Closed operator ideals on the Banach space of continuous functions on the first uncountable ordinal

Niels Jakob Laustsen

Lancaster University

BIRS, 6$^{\text{th}}$ March 2012

Joint work with Tomasz Kania

C(K)-spaces

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}.$$

C(K)-spaces

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}.$$

Fact. $C(K)$ separable $\iff$ K metrizable.

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}.$$

Fact. $C(K)$ separable $\iff$ K metrizable.

Classification. Let K be a compact metric space. Then:

(i) K has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell_\infty^n$;

C(K)-spaces

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}.$$

Fact. $C(K)$ separable $\iff$ K metrizable.

Classification. Let K be a compact metric space. Then:

(i) K has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell_\infty^n$;

(ii) (Milutin) K is uncountable $\iff$ $C(K) \cong C[0,1]$;

C(K)-spaces

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}.$$

Fact. $C(K)$ separable $\iff$ K metrizable.

Classification. Let K be a compact metric space. Then:

(i) K has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell_\infty^n$;

(ii) (Milutin) K is uncountable $\iff$ $C(K) \cong C[0,1]$;

(iii) (Bessaga and Pełczyński) K is countably infinite $\iff$ $C(K) \cong C[0, \omega^{\omega^\alpha}]$ for a unique countable ordinal α.

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}.$$

Fact. $C(K)$ separable $\iff$ K metrizable.

Classification. Let K be a compact metric space. Then:

(i) K has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell_\infty^n$;

(ii) (Milutin) K is uncountable $\iff$ $C(K) \cong C[0,1]$;

(iii) (Bessaga and Pełczyński) K is countably infinite $\iff$ $C(K) \cong C[0, \omega^{\omega^\alpha}]$ for a unique countable ordinal α.

Here, for an ordinal σ,

$$[0, \sigma] = \{\alpha \text{ ordinal} : \alpha \leqslant \sigma\}$$

is equipped with the *order topology*, which is determined by the basis

$$[0, \beta), \qquad (\alpha, \beta), \qquad (\alpha, \sigma] \qquad (0 \leqslant \alpha < \beta \leqslant \sigma).$$

Introducing our main character: the Loy–Willis ideal

Let ω_1 be the first uncountable ordinal, so that $C[0,\omega_1]$ is the "next" $C(K)$-space after the separable ones $C[0,\omega^{\omega^\alpha}]$ for countable α.

Let ω_1 be the first uncountable ordinal, so that $C[0,\omega_1]$ is the "next" $C(K)$-space after the separable ones $C[0,\omega^{\omega^\alpha}]$ for countable α.

Theorem (Semadeni 1960). The Banach space $C[0,\omega_1]$ is not isomorphic to its square $C[0,\omega_1] \oplus C[0,\omega_1]$.

Let ω_1 be the first uncountable ordinal, so that $C[0, \omega_1]$ is the "next" $C(K)$-space after the separable ones $C[0, \omega^{\omega^\alpha}]$ for countable α.

Theorem (Semadeni 1960). The Banach space $C[0, \omega_1]$ is not isomorphic to its square $C[0, \omega_1] \oplus C[0, \omega_1]$.

Theorem (Loy and Willis 1989). The Banach algebra $\mathscr{B}(C[0, \omega_1])$ of (bounded) operators on $C[0, \omega_1]$ contains a maximal ideal $\mathscr{M}$ of codimension one.

Introducing our main character: the Loy–Willis ideal

Let ω_1 be the first uncountable ordinal, so that $C[0, \omega_1]$ is the "next" $C(K)$-space after the separable ones $C[0, \omega^{\omega^\alpha}]$ for countable α.

Theorem (Semadeni 1960). The Banach space $C[0, \omega_1]$ is not isomorphic to its square $C[0, \omega_1] \oplus C[0, \omega_1]$.

Theorem (Loy and Willis 1989). The Banach algebra $\mathscr{B}(C[0, \omega_1])$ of (bounded) operators on $C[0, \omega_1]$ contains a maximal ideal $\mathscr{M}$ of codimension one.

We call $\mathscr{M}$ the *Loy–Willis ideal*.

Introducing our main character: the Loy–Willis ideal

Let ω_1 be the first uncountable ordinal, so that $C[0,\omega_1]$ is the "next" $C(K)$-space after the separable ones $C[0,\omega^{\omega^\alpha}]$ for countable α.

Theorem (Semadeni 1960). The Banach space $C[0,\omega_1]$ is not isomorphic to its square $C[0,\omega_1] \oplus C[0,\omega_1]$.

Theorem (Loy and Willis 1989). The Banach algebra $\mathscr{B}(C[0,\omega_1])$ of (bounded) operators on $C[0,\omega_1]$ contains a maximal ideal $\mathscr{M}$ of codimension one.

We call $\mathscr{M}$ the *Loy–Willis ideal*.

It is defined using a representation of operators on $C[0,\omega_1]$ as scalar-valued $[0,\omega_1] \times [0,\omega_1]$-matrices; an operator belongs to $\mathscr{M}$ if and only if its final column is continuous. The precise definition will follow later.

Let ω_1 be the first uncountable ordinal, so that $C[0,\omega_1]$ is the "next" $C(K)$-space after the separable ones $C[0,\omega^{\omega^\alpha}]$ for countable α.

Theorem (Semadeni 1960). The Banach space $C[0,\omega_1]$ is not isomorphic to its square $C[0,\omega_1] \oplus C[0,\omega_1]$.

Theorem (Loy and Willis 1989). The Banach algebra $\mathscr{B}(C[0,\omega_1])$ of (bounded) operators on $C[0,\omega_1]$ contains a maximal ideal $\mathscr{M}$ of codimension one.

We call $\mathscr{M}$ the *Loy–Willis ideal*.

It is defined using a representation of operators on $C[0,\omega_1]$ as scalar-valued $[0,\omega_1] \times [0,\omega_1]$-matrices; an operator belongs to $\mathscr{M}$ if and only if its final column is continuous. The precise definition will follow later.

Motivation. Loy and Willis' aim was to show that each derivation from $\mathscr{B}(C[0,\omega_1])$ into a Banach $\mathscr{B}(C[0,\omega_1])$-bimodule is automatically continuous.

