The Bourgain–Delbaen construction and its applications

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Banff 2012

Introduction

In 1980 Bourgain and Delbaen introduced a new class of $\mathscr{L}^\infty$-spaces providing counterexamples to a number of outstanding conjectures.

More recently, variants of this construction have been used to address other problems,in particular the following.

(Argyros–Haydon, Acta Math. 2011)
There is a Banach space on which every operator has the form $\lambda I + K$, with λ a scalar and K compact .

(Freeman–Odell–Schlumprecht, Math. Ann. 2011)
Every Banach space with separable dual embeds in a space with dual isomorphic to ℓ_1.

Other speakers at this meeting will be presenting further results that use the same basic framework. My hope in this talk is to prepare the way, as well as giving a sketch of the scalar-plus-compact space.

A word about notation

The spaces X that we construct will be subspaces of ℓ_∞ (or $\ell_\infty(\Gamma)$ for various countable sets Γ). Depending on the details of the construction, ℓ_1 will be naturally identifiable with the dual space X^*, or with a subspace of that dual.

For this reason, we shall think of elements of ℓ_1 as *functionals*, using star notation f^* to remind ourselves of this. Elements of ℓ_∞ will be called *vectors*. We use angle brackets for the action of a functional on a vector:

$$\langle f^*, x \rangle = \sum_{n \in \mathbb{N}} f^*(n) x(n).$$

Notice the notation $x(n)$ (resp. $f^*(n)$) for the n^{th} coordinate of x (resp f^*).

More notation

The usual unit vector $(0, 0, \dots, 0, 1, 0, \dots)$ may be denoted either by e_n or by e_n^*, depending on whether we are thinking of it as a vector or as a functional. In the latter case, it is the *evaluation functional* satisfying

$$\langle e_n^*, x \rangle = x(n).$$

If we are working with a countable set Γ, rather than with the natural numbers, then e_γ^* etc will have the obvious meanings.

For the moment, however, let's stay with $\mathbb{N}$.

The BD framework

It may help to start by considering a very general situation. Let $(d_n^*)_{n \in \mathbb{N}}$ be a Schauder basis for $\ell_1(\mathbb{N})$, and let (d_n) be the biorthogonal sequence of vectors in ℓ_∞.

The closed linear span $X = \overline{\mathrm{sp}}\langle d_n : n \in \mathbb{N} \rangle$ is a separable $\mathscr{L}_\infty$-space whose structure (as it turns out) can be quite exotic.

Of course, if we take d_n^* to be the usual unit vector e_n^* the biorthogonal vectors are just e_n and $X = c_0$.

If we go a little further and make a *small* perturbation, setting $d_n^* = e_n^* - c_n^*$ where $\sup_n \|c_n^*\|_1 < 1$, then the sequence (d_n^*) is still equivalent to the unit vector basis of ℓ_1 and so X is still isomorphic to c_0.

To get something new and exciting, we shall look at a class of *large* perturbations.

BD-structures on countable sets

Let Γ be a countable set and let $\operatorname{rank} : \Gamma \to \mathbb{N}$ be a function such that each of the sets

$$\Delta_n = \{\gamma \in \Gamma : \operatorname{rank}\gamma = n\} \quad \text{is finite.}$$

Write $\Gamma_n = \bigcup_{k \le n} \Delta_k = \{\gamma \in \Gamma : \operatorname{rank}\gamma \le n\}$,
$\Gamma^+ = \Gamma \setminus \Gamma_1 = \{\gamma \in \Gamma : \operatorname{rank}\gamma > 1\}$, and let

$$\begin{aligned} \operatorname{weight} &: \Gamma^+ \to [0,1) \\ \operatorname{top} &: \Gamma^+ \to \operatorname{ball}\ell_1(\Gamma) \\ \operatorname{base} &: \Gamma^+ \to \Gamma \cup \{\text{undefined}\} \end{aligned}$$

be further functions.

