi_\lambda^{(k)}(\mu) p_\mu$$

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