Introducing our main character: the Loy–Willis ideal

Let ω_1 be the first uncountable ordinal, so that $C[0, \omega_1]$ is the "next" $C(K)$-space after the separable ones $C[0, \omega^{\omega^\alpha}]$ for countable α.

Theorem (Semadeni 1960). The Banach space $C[0, \omega_1]$ is not isomorphic to its square $C[0, \omega_1] \oplus C[0, \omega_1]$.

Theorem (Loy and Willis 1989). The Banach algebra $\mathscr{B}(C[0, \omega_1])$ of (bounded) operators on $C[0, \omega_1]$ contains a maximal ideal $\mathscr{M}$ of codimension one.

We call $\mathscr{M}$ the *Loy–Willis ideal*.

It is defined using a representation of operators on $C[0, \omega_1]$ as scalar-valued $[0, \omega_1] \times [0, \omega_1]$-matrices; an operator belongs to $\mathscr{M}$ if and only if its final column is continuous. The precise definition will follow later.

Motivation. Loy and Willis' aim was to show that each derivation from $\mathscr{B}(C[0, \omega_1])$ into a Banach $\mathscr{B}(C[0, \omega_1])$-bimodule is automatically continuous.

Key step: $\mathscr{M}$ has a bounded right approximate identity.

Main result: a coordinate-free characterization of $\mathscr{M}$

Theorem (Kania+NJL 2011). An operator on $C[0,\omega_1]$ belongs to the Loy–Willis ideal if and only if the identity operator on $C[0,\omega_1]$ does not factor through it;

$$\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : \forall\, R, S \in \mathscr{B}(C[0,\omega_1])\colon I \neq STR\}.$$

Main result: a coordinate-free characterization of $\mathscr{M}$

Theorem (Kania+NJL 2011). An operator on $C[0,\omega_1]$ belongs to the Loy–Willis ideal if and only if the identity operator on $C[0,\omega_1]$ does not factor through it;

$$\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : \forall\, R, S \in \mathscr{B}(C[0,\omega_1]) \colon I \neq STR\}.$$

Corollary. The Loy–Willis ideal is the unique maximal ideal of $\mathscr{B}(C[0,\omega_1])$.

Main result: a coordinate-free characterization of $\mathscr{M}$

Theorem (Kania+NJL 2011). An operator on $C[0, \omega_1]$ belongs to the Loy–Willis ideal if and only if the identity operator on $C[0, \omega_1]$ does not factor through it;

$$\mathscr{M} = \{T \in \mathscr{B}(C[0, \omega_1]) : \forall\, R, S \in \mathscr{B}(C[0, \omega_1]) \colon I \neq STR\}.$$

Corollary. The Loy–Willis ideal is the unique maximal ideal of $\mathscr{B}(C[0, \omega_1])$.

Proof. The theorem implies that the identity operator belongs to the ideal generated by any operator not in $\mathscr{M}$. $\square$

Main result: a coordinate-free characterization of $\mathscr{M}$

Theorem (Kania+NJL 2011). An operator on $C[0,\omega_1]$ belongs to the Loy–Willis ideal if and only if the identity operator on $C[0,\omega_1]$ does not factor through it;

$$\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : \forall\, R, S \in \mathscr{B}(C[0,\omega_1]) \colon I \neq STR\}.$$

Corollary. The Loy–Willis ideal is the unique maximal ideal of $\mathscr{B}(C[0,\omega_1])$.

Proof. The theorem implies that the identity operator belongs to the ideal generated by any operator not in $\mathscr{M}$. $\square$

Remark. Many Banach spaces X share with $C[0,\omega_1]$ the property that

$$\mathscr{M}_X := \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$$

is the unique maximal ideal of $\mathscr{B}(X)$.

Main result: a coordinate-free characterization of $\mathscr{M}$

Theorem (Kania+NJL 2011). An operator on $C[0,\omega_1]$ belongs to the Loy–Willis ideal if and only if the identity operator on $C[0,\omega_1]$ does not factor through it;

$$\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : \forall\, R, S \in \mathscr{B}(C[0,\omega_1]) \colon I \neq STR\}.$$

Corollary. The Loy–Willis ideal is the unique maximal ideal of $\mathscr{B}(C[0,\omega_1])$.

Proof. The theorem implies that the identity operator belongs to the ideal generated by any operator not in $\mathscr{M}$. $\square$

Remark. Many Banach spaces X share with $C[0,\omega_1]$ the property that

$$\mathscr{M}_X := \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$$

is the unique maximal ideal of $\mathscr{B}(X)$.

Fact (Dosev and Johnson 2010). Suppose that $\mathscr{M}_X$ is closed under addition. Then $\mathscr{M}_X$ is the unique maximal ideal of $\mathscr{B}(X)$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}.$

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}.$

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}.$

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

(iv) $X = \ell_\infty / c_0$ (follows from Drewnowski and Roberts 1991);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

(iv) $X = \ell_\infty / c_0$ (follows from Drewnowski and Roberts 1991);

(v) $X = d_{w,p}$, the Lorentz sequence space determined by a decreasing, non-summable sequence $w = (w_n)$ in $(0,1]$ and $p \in [1,\infty)$
(Kamińska, Popov, Spinu, Tcaciuc and Troitsky 2011);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}.$

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

(iv) $X = \ell_\infty / c_0$ (follows from Drewnowski and Roberts 1991);

(v) $X = d_{w,p}$, the Lorentz sequence space determined by a decreasing, non-summable sequence $w = (w_n)$ in $(0,1]$ and $p \in [1, \infty)$
(Kamińska, Popov, Spinu, Tcaciuc and Troitsky 2011);

(vi) $X = \left(\bigoplus \ell_2^n\right)_{c_0}$
(NJL, Loy and Read 2004;

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

(iv) $X = \ell_\infty / c_0$ (follows from Drewnowski and Roberts 1991);

(v) $X = d_{w,p}$, the Lorentz sequence space determined by a decreasing, non-summable sequence $w = (w_n)$ in $(0,1]$ and $p \in [1, \infty)$
(Kamińska, Popov, Spinu, Tcaciuc and Troitsky 2011);