For $\gamma \in \Gamma^+$ define $d^*_\gamma = e^*_\gamma - c^*_\gamma$, where

$$c^*_\gamma = e^*_\xi + \theta b^*, \quad \text{resp.} \quad c^*_\gamma = \theta b^*,$$

when $\theta = \operatorname{weight}\gamma$, $b^* = \operatorname{top}\gamma$ and $\xi = \operatorname{base}\gamma \in \Gamma$ (resp. when $\operatorname{base}\gamma$ is undefined.)

The BD conditions

We shall say that the functions rank, weight, top, base form a BD-structure on Γ if :

1. $\sup_{\gamma\in\Gamma+} \mathrm{weight}\,\gamma < 1$;
2. for all $\gamma \in \Gamma^+$, $\mathrm{rank}\,\mathrm{base}\,\gamma < \mathrm{rank}\,\gamma$ (if $\mathrm{base}\,\gamma$ is defined);
3. for all $\gamma \in \Gamma^+$, $\mathrm{top}\,\gamma \in \mathrm{sp}\langle d^*_\eta : \mathrm{rank}\,\mathrm{base}\,\gamma < \mathrm{rank}\,\eta < \mathrm{rank}\,\gamma\rangle$.

A set Γ equipped with a structure of this kind will be called a BD-set.

Theorem

If Γ is a BD-set, then the functionals d^*_γ form a basis of $\ell_1(\Gamma)$ and the biorthogonal vectors d_γ form a basis for a $\mathscr{L}_\infty$ subspace $X(\Gamma)$ of $\ell_\infty(\Gamma)$.

We shall see in due course conditions under which the dual of $X(\Gamma)$ is naturally isomorphic to $\ell_1(\Gamma)$.

The finite-dimensional decompositions

Although, by the above theorem, our spaces have Schauder bases, what enters most naturally into our calculations is a finite-dimensional decomposition of $X(\Gamma)$, induced by the partition of Γ into the strata Δ_n, and the dual f.d.d. on $\ell_1(\Gamma)$.

We write $P_{[1,n]}$ and $P^*_{[1,n]}$ for the projections associated with this f.d.d., which may be defined on $X(\Gamma)$ and $\ell_1(\Gamma)$ as the bounded linear operators satisfying

$$P_{[1,n]}(d_\gamma) = \begin{cases} d_\gamma & \text{if } \mathrm{rank}\,\gamma \le n \\ 0 & \text{otherwise,} \end{cases}$$

$$P^*_{[1,n]}(d^*_\gamma) = \begin{cases} d^*_\gamma & \text{if } \mathrm{rank}\,\gamma \le n \\ 0 & \text{otherwise,} \end{cases}$$

It follows from the proof of the BD theorem that $\|P_n\| \le (1-\theta)^{-1}$ where $\theta = \max_\gamma \mathrm{weight}\,\gamma$.

Support versus range

For both functionals $f^* \in \ell_1(\Gamma)$ and vectors $x \in \ell_\infty(\Gamma)$ we have a notion of *support*, defined as usual to be the set of γ for which $f^*(\gamma)$, resp $x(\gamma)$, is non-zero.

There is another notion of "support with respect to the f.d.d." To avoid confusion, we call this notion *range* and, for $x \in X(\Gamma)$, resp. $f^* \in \ell_1$, write $\operatorname{ran} x$, resp. $\operatorname{ran} f^*$, for the minimal interval I such that $x \in \operatorname{sp}\langle d_\gamma : \operatorname{rank}\gamma \in I\rangle$, resp. $f^* \in \operatorname{sp}\langle d^*_\gamma : \operatorname{rank}\gamma \in I\rangle$.

Note that if $\operatorname{rank}\gamma = n$ then $\operatorname{ran} d^*_\gamma = \{n\}$ whilst there is no reason for the support of d^*_γ to be contained in Δ_n; all we can say is that

$$\operatorname{supp} d^*_\gamma \subseteq \{\gamma\} \cup \{\eta \in \Gamma : \operatorname{rank}\eta < n\}.$$

Dually, $$\operatorname{supp} d_\gamma \subseteq \{\gamma\} \cup \{\delta \in \Gamma : \operatorname{rank}\delta > n\}.$$