(vi) $X = \left(\bigoplus \ell_2^n\right)_{c_0}$ and $X = \left(\bigoplus \ell_2^n\right)_{\ell_1}$
(NJL, Loy and Read 2004; NJL, Schlumprecht and Zsák 2006);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

(iv) $X = \ell_\infty / c_0$ (follows from Drewnowski and Roberts 1991);

(v) $X = d_{w,p}$, the Lorentz sequence space determined by a decreasing, non-summable sequence $w = (w_n)$ in $(0,1]$ and $p \in [1,\infty)$
(Kamińska, Popov, Spinu, Tcaciuc and Troitsky 2011);

(vi) $X = \left(\bigoplus \ell_2^n\right)_{c_0}$ and $X = \left(\bigoplus \ell_2^n\right)_{\ell_1}$
(NJL, Loy and Read 2004; NJL, Schlumprecht and Zsák 2006);

(vii) $X = \left(\bigoplus_{\mathbb{N}} \ell_q\right)_{\ell_p}$ for $1 \leqslant q < p < \infty$ (Chen, Johnson and Zheng 2011);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

(iv) $X = \ell_\infty / c_0$ (follows from Drewnowski and Roberts 1991);

(v) $X = d_{w,p}$, the Lorentz sequence space determined by a decreasing, non-summable sequence $w = (w_n)$ in $(0,1]$ and $p \in [1,\infty)$
(Kamińska, Popov, Spinu, Tcaciuc and Troitsky 2011);

(vi) $X = \left(\bigoplus \ell_2^n\right)_{c_0}$ and $X = \left(\bigoplus \ell_2^n\right)_{\ell_1}$
(NJL, Loy and Read 2004; NJL, Schlumprecht and Zsák 2006);

(vii) $X = \left(\bigoplus_{\mathbb{N}} \ell_q\right)_{\ell_p}$ for $1 \leqslant q < p < \infty$ (Chen, Johnson and Zheng 2011);

(viii) $X = C[0,1]$ (Brooker 2010, using Pełczyński and Rosenthal);

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

(iv) $X = \ell_\infty / c_0$ (follows from Drewnowski and Roberts 1991);

(v) $X = d_{w,p}$, the Lorentz sequence space determined by a decreasing, non-summable sequence $w = (w_n)$ in $(0,1]$ and $p \in [1,\infty)$
(Kamińska, Popov, Spinu, Tcaciuc and Troitsky 2011);

(vi) $X = \left(\bigoplus \ell_2^n\right)_{c_0}$ and $X = \left(\bigoplus \ell_2^n\right)_{\ell_1}$
(NJL, Loy and Read 2004; NJL, Schlumprecht and Zsák 2006);

(vii) $X = \left(\bigoplus_{\mathbb{N}} \ell_q\right)_{\ell_p}$ for $1 \leqslant q < p < \infty$ (Chen, Johnson and Zheng 2011);

(viii) $X = C[0,1]$ (Brooker 2010, using Pełczyński and Rosenthal);

(ix) $X = C[0,\omega^\omega]$ and $X = C[0,\omega^\alpha]$, where α is a countable epsilon number, that is, a countable ordinal satisfying $\alpha = \omega^\alpha$
(Brooker (unpublished), using Bourgain and Pełczyński).

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal

(i) $X = \ell_p$ for $1 \leqslant p < \infty$ and $X = c_0$ (Gohberg, Markus and Feldman 1960);

(ii) $X = L_p[0,1]$ for $1 \leqslant p < \infty$
(Dosev, Johnson and Schechtman 2011; known implicitly before);

(iii) $X = \ell_\infty \cong L_\infty[0,1]$ (NJL and Loy 2005, using Pełczyński and Rosenthal);

(iv) $X = \ell_\infty / c_0$ (follows from Drewnowski and Roberts 1991);

(v) $X = d_{w,p}$, the Lorentz sequence space determined by a decreasing, non-summable sequence $w = (w_n)$ in $(0,1]$ and $p \in [1, \infty)$
(Kamińska, Popov, Spinu, Tcaciuc and Troitsky 2011);

(vi) $X = \left(\bigoplus \ell_2^n\right)_{c_0}$ and $X = \left(\bigoplus \ell_2^n\right)_{\ell_1}$
(NJL, Loy and Read 2004; NJL, Schlumprecht and Zsák 2006);

(vii) $X = \left(\bigoplus_{\mathbb{N}} \ell_q\right)_{\ell_p}$ for $1 \leqslant q < p < \infty$ (Chen, Johnson and Zheng 2011);

(viii) $X = C[0,1]$ (Brooker 2010, using Pełczyński and Rosenthal);

(ix) $X = C[0, \omega^\omega]$ and $X = C[0, \omega^\alpha]$, where α is a countable epsilon number, that is, a countable ordinal satisfying $\alpha = \omega^\alpha$
(Brooker (unpublished), using Bourgain and Pełczyński).

Note: $C[0, \omega_1]$ differs from all these Banach spaces because

$$C[0, \omega_1] \ncong C[0, \omega_1] \oplus C[0, \omega_1].$$

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}.$

Theorem (Kania+NJL 2011). The following are equivalent for $T \in \mathscr{B}(C[0,\omega_1])$:

(a) T has separable range,

Operators on $C[0,\omega_1]$ with separable range

Theorem (Kania+NJL 2011). The following are equivalent for $T \in \mathscr{B}(C[0,\omega_1])$:

(a) T has separable range,

(b) T does not fix a copy of the Banach space

$$c_0(\omega_1) = \{f\colon [0,\omega_1) \to \mathbb{C} : \{\alpha \in [0,\omega_1) : |f(\alpha)| \geqslant \varepsilon\} \text{ is finite for each } \varepsilon > 0\},$$

Operators on $C[0,\omega_1]$ with separable range

Theorem (Kania+NJL 2011). The following are equivalent for $T \in \mathscr{B}(C[0,\omega_1])$:

(a) T has separable range,

(b) T does not fix a copy of the Banach space

$$c_0(\omega_1) = \{f\colon [0,\omega_1) \to \mathbb{C} : \{\alpha \in [0,\omega_1) : |f(\alpha)| \geqslant \varepsilon\} \text{ is finite for each } \varepsilon > 0\},$$