FDD projections and extensions

There are the following explicit formulas for the f.d.d. projections introduced earlier:

$$P^*_{[1,n]}(f^*) = \sum_{\gamma\in\Gamma_n} \langle f^*, d_\gamma\rangle d^*_\gamma$$
$$P_{[1,n]}(x) = \sum_{\gamma\in\Gamma_n} \langle d^*_\gamma, x\rangle d_\gamma.$$

Because the support of d^*_γ is contained in Γ_n whenever $\gamma \in \Gamma_n$, the value of $P_{[1,n]}(x)$ is determined by the restriction of x to Γ_n. We can therefore use the same formula to define an *extension operator* J_n from the finite-dimensional space $\ell_\infty(\Gamma_n)$ to $X(\Gamma)$:

$$J_n(u) = \sum_{\gamma\in\Gamma_n} \langle d^*_\gamma, u\rangle d_\gamma \quad (u \in \ell_\infty(\Gamma_n)).$$

These extension operators will perhaps be familiar from earlier presentations of the BD construction.

How to construct BD-sets

Typically, we construct a BD set by recursion, starting with a finite set Δ_1. The elements of Δ_1 have rank 1 and we do not have to define anything else.
Subsequently, if we have defined $\Gamma_n = \bigcup_{k \le n} \Delta_k$, as well as the associated c^*_γ, we need to decide for which triples (θ, ξ, b^*) we shall admit into Δ_{n+1} an element δ with

$$\operatorname{weight} \delta = \theta, \quad \operatorname{top} \delta = b^*, \quad \operatorname{base} \delta = \xi.$$

Sometimes it is convenient to use a notation that automatically codes the above data, writing

$$\delta = (n+1, \xi, \theta, b^*)$$

for for an element as above. Of course, we need a modification (simply leaving out the "ξ") if $\operatorname{base} \delta$ is undefined.

Regular BD-sets

We shall say that a BD-set is *regular* if the weight of the base of γ (when this is defined) is always equal to the weight of γ.

We shall work only with BD-sets of this kind,and shall assume moreover that the weights of elements of Γ are of the form $\theta = m_i^{-1}$, where $(m_i)_{i\in\mathbb{N}}$ is a fairly fast-growing sequence of natural numbers.

$$m_i = 2^{2^i} \text{ will do fine.}$$

We assume in particular that $m_1 \geq 4$, so that the norms of the operators $P_{[1,n]}$, $P^*_{[1,n]}$ and J_n are all at most $4/3$.

The evaluation analysis

An important tool for norm estimates is a formula that expresses the evaluation functionals e^*_γ in terms of the basis elements d^*_γ.
By our definitions, we have

$$e^*_\gamma = c^*_\gamma + d^*_\gamma = e^*_\xi + \theta b^* + d^*_\gamma,$$

whenever the base ξ is defined. If we repeat this operation and continue until we meet an element whose base is undefined we obtain

$$e^*_\gamma = \theta b^*_1 + d^*_{\xi_1} + \theta b^*_2 + d^*_{\xi_2} + \cdots + \theta b^*_a + d^*_{\xi_a},$$

where $\xi_a = \gamma$, $\theta = \mathrm{weight}\,\gamma$, $b^*_j = \mathrm{top}\,\xi_j$ and $\xi_j = \mathrm{base}\,\xi_{j+1}$
We call this the *evaluation analysis.* The natural number a is called the *age* of γ.

A criterion for $X(\Gamma)$ to be a predual of $\ell_1(\Gamma)$

Since $X(\Gamma)$ is a subspace of $\ell_\infty(\Gamma)$ there is a natural mapping $\ell_1(\Gamma) \to X(\Gamma)^*$ and it follows from our construction that this is always an isomorphic embedding. If it is *surjective*, we shall that the dual of $X(\Gamma)$ is *naturally isomorphic* to $\ell_1(\Gamma)$. There are obvious criteria for this expressed in terms of boundedly complete and shrinking bases/f.d.d.'s, but the following is also very useful.

Theorem

The following are equivalent:

1. *$X(\Gamma)^*$ is naturally isomorphic to $\ell_1(\Gamma)$;*
2. *there is no infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $\gamma_n = \mathrm{base}\,\gamma_{n+1}$ for all n.*

Age and History

If we are building a BD-set and want to be sure of ending up with a natural predual of ℓ_1 then we have to stop the growth of infinite branches (γ_n) with $\gamma_n = \operatorname{base} \gamma_{n+1}$.

The approach adopted in our first paper on the scalar-plus-compact problem was to fix a second sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers and demand that an element γ of weight m_i^{-1} may not have age greater than n_i. The sequence (n_i) needs to grow a bit faster than (m_i),

$$n_i = 2^{2^{i^2+1}} \text{ will do.}$$

Subsequent applications oblige us to work with something a little more complicated than age. We define the *history* $\operatorname{hist} \gamma$ by recursion:

$$\operatorname{hist} \gamma = \begin{cases} \{\min \operatorname{ran} \operatorname{top} \gamma\} & \text{if } \operatorname{base} \gamma \text{ is undefined} \\ \operatorname{hist} \operatorname{base} \gamma \cup \{\min \operatorname{ran} \operatorname{top} \gamma\} & \text{otherwise.} \end{cases}$$

Regular families of sets

Recall that a collection $\mathscr{N}$ of finite subsets of $\mathbb{N}$ is said to be *spreading* if $\{n_1, \ldots, n_r\} \in \mathscr{N}$ whenever $\{m_1, \ldots, m_r\} \in \mathscr{N}$ and $m_i \leq n_i$ for all i. Such a collection is called a *regular family* if it is also *compact* for the pointwise topology on $[\mathbb{N}]^{<\omega} \subset \{0, 1\}^{\mathbb{N}}$.

A more general (and in a sense the *most* general) way to ensure that our BD space is a predual of ℓ_1 is to demand that there exist regular families $\mathscr{N}_i$ such that the history of an element γ of weight m_i^{-1} is always in $\mathscr{N}_i$.

The "age-oriented" approach mentioned above corresponds to taking $\mathscr{N}_i = \mathscr{A}_{n_i}$.

Where have we got to?

We are by now considering a regular BD-set Γ with weights $(m_i^{-1})_{i\in\mathbb{N}}$ and having the property that the history of any element of weight m_i^{-1} is in the family $\mathscr{N}_i$. For the moment, we shall make no assumptions about these families.

We have two further small assumptions to make, both related to the fact that in a BD-set Γ each of the strata Δ_n has to be finite.

We thus assume that an element γ of weight m_i^{-1} must have rank at least i, and that the top b^* of an element $\gamma \in \Delta_n$ must be a *rational* linear combination of d_η^*'s in which the denominators of the coefficients all divide some suitably large natural number $N_n!$.

We shall write B_n for the set of all linear combinations of this type.

Eligibility

A tuple $(n+1, m_i^{-1}, b^*, \xi)$ is thus *eligible* to be an element of Δ_{n+1} if

1. $i \leq n+1$, $\xi \in \Gamma_n$;
2. b^* is a linear combination

$$b^* = \sum_{\mathrm{rank}\, \xi < \mathrm{rank}\, \eta \leq n} \alpha_\eta d_\eta^*,$$

   where $N_{n+1}! \alpha_\eta \in \mathbb{Z}$ for all η.;
3. $\mathrm{hist}\, \xi \cup \{\min \mathrm{ran}\, b^*\} \in \mathscr{N}_i$.

In the sort of construction we are interested in, it is usual to arrange that *all* eligible tuples of "even weight", that is to say of weight m_i^{-1} with i even, do belong to Δ_{n+1}. The careful selection of the odd-weight elements introduces the more subtle structure into the examples.

An easy lower estimate

We have not yet mentioned mixed Tsirelson spaces, but we get our first idea that they will have a role to play by noting an easy lower estimate. Notice the use made of the richness of even-weight elements in Γ.