(c) $T = P_\sigma T P_\sigma$ for some $\sigma \in [0,\omega_1)$, where

$$(P_\sigma f)(\alpha) = \begin{cases} f(\alpha) & \text{for } \alpha \in [0,\sigma] \\ f(\omega_1) & \text{for } \alpha \in [\sigma+1,\omega_1] \end{cases} \qquad (f \in C[0,\omega_1]),$$

Operators on $C[0,\omega_1]$ with separable range

Theorem (Kania+NJL 2011). The following are equivalent for $T \in \mathscr{B}(C[0,\omega_1])$:

(a) T has separable range,

(b) T does not fix a copy of the Banach space

$$c_0(\omega_1) = \big\{f : [0,\omega_1) \to \mathbb{C} : \{\alpha \in [0,\omega_1) : |f(\alpha)| \geqslant \varepsilon\} \text{ is finite for each } \varepsilon > 0\big\},$$

(c) $T = P_\sigma T P_\sigma$ for some $\sigma \in [0,\omega_1)$, where

$$(P_\sigma f)(\alpha) = \begin{cases} f(\alpha) & \text{for } \alpha \in [0,\sigma] \\ f(\omega_1) & \text{for } \alpha \in [\sigma+1,\omega_1] \end{cases} \qquad (f \in C[0,\omega_1]),$$

(d) $T \in \mathscr{G}_{C[0,\sigma]}(C[0,\omega_1])$ for some $\sigma \in [0,\omega_1)$,

where, for Banach spaces X and Y,

$$\mathscr{G}_Y(X) := \operatorname{span}\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$$

This is always an ideal of $\mathscr{B}(X)$

Theorem (Kania+NJL 2011). The following are equivalent for $T \in \mathscr{B}(C[0,\omega_1])$:

(a) T has separable range,

(b) T does not fix a copy of the Banach space

$$c_0(\omega_1) = \big\{f\colon [0,\omega_1) \to \mathbb{C} : \{\alpha \in [0,\omega_1) : |f(\alpha)| \geqslant \varepsilon\} \text{ is finite for each } \varepsilon > 0\big\},$$

(c) $T = P_\sigma T P_\sigma$ for some $\sigma \in [0,\omega_1)$, where

$$(P_\sigma f)(\alpha) = \begin{cases} f(\alpha) & \text{for } \alpha \in [0,\sigma] \\ f(\omega_1) & \text{for } \alpha \in [\sigma+1,\omega_1] \end{cases} \qquad (f \in C[0,\omega_1]),$$

(d) $T \in \mathscr{G}_{C[0,\sigma]}(C[0,\omega_1])$ for some $\sigma \in [0,\omega_1)$,

where, for Banach spaces X and Y,

$$\mathscr{G}_Y(X) := \operatorname{span}\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$$

This is always an ideal of $\mathscr{B}(X)$

Note: if Y contains a complemented copy of $Y \oplus Y$, then the 'span' is not needed; $\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$ is already closed under addition.

Theorem (Kania+NJL 2011). The following are equivalent for $T \in \mathscr{B}(C[0,\omega_1])$:

(a) T has separable range,

(b) T does not fix a copy of the Banach space

$$c_0(\omega_1) = \big\{f\colon [0,\omega_1) \to \mathbb{C} : \{\alpha \in [0,\omega_1) : |f(\alpha)| \geqslant \varepsilon\} \text{ is finite for each } \varepsilon > 0\big\},$$

(c) $T = P_\sigma T P_\sigma$ for some $\sigma \in [0,\omega_1)$, where

$$(P_\sigma f)(\alpha) = \begin{cases} f(\alpha) & \text{for } \alpha \in [0,\sigma] \\ f(\omega_1) & \text{for } \alpha \in [\sigma+1,\omega_1] \end{cases} \qquad (f \in C[0,\omega_1]),$$

(d) $T \in \mathscr{G}_{C[0,\sigma]}(C[0,\omega_1])$ for some $\sigma \in [0,\omega_1)$,

(e) $T \in \overline{\mathscr{G}}_{C[0,\sigma]}(C[0,\omega_1])$ for some $\sigma \in [0,\omega_1)$,

where, for Banach spaces X and Y,

$$\mathscr{G}_Y(X) := \operatorname{span}\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$$

This is always an ideal of $\mathscr{B}(X)$, and $\overline{\mathscr{G}}_Y(X)$ is its closure.
Note: if Y contains a complemented copy of $Y \oplus Y$, then the 'span' is not needed; $\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$ is already closed under addition.

Theorem (Kania+NJL 2011). The following are equivalent for $T \in \mathscr{B}(C[0,\omega_1])$:

(a) T has separable range,

(b) T does not fix a copy of the Banach space

$$c_0(\omega_1) = \{f\colon [0,\omega_1) \to \mathbb{C} : \{\alpha \in [0,\omega_1) : |f(\alpha)| \geqslant \varepsilon\} \text{ is finite for each } \varepsilon > 0\},$$

(c) $T = P_\sigma T P_\sigma$ for some $\sigma \in [0,\omega_1)$, where

$$(P_\sigma f)(\alpha) = \begin{cases} f(\alpha) & \text{for } \alpha \in [0,\sigma] \\ f(\omega_1) & \text{for } \alpha \in [\sigma+1,\omega_1] \end{cases} \qquad (f \in C[0,\omega_1]),$$

(d) $T \in \mathscr{G}_{C[0,\sigma]}(C[0,\omega_1])$ for some $\sigma \in [0,\omega_1)$,

(e) $T \in \overline{\mathscr{G}}_{C[0,\sigma]}(C[0,\omega_1])$ for some $\sigma \in [0,\omega_1)$,

where, for Banach spaces X and Y,

$$\mathscr{G}_Y(X) := \operatorname{span}\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$$

This is always an ideal of $\mathscr{B}(X)$, and $\overline{\mathscr{G}}_Y(X)$ is its closure.
Note: if Y contains a complemented copy of $Y \oplus Y$, then the 'span' is not needed; $\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$ is already closed under addition.

Warning! This theorem does *not* imply that the ideal $\mathscr{G}_{C[0,\sigma]}(C[0,\omega_1])$ is closed for each $\sigma \in [0,\omega_1)$, despite the equivalence of (d) and (e).