Lemma

Let $(x_k)k \in \mathbb{N}$ be a skipped-block sequence in $X(\Gamma)$ and let i be a natural number. Write $\nu_k = \min \operatorname{ran} x_k$ and assume that $\{\nu_1, \ldots, \nu_a\} \in \mathscr{N}_{2i}$. Then

$$\|\sum_k x_k\|_{X(\Gamma)} \geq \tfrac{1}{4} m_{2i}^{-1} \sum_{k=1}^{a} \|x_k\|.$$

Sketch Proof. Our skipped-block assumption is that there exist p_k such that

$$\operatorname{ran} x_1 < p_1 < \operatorname{ran} x_2 < p_2 < \cdots < \operatorname{ran} x_a < p_a.$$

An easy proof (continued)

We want to find an element γ whose evaluation analysis will "pick up" a good contribution from each of the x_k. For each k we can find $b_k^* \in B_{p_k}$ with $\operatorname{ran} b_k^* \subseteq (p_{k-1}, p_k)$ and $\langle b_k^*, x_k \rangle$ close to $\frac{3}{8}\|x_k\|$.

To simplify things, assume that $p_1 \geq 2i$: in this case there are elements $\xi_k \in \Delta_{p_k}$ $(1 \leq k \leq a)$ such that the evaluation analysis of $\gamma = \xi_a$ is

$$e_\gamma^* = \sum_{k=1}^{a} (m_{2i}^{-1} b_k^* + d_{\xi_k}^*).$$

We see that

$$\langle e_\gamma^*, \sum x_k \rangle = \sum \langle b_k^*, x_k \rangle \approx \frac{3}{8} \sum \|x_k\|.$$

If $p_1 < 2i$ then there a few extra terms to deal with.

Coding and odd-weight elements

We introduce a coding function σ which maps Γ injectively into $\mathbb{N}$; actually, if we are constructing Γ recursively then we define σ "as we go along".

The rules for admission of an eligible odd-weight tuple $(n+1, m_{2j-1}^{-1}, b^*)$ into Δ_{n+1} are that b^* must have the special form e_η^* where $p < \operatorname{rank} \eta \leq n$, and $\operatorname{weight} \eta$ is of the form m_{4i-2}^{-1} with $i > \frac{1}{2}j$.

For a tuple $(n+1, \xi, m_{2j-1}^{-1}, b^*)$ we are even more demanding: b^* must have the form e_η^* where $\operatorname{rank} \xi < \operatorname{ran} e^* \eta \leq n$ and the weight of η is exactly $m_{4\sigma(\xi)}^{-1}$.

Thinking back to the lemma on the previous slide, we can see, at least intuitively, that it will only be in exceptional circumstances that a skipped block sequence will satisfy a lower estimate of a similar kind with weight m_{2j-1}^{-1}.

Mixed Tsirelson spaces

We recall the definition of the mixed Tsirelson space

$$T[(m_i^{-1}, \mathscr{N}_i)_{i\in\mathbb{N}}],$$

starting with the recursive definition of the *norming set*

$$W[(m_i^{-1}, \mathscr{N}_i)_{i\in\mathbb{N}}].$$

This is defined to be the smallest subset W of $c_{00}(\mathbb{N})$ that contains all $\pm e_n^*$ and also has the property that $m_i^{-1}\sum_{r=1}^{a} f_r^* \in W$ whenever the successive functionals f_r^* are all in W and the set $\{\min\operatorname{supp} f_r^* : 1 \le r \le a\}$ is in $\mathscr{N}_i$.
A functional of the form $f^* = m_i^{-1}\sum_{r=1}^{a} f_r^*$ is said to have weight m_i^{-1}.
The space $T[(m_i^{-1}, \mathscr{N}_i)_{i\in\mathbb{N}}]$ is defined to be the completion of $c_{00}(\mathbb{N})$ for the norm defined by

$$\|x\| = \sup_{f^*\in W[(m_i^{-1}, \mathscr{N}_i)_{i\in\mathbb{N}}]} \langle f^*, x\rangle.$$

Our assumptions about $\mathscr{N}_i$

At this point introduce some assumptions about the families $\mathscr{N}_j$. We require them to be regular, and $\mathscr{N}_1$ can be any regular family. Thereafter, we require them to grow very fast. A little more precisely, we want every maximal N in $\mathscr{N}_{j+1}$ to be the support of a convex vector a that is extremely small with respect to $\mathscr{N}_j$.

For those desperate for precision what we actually require of the convex vector a is that

$$\sum_{m \in \mathscr{M}} a(m) < m_{j+1}^{-1},$$

for every set M in $(\mathscr{N}_j{}')^{*l_{j+1}}$, where $\mathscr{N}_j{}' = \mathscr{A}_3 * \mathscr{N}_j$ and $l_{j+1} = \log_2 m_{j+1}$.

Special convex vectors

A vector $a \in c_{00}(\mathbb{N})$ with the property set out in the previous slide is called a $(j+1)$-special convex vector.
The following norm estimates play an important role.

Lemma

Let $T = T[(\mathscr{N}_i, m_i^{-1})_{i\in\mathbb{N}}]$, $T' = T[(\mathscr{N}_i', m_i^{-1})_{i\in\mathbb{N}}]$ and $T'' = T[(\mathscr{N}_i', m_i^{-1})_{i\neq j+1}]$. If a is a $(j+1)$-special convex vector then

$$\|a\|_T = \|a\|_{T'} = m_{j+1}^{-1}, \text{ while } \|a\|_{T''} \leq m_{j+1}^{-2}.$$

RIS and the Basic Inequality

A *rapidly increasing sequence*, or RIS, will be a block sequence in X for which we have upper mixed-Tsirelson estimates. These estimates, together with facts about special convex vectors in mT-spaces, will give us strong norm estimates for certain vectors in X.

Definition

Let $(x_k)_{k\in\mathbb{N}}$ be a block sequence in $X(\Gamma)$. We shall say that (x_k) is a C-RIS if

1. $\|x_k\| \le C/2$ for all k;
   and there exist natural numbers $j_1 \le j_1' < j_2 \le j_2' < \dots$ such that
2. $|x_k(\gamma)| \le C/m_h \quad$ if $\mathrm{weight}\,\gamma = m_h^{-1}$ with $h < j_k$;
3. $|x_k(\gamma)| \le C m_{j_k}/m_h$ if $\mathrm{weight}\,\gamma = m_h^{-1}$ with $h > j_k'$.

There are plenty of RIS in the space $X(\Gamma)$.

Lemma

If (w_j) is a block sequence in $X(\Gamma)$ then there is a normalized block-subsequence (x_i) that is a 2-RIS.

The next lemma, which seems to peculiar to constructions using the BD method, shows that the behaviour of arbitrary block sequences is determined by that of RIS.

Lemma

Let Y be a Banach space and let $T : X \to Y$ be a bounded linear operator. If $\|Tx_n\| \to 0$ for every RIS then $\|T(x_n)\| \to 0$ for every bounded block sequence, and hence T is compact.

The relevance to the scalar-plus-compact problem should be obvious.

The Basic Inequality

Theorem (A simple version)

Let (x_k) be a C-RIS in $X(\Gamma)$, let $\nu_k = \min\operatorname{ran} x_k$ and let t_k be the unit vector e_{ν_k} in the mixed Tsirelson space $T' = T[(m_i^{-1}, \mathscr{N}_i')_{i\in\mathbb{N}}]$. Then for all l and all scalars α_k

$$\|\sum_{k=1}^{l} \alpha_k x_k\|_X \leq 2C\|\sum_{k=1}^{l} \alpha_k t_k\|_{T'}.$$

The family $\mathscr{N}_i'$ can often be taken to be the same as $\mathscr{N}_i$, but for the moment we are trying not to make any special assumptions about the $\mathscr{N}_i$. In any case $\mathscr{N}_i'$ does not need to be *much* bigger than $\mathscr{N}_i$. We may take $\mathscr{N}_i' = \mathscr{A}_3 * \mathscr{N}_i$.

The Basic Inequality (continued)

Theorem (A more technical version)

Let (x_k), (t_k) and α_k be as before. Let $I \subset \mathbb{N}$ be an interval and define $x_I = \sum_{k \in I} \alpha_k x_k$, $t_I = \sum_{k \in I} \alpha_k t_k$. Let weight $\gamma = m_h^{-1}$.