Operators on $C[0,\omega_1]$ with separable range

Theorem (Kania+NJL 2011). The following are equivalent for $T \in \mathscr{B}(C[0,\omega_1])$:

(a) T has separable range,

(b) T does not fix a copy of the Banach space

$$c_0(\omega_1) = \big\{f\colon [0,\omega_1) \to \mathbb{C} : \{\alpha \in [0,\omega_1) : |f(\alpha)| \geqslant \varepsilon\} \text{ is finite for each } \varepsilon > 0\big\},$$

(c) $T = P_\sigma T P_\sigma$ for some $\sigma \in [0,\omega_1)$, where

$$(P_\sigma f)(\alpha) = \begin{cases} f(\alpha) & \text{for } \alpha \in [0,\sigma] \\ f(\omega_1) & \text{for } \alpha \in [\sigma+1,\omega_1] \end{cases} \qquad (f \in C[0,\omega_1]),$$

(d) $T \in \mathscr{G}_{C[0,\sigma]}(C[0,\omega_1])$ for some $\sigma \in [0,\omega_1)$,

(e) $T \in \overline{\mathscr{G}}_{C[0,\sigma]}(C[0,\omega_1])$ for some $\sigma \in [0,\omega_1)$,

where, for Banach spaces X and Y,

$$\mathscr{G}_Y(X) := \operatorname{span}\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$$

This is always an ideal of $\mathscr{B}(X)$, and $\overline{\mathscr{G}}_Y(X)$ is its closure.
Note: if Y contains a complemented copy of $Y \oplus Y$, then the 'span' is not needed; $\{TS : S \in \mathscr{B}(X,Y),\ T \in \mathscr{B}(Y,X)\}$ is already closed under addition.

Warning! This theorem does *not* imply that the ideal $\mathscr{G}_{C[0,\sigma]}(C[0,\omega_1])$ is closed for each $\sigma \in [0,\omega_1)$, despite the equivalence of (d) and (e).
Reason: for given $\tau \in [0,\omega_1)$ and $T \in \overline{\mathscr{G}}_{C[0,\tau]}(C[0,\omega_1])$, the ordinal σ such that (d) holds may be much larger than τ and depend on T.

Partial structure of the lattice of closed ideals of $\mathscr{B} = \mathscr{B}(C[0,\omega_1])$

$$
\begin{tikzcd}
 & & & \mathscr{M} \arrow[r, hook, two heads] & \mathscr{B} \\
 & & \mathscr{X}+\overline{\mathscr{G}}_{c_0(\omega_1)} \arrow[ur, hook] & & \\
 & \mathscr{X} \arrow[ur, hook] & \vdots \arrow[u, hook] & & \\
 & \vdots \arrow[u, hook] & \overline{\mathscr{G}}_{C(K_{\alpha+1})\oplus c_0(\omega_1)} \arrow[u, hook] & & \\
 & \overline{\mathscr{G}}_{C(K_{\alpha+1})} \arrow[u, hook] \arrow[ur, hook] & \overline{\mathscr{G}}_{C(K_\alpha)\oplus c_0(\omega_1)} \arrow[u, hook] & & \\
 & \overline{\mathscr{G}}_{C(K_\alpha)} \arrow[u, hook] \arrow[ur, hook] & \vdots \arrow[u, hook] & & \\
 & \vdots \arrow[u, hook] & \overline{\mathscr{G}}_{C(K_1)\oplus c_0(\omega_1)} \arrow[u, hook] & & \\
 & \overline{\mathscr{G}}_{C(K_1)} \arrow[u, hook] \arrow[ur, hook] & & \overline{\mathscr{G}}_{c_0(\omega_1)} \arrow[ul, hook] & \\
 & & \overline{\mathscr{G}}_{c_0} \arrow[ul, hook] \arrow[ur, hook, two heads] & & \\
\{0\} \arrow[r, hook, two heads] & \mathscr{K} \arrow[ur, hook, two heads] & & &
\end{tikzcd}
$$

Partial structure of the lattice of closed ideals of $\mathscr{B} = \mathscr{B}(C[0, \omega_1])$

$$
\begin{tikzcd}
 & & & \mathscr{M} \arrow[r, hook, two heads] & \mathscr{B} \\
 & & \mathscr{X} + \overline{\mathscr{G}}_{c_0(\omega_1)} \arrow[ur, hook] & & \\
 & \mathscr{X} \arrow[ur, hook] & & & \\
 & \vdots \arrow[u, hook] & \vdots \arrow[uu, hook] & & \\
 & & \overline{\mathscr{G}}_{C(K_{\alpha+1}) \oplus c_0(\omega_1)} \arrow[u, hook] & & \\
 & \overline{\mathscr{G}}_{C(K_{\alpha+1})} \arrow[uu, hook] \arrow[ur, hook] & & & \\
 & & \overline{\mathscr{G}}_{C(K_{\alpha}) \oplus c_0(\omega_1)} \arrow[uu, hook] & & \\
 & \overline{\mathscr{G}}_{C(K_{\alpha})} \arrow[uu, hook] \arrow[ur, hook] & & & \\
 & \vdots \arrow[u, hook] & \vdots \arrow[uu, hook] & & \\
 & & \overline{\mathscr{G}}_{C(K_1) \oplus c_0(\omega_1)} \arrow[u, hook] & & \\
 & \overline{\mathscr{G}}_{C(K_1)} \arrow[uu, hook] \arrow[ur, hook] & & \overline{\mathscr{G}}_{c_0(\omega_1)} \arrow[ul, hook] & \\
 & & \overline{\mathscr{G}}_{c_0} \arrow[ul, hook] \arrow[ur, hook, two heads] & & \\
\{0\} \arrow[r, hook, two heads] & \mathscr{K} \arrow[ur, hook, two heads] & & &
\end{tikzcd}
$$

$$K_\alpha = [0, \omega^{\omega^\alpha}]$$

Conventions

(i) We suppress $C[0,\omega_1]$ everywhere, thus writing $\mathscr{K}$ instead of $\mathscr{K}(C[0,\omega_1])$ for the ideal of compact operators on $C[0,\omega_1]$, *etc.*;

(ii) $\mathscr{I} \hookrightarrow \mathscr{J}$ means that the ideal $\mathscr{I}$ is properly contained in the ideal $\mathscr{J}$;

(iii) a double-headed arrow indicates that there are no closed ideals between $\mathscr{I}$ and $\mathscr{J}$;

(iv) α denotes a countable ordinal; and

(v) $K_\alpha = [0, \omega^{\omega^\alpha}]$.

The definition of the Loy–Willis ideal

Fact. $[0, \omega_1]$ is *scattered:* each non-empty subset contains an isolated point.

The definition of the Loy–Willis ideal

Fact. $[0,\omega_1]$ is *scattered:* each non-empty subset contains an isolated point.

Theorem (Rudin 1957). $C[0,\omega_1]^* \cong \ell_1[0,\omega_1]$.

The definition of the Loy–Willis ideal

Fact. $[0,\omega_1]$ is *scattered:* each non-empty subset contains an isolated point.

Theorem (Rudin 1957). $C[0,\omega_1]^* \cong \ell_1[0,\omega_1]$.

More precisely, for each $\mu \in C[0,\omega_1]^*$, there are unique scalars (c_α) such that

$$\|\mu\| = \sum_{\alpha\in[0,\omega_1]} |c_\alpha| < \infty \qquad \text{and} \qquad \mu = \sum_{\alpha\in[0,\omega_1]} c_\alpha\delta_\alpha,$$

where δ_α is the evaluation map at α, that is, $\delta_\alpha(f) = f(\alpha)$.

Fact. $[0,\omega_1]$ is *scattered:* each non-empty subset contains an isolated point.

Theorem (Rudin 1957). $C[0,\omega_1]^* \cong \ell_1[0,\omega_1]$.

More precisely, for each $\mu \in C[0,\omega_1]^*$, there are unique scalars (c_α) such that

$$\|\mu\| = \sum_{\alpha\in[0,\omega_1]} |c_\alpha| < \infty \qquad \text{and} \qquad \mu = \sum_{\alpha\in[0,\omega_1]} c_\alpha \delta_\alpha,$$

where δ_α is the evaluation map at α, that is, $\delta_\alpha(f) = f(\alpha)$.

Corollary. For each $T \in \mathscr{B}(C[0,\omega_1])$, there is a unique scalar-valued matrix $(T_{\alpha,\beta})_{\alpha,\beta\in[0,\omega_1]}$ such that

$$\sum_{\beta\in[0,\omega_1]} |T_{\alpha,\beta}| < \infty \qquad \text{and} \qquad Tf(\alpha) = \sum_{\beta\in[0,\omega_1]} T_{\alpha,\beta} f(\beta)$$

for each $f \in C[0,\omega_1]$ and $\alpha \in [0,\omega_1]$.

Fact. $[0,\omega_1]$ is *scattered:* each non-empty subset contains an isolated point.

Theorem (Rudin 1957). $C[0,\omega_1]^* \cong \ell_1[0,\omega_1]$.

More precisely, for each $\mu \in C[0,\omega_1]^*$, there are unique scalars (c_α) such that

$$\|\mu\| = \sum_{\alpha\in[0,\omega_1]} |c_\alpha| < \infty \qquad \text{and} \qquad \mu = \sum_{\alpha\in[0,\omega_1]} c_\alpha\delta_\alpha,$$

where δ_α is the evaluation map at α, that is, $\delta_\alpha(f) = f(\alpha)$.

Corollary. For each $T \in \mathscr{B}(C[0,\omega_1])$, there is a unique scalar-valued matrix $(T_{\alpha,\beta})_{\alpha,\beta\in[0,\omega_1]}$ such that

$$\sum_{\beta\in[0,\omega_1]} |T_{\alpha,\beta}| < \infty \qquad \text{and} \qquad Tf(\alpha) = \sum_{\beta\in[0,\omega_1]} T_{\alpha,\beta}f(\beta)$$

for each $f \in C[0,\omega_1]$ and $\alpha \in [0,\omega_1]$.

Notation. For $T \in \mathscr{B}(C[0,\omega_1])$ and $\beta \in [0,\omega_1]$, let $k_\beta^T\colon [0,\omega_1] \to \mathbb{C}$ denote the β^{th} column of the matrix of T, that is, $k_\beta^T(\alpha) = T_{\alpha,\beta}$.

Fact. $[0,\omega_1]$ is *scattered:* each non-empty subset contains an isolated point.

Theorem (Rudin 1957). $C[0,\omega_1]^* \cong \ell_1[0,\omega_1]$.

More precisely, for each $\mu \in C[0,\omega_1]^*$, there are unique scalars (c_α) such that

$$\|\mu\| = \sum_{\alpha\in[0,\omega_1]} |c_\alpha| < \infty \qquad \text{and} \qquad \mu = \sum_{\alpha\in[0,\omega_1]} c_\alpha\delta_\alpha,$$

where δ_α is the evaluation map at α, that is, $\delta_\alpha(f) = f(\alpha)$.

Corollary. For each $T \in \mathscr{B}(C[0,\omega_1])$, there is a unique scalar-valued matrix $(T_{\alpha,\beta})_{\alpha,\beta\in[0,\omega_1]}$ such that

$$\sum_{\beta\in[0,\omega_1]} |T_{\alpha,\beta}| < \infty \qquad \text{and} \qquad Tf(\alpha) = \sum_{\beta\in[0,\omega_1]} T_{\alpha,\beta} f(\beta)$$

for each $f \in C[0,\omega_1]$ and $\alpha \in [0,\omega_1]$.

Notation. For $T \in \mathscr{B}(C[0,\omega_1])$ and $\beta \in [0,\omega_1]$, let $k_\beta^T : [0,\omega_1] \to \mathbb{C}$ denote the β^{th} column of the matrix of T, that is, $k_\beta^T(\alpha) = T_{\alpha,\beta}$.

Theorem (Loy and Willis 1989). The set

$$\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k_{\omega_1}^T \text{ is continuous at } \omega_1\}$$

is a maximal ideal of codimension one in $\mathscr{B}(C[0,\omega_1])$.

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1}$ is continuous at $\omega_1\}$.