1. *There exists $g^* \in c_{00}(\mathbb{N})$ satisfying*

$$|x_I(\gamma)| \leq C\langle g^*, t_I \rangle,$$

   such that $g^* = \pm t^*_{k_0} + f^*$, for suitably chosen $k_0 \in I$ and f^* that is either 0 or a weight-m_h^{-1} element of $W[(\mathscr{N}_j', m_j^{-1})_{j \in \mathbb{N}}]$ with $\nu_{k_0} < \operatorname{supp} f^*$.

2. *If the scalar sequence (α_k) has the property that $|x_J(\eta)| \leq C m_{j_0}^{-1}$ for every subinterval J of I and every $\eta \in \Gamma$ of weight $m_{j_0}^{-1}$, then the functional f^* (when not zero) may be chosen to lie in $W[(\mathscr{N}_j', m_j^{-1})_{j \neq j_0}]$. In this case, $\|x_I\| \leq 2C \|t_I\|_{T[(\mathscr{N}_j', m_j^{-1})_{j \neq j_0}]}$.*

Special convex combinations

Let $(x_k)_{k\in\mathbb{N}}$ be a C-RIS and let t_k be as above. If $a = \sum_k \alpha_k t_k$ is a $(j+1)$-special convex vector we shall say that $\sum_k \alpha_k x_k$ is a $(j+1)$-*special convex combination*.

The Basic Inequality and norm estimates for special convex vectors yield the following

Lemma

If y is a $(j+1)$-special s.c.c. of a C-RIS (x_k) then

$$\|y\| \leq 2Cm_{j+1}^{-1}.$$

If $|\sum_{k\in J} \alpha_k x_k(\eta)| \leq Cm_{j+1}^{-1}$ for every interval J and every η of weight m_{j+1}^{-1} then

$$\|y\| \leq 2Cm_{j+1}^{2}.$$

The scalar-plus-compact property

The key is the following lemma.

Lemma

Let T be a bounded linear operator on $X(\Gamma)$. If (x_k) is a RIS then $\operatorname{dist}(Tx_k, \mathbb{R}x_k) \to 0$ as $k \to \infty$.

Let's see first how this leads to what we want. First of all, a couple of easy steps show that there exists a scalar λ such that $\|Tx_k - \lambda x_k\| \to 0$ for every RIS.

But as we noted earlier, this implies that $T - \lambda I$ is compact.

Now, without getting too technical, we shall try to sketch a proof of the Key Lemma.

Assume that (x_k) is a C-RIS and that $\operatorname{dist}(T(x_k), \mathbb{R}x_k) > 1$ for all k.

The Key Lemma

By taking subsequences and small perturbations, we may suppose that there are natural numbers $p_0 < p_1 < \dots$ and functionals b_k^*, of norm 1 such that $p_{j-1} < ranx_k, \mathrm{ran}\, Tx_k, \mathrm{ran}\, b_k^*$ and $\langle b_k^*, x_k \rangle = 0$, $\langle b_k^*, Tx_k \rangle > \frac{1}{4}$.

The next step is to consider a 2j-special convex combinations $y = \sum_{k \in I} \alpha_k x_k$ and an element η of Γ, of weight m_{2j}^{-1} in whose evaluation analysis the "b^*"s are exactly b_k^* ($k \in I$). This will satisfy $\langle e_\eta^*, T(y) \rangle \geq \frac{1}{4} m_{2j}^{-1}$ and $\langle e_\eta^*, y \rangle = 0$.

The above can be done for each j yielding a block subsequence (y_j) with associated η_j. If we seminormalize y_j, setting $z_j = m_{2j} y_j$, we have another RIS!

The Key Lemma (continued)

Now we work with an odd weight m_{2i-1}^{-1} and, taking some care with coding, find a γ of that weight in whose evaluation analysis the "b_k^*"s form some subsequence of $(e_{\eta_j}^*)$.

For a suitably chosen $(2i-1)$-s.c.c. w of the RIS (z_j), evaluation at γ witnesses that

$$\|T(w)\| \geq \tfrac{1}{4} m_{2i-1}^{-1}.$$

But, because of the rigidity imposed by the coding function, it turns out that

$$\|Tw\| \leq C m_{2i-1}^{-2}.$$

For a suitably large i this contradicts boundedness of T.