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k_{\omega_1}^T$ is continuous at $\omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k_{\omega_1}^S$ to $[0,\omega_1)$ is continuous

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1}$ is continuous at $\omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\omega_1}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k^S_{\omega_1}(\alpha)$ exists.

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1} \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\omega_1}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k^S_{\omega_1}(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,

$$k^{ST}_{\omega_1} = S(k^T_{\omega_1})$$

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1} \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\omega_1}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k^S_{\omega_1}(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,

$$k^{ST}_{\omega_1} = S(k^T_{\omega_1}) \in C[0,\omega_1].$$

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\boldsymbol{\omega_1}} \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\boldsymbol{\omega_1}}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\boldsymbol{\omega_1}} k^S_{\boldsymbol{\omega_1}}(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,

$$k^{ST}_{\boldsymbol{\omega_1}} = S(k^T_{\boldsymbol{\omega_1}}) \in C[0,\omega_1].$$

- $\mathscr{M}$ is *proper* because $k^I_{\boldsymbol{\omega_1}} = \mathbf{1}_{\{\boldsymbol{\omega_1}\}}$ is discontinuous, so $I \notin \mathscr{M}$.

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1} \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\omega_1}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k^S_{\omega_1}(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,

$$k^{ST}_{\omega_1} = S(k^T_{\omega_1}) \in C[0,\omega_1].$$

- $\mathscr{M}$ is *proper* because $k^I_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$ is discontinuous, so $I \notin \mathscr{M}$.
- $\mathscr{M}$ has *codimension one*. Given $S \in \mathscr{B}(C[0,\omega_1])$, define

$$c = \lim_{\alpha\to\omega_1} S_{\alpha,\omega_1} - S_{\omega_1,\omega_1} \qquad \text{and} \qquad T = c\cdot I + S.$$

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1} \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\omega_1}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k^S_{\omega_1}(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,
$$k^{ST}_{\omega_1} = S(k^T_{\omega_1}) \in C[0,\omega_1].$$
- $\mathscr{M}$ is *proper* because $k^I_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$ is discontinuous, so $I \notin \mathscr{M}$.
- $\mathscr{M}$ has *codimension one*. Given $S \in \mathscr{B}(C[0,\omega_1])$, define
$$c = \lim_{\alpha\to\omega_1} S_{\alpha,\omega_1} - S_{\omega_1,\omega_1} \qquad \text{and} \qquad T = c \cdot I + S.$$
Then $T \in \mathscr{M}$ because $k^T_{\omega_1}$ is continuous at ω_1:
$$k^T_{\omega_1}(\alpha) = ck^I_{\omega_1}(\alpha) + k^S_{\omega_1}(\alpha)$$

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1} \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\omega_1}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k^S_{\omega_1}(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,
$$k^{ST}_{\omega_1} = S(k^T_{\omega_1}) \in C[0,\omega_1].$$
- $\mathscr{M}$ is *proper* because $k^I_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$ is discontinuous, so $I \notin \mathscr{M}$.
- $\mathscr{M}$ has *codimension one*. Given $S \in \mathscr{B}(C[0,\omega_1])$, define
$$c = \lim_{\alpha\to\omega_1} S_{\alpha,\omega_1} - S_{\omega_1,\omega_1} \qquad \text{and} \qquad T = c\cdot I + S.$$
Then $T \in \mathscr{M}$ because $k^T_{\omega_1}$ is continuous at ω_1:
$$k^T_{\omega_1}(\alpha) = ck^I_{\omega_1}(\alpha) + k^S_{\omega_1}(\alpha) = \begin{cases} S_{\alpha,\omega_1} & \text{for } \alpha < \omega_1 \\ c + S_{\omega_1,\omega_1} & \text{for } \alpha = \omega_1. \end{cases}$$

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k_{\omega_1}^T \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k_{\omega_1}^S$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k_{\omega_1}^S(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,
$$k_{\omega_1}^{ST} = S(k_{\omega_1}^T) \in C[0,\omega_1].$$
- $\mathscr{M}$ is *proper* because $k_{\omega_1}^I = \mathbf{1}_{\{\omega_1\}}$ is discontinuous, so $I \notin \mathscr{M}$.
- $\mathscr{M}$ has *codimension one*. Given $S \in \mathscr{B}(C[0,\omega_1])$, define
$$c = \lim_{\alpha\to\omega_1} S_{\alpha,\omega_1} - S_{\omega_1,\omega_1} \qquad \text{and} \qquad T = c\cdot I + S.$$
Then $T \in \mathscr{M}$ because $k_{\omega_1}^T$ is continuous at ω_1:
$$k_{\omega_1}^T(\alpha) = ck_{\omega_1}^I(\alpha) + k_{\omega_1}^S(\alpha) = \begin{cases} S_{\alpha,\omega_1} & \text{for } \alpha < \omega_1 \\ c + S_{\omega_1,\omega_1} = \lim\limits_{\beta\to\omega_1} S_{\beta,\omega_1} & \text{for } \alpha = \omega_1. \end{cases}$$

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1} \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\omega_1}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k^S_{\omega_1}(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,
$$k^{ST}_{\omega_1} = S(k^T_{\omega_1}) \in C[0,\omega_1].$$
- $\mathscr{M}$ is *proper* because $k^I_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$ is discontinuous, so $I \notin \mathscr{M}$.
- $\mathscr{M}$ has *codimension one*. Given $S \in \mathscr{B}(C[0,\omega_1])$, define
$$c = \lim_{\alpha\to\omega_1} S_{\alpha,\omega_1} - S_{\omega_1,\omega_1} \qquad \text{and} \qquad T = c\cdot I + S.$$
Then $T \in \mathscr{M}$ because $k^T_{\omega_1}$ is continuous at ω_1:
$$k^T_{\omega_1}(\alpha) = ck^I_{\omega_1}(\alpha) + k^S_{\omega_1}(\alpha) = \begin{cases} S_{\alpha,\omega_1} & \text{for } \alpha < \omega_1 \\ c + S_{\omega_1,\omega_1} = \lim\limits_{\beta\to\omega_1} S_{\beta,\omega_1} & \text{for } \alpha = \omega_1. \end{cases}$$
Hence $S = T - c\cdot I \in \mathscr{M} + \mathbb{C}\cdot I$.

Sketch proof: $\mathscr{M}$ is a maximal ideal of codimension one

Recall: $\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : k^T_{\omega_1} \text{ is continuous at } \omega_1\}$.

Loy and Willis' Key Lemma. For each $S \in \mathscr{B}(C[0,\omega_1])$, the restriction of $k^S_{\omega_1}$ to $[0,\omega_1)$ is continuous, and $\lim_{\alpha\to\omega_1} k^S_{\omega_1}(\alpha)$ exists.

- $\mathscr{M}$ is a *left ideal* because, for $S \in \mathscr{B}(C[0,\omega_1])$ and $T \in \mathscr{M}$,

$$k^{ST}_{\omega_1} = S(k^T_{\omega_1}) \in C[0,\omega_1].$$

- $\mathscr{M}$ is *proper* because $k^I_{\omega_1} = \mathbf{1}_{\{\omega_1\}}$ is discontinuous, so $I \notin \mathscr{M}$.
- $\mathscr{M}$ has *codimension one*. Given $S \in \mathscr{B}(C[0,\omega_1])$, define

$$c = \lim_{\alpha\to\omega_1} S_{\alpha,\omega_1} - S_{\omega_1,\omega_1} \qquad \text{and} \qquad T = c\cdot I + S.$$

  Then $T \in \mathscr{M}$ because $k^T_{\omega_1}$ is continuous at ω_1:

$$k^T_{\omega_1}(\alpha) = ck^I_{\omega_1}(\alpha) + k^S_{\omega_1}(\alpha) = \begin{cases} S_{\alpha,\omega_1} & \text{for } \alpha < \omega_1 \\ c + S_{\omega_1,\omega_1} = \lim\limits_{\beta\to\omega_1} S_{\beta,\omega_1} & \text{for } \alpha = \omega_1. \end{cases}$$

  Hence $S = T - c\cdot I \in \mathscr{M} + \mathbb{C}\cdot I$.

- $\mathscr{M}$ is a *right ideal* and *maximal:* automatic!

Further work (in progress with Kania and Piotr Koszmider)

Let L_0 be the one-point compactification of the disjoint union of the intervals $[0, \sigma]$, where $\sigma \in [0, \omega_1)$.

Let L_0 be the one-point compactification of the disjoint union of the intervals $[0, \sigma]$, where $\sigma \in [0, \omega_1)$.

Theorem (Kania, Koszmider and NJL). $\mathscr{M} = \mathscr{G}_{C(L_0)}(C[0, \omega_1])$;

Further work (in progress with Kania and Piotr Koszmider)

Let L_0 be the one-point compactification of the disjoint union of the intervals $[0, \sigma]$, where $\sigma \in [0, \omega_1)$.

Theorem (Kania, Koszmider and NJL). $\mathscr{M} = \mathscr{G}_{C(L_0)}(C[0, \omega_1])$; that is,

$$T \in \mathscr{M} \iff \exists\, V \in \mathscr{B}(C[0, \omega_1], C(L_0)),\ U \in \mathscr{B}(C(L_0), C[0, \omega_1]) \colon\ T = UV.$$

Further work (in progress with Kania and Piotr Koszmider)

Let L_0 be the one-point compactification of the disjoint union of the intervals $[0,\sigma]$, where $\sigma \in [0,\omega_1)$.

Theorem (Kania, Koszmider and NJL). $\mathscr{M} = \mathscr{G}_{C(L_0)}(C[0,\omega_1])$; that is,

$$T \in \mathscr{M} \iff \exists\, V \in \mathscr{B}(C[0,\omega_1], C(L_0)),\ U \in \mathscr{B}(C(L_0), C[0,\omega_1]) \colon\ T = UV.$$

A topological space is *Eberlein compact* if it is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some index set Γ.

Further work (in progress with Kania and Piotr Koszmider)

Let L_0 be the one-point compactification of the disjoint union of the intervals $[0, \sigma]$, where $\sigma \in [0, \omega_1)$.

Theorem (Kania, Koszmider and NJL). $\mathscr{M} = \mathscr{G}_{C(L_0)}(C[0, \omega_1])$; that is,

$$T \in \mathscr{M} \iff \exists\, V \in \mathscr{B}(C[0, \omega_1], C(L_0)),\ U \in \mathscr{B}(C(L_0), C[0, \omega_1]) : \ T = UV.$$

A topological space is *Eberlein compact* if it is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some index set Γ.

Fact. L_0 is Eberlein compact (Lindenstrauss)

Further work (in progress with Kania and Piotr Koszmider)

Let L_0 be the one-point compactification of the disjoint union of the intervals $[0, \sigma]$, where $\sigma \in [0, \omega_1)$.

Theorem (Kania, Koszmider and NJL). $\mathscr{M} = \mathscr{G}_{C(L_0)}(C[0, \omega_1])$; that is,

$$T \in \mathscr{M} \iff \exists\, V \in \mathscr{B}(C[0, \omega_1], C(L_0)),\ U \in \mathscr{B}(C(L_0), C[0, \omega_1]) \colon\ T = UV.$$

A topological space is *Eberlein compact* if it is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some index set Γ.

Fact. L_0 is Eberlein compact (Lindenstrauss), whereas $[0, \omega_1]$ is not.

Further work (in progress with Kania and Piotr Koszmider)

Let L_0 be the one-point compactification of the disjoint union of the intervals $[0, \sigma]$, where $\sigma \in [0, \omega_1)$.

Theorem (Kania, Koszmider and NJL). $\mathscr{M} = \mathscr{G}_{C(L_0)}(C[0, \omega_1]);$ that is,

$$T \in \mathscr{M} \iff \exists\, V \in \mathscr{B}(C[0, \omega_1], C(L_0)),\ U \in \mathscr{B}(C(L_0), C[0, \omega_1]):\ T = UV.$$

A topological space is *Eberlein compact* if it is homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some index set Γ.

Fact. L_0 is Eberlein compact (Lindenstrauss), whereas $[0, \omega_1]$ is not.

Theorem (Amir and Lindenstrauss 1968). A compact Hausdorff space K is Eberlein compact if and only if $C(K)$ is weakly compactly generated (that is, $C(K) = \overline{\operatorname{span}}\, W$ for some weakly compact subset W).