Banach Spaces with Few Operators

Matthew Tarbard

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Definition

- X has *few operators* if every operator from X to itself is of the form $\lambda I + S$, with S strictly singular.
- X has *very few operators* if every operator from X to itself is of the form $\lambda I + K$, with K compact.

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Notation

- $\mathbb{N}_0$
- $e_m^* = (0, 0, \ldots, 1, 0 \ldots)$.
- $\mathcal{L}(X)$
- $\mathcal{K}(X)$
- $\mathcal{SS}(X)$

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- However, proof $\mathfrak{X}_{\mathrm{K}}$ has very few operators does not use the Schur property of ℓ_1.

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- If X is an ℓ_1 predual, is $\mathcal{K}(X) = \mathcal{SS}(X)$?
- If X satisfies the properties of $\mathfrak{X}_{\mathrm{K}}$ and has few operators, must it have very few operators?

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Given $k \in \mathbb{N}, k \geq 2$, there is a (HI), separable $\mathscr{L}_\infty$ space $\mathfrak{X}_k$ with a basis such that

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2. There is a non-compact, strictly singular operator $S\colon \mathfrak{X}_k \to \mathfrak{X}_k$ with $S^j \neq 0$ for $1 \leq j \leq k-1$ and $S^k = 0$.

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2. There is a non-compact, strictly singular operator $S\colon \mathfrak{X}_k \to \mathfrak{X}_k$ with $S^j \neq 0$ for $1 \leq j \leq k-1$ and $S^k = 0$.
3. Moreover S^j $(0 \leq j \leq k-1)$ is not a compact perturbation of any linear combination of the operators $S^l, l \neq j$. Equivalently, $[S^j]_{j=0}^{k-1}$ are linearly independent vectors in $\mathcal{L}(\mathfrak{X}_k)/\mathcal{K}(\mathfrak{X}_k)$.

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4. Whenever $T\colon \mathfrak{X}_k \to \mathfrak{X}_k$ is an operator on $\mathfrak{X}_k$, there are (unique) $\lambda_i \in \mathbb{R}$ and a compact operator $K \in \mathcal{K}(\mathfrak{X}_k)$ such that
$$T = \sum_{i=0}^{k-1} \lambda_i S^i + K$$

One further 'generalisation'

Theorem

There is a separable $\mathscr{L}_\infty$ space with a basis, $\mathfrak{X}_\infty$; the space has ℓ_1 dual and there exists a non-compact operator $S\colon \mathfrak{X}_\infty \to \mathfrak{X}_\infty$ satisfying the following properties:

- *The sequence of vectors $\left([S^j]\right)_{j=0}^\infty$ is a basic sequence in the Calkin algebra isometrically equivalent to the canonical basis of $\ell_1(\mathbb{N}_0)$.*
- *If $T \in \mathcal{L}(\mathfrak{X}_\infty)$ then there are unique scalars $(\lambda_i)_{i=0}^\infty$ and a compact operator $K \in \mathcal{L}(\mathfrak{X}_\infty)$ with $\sum_{i=0}^\infty |\lambda_i| < \infty$ and*

$$T = \sum_{i=0}^{\infty} \lambda_i S^i + K$$

- Note the Calkin algebra $\mathcal{L}(\mathfrak{X}_k)/\mathcal{K}(\mathfrak{X}_k)$ is isomorphic to the algebra $\mathcal{A}$ of $k \times k$ upper-triangular-Toeplitz matrices.

Calkin Algebras

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- Explicit isomorphism given by $\psi\colon \mathcal{L}(\mathfrak{X}_k)/\mathcal{K}(\mathfrak{X}_k) \to \mathcal{A}$

$$\sum_{j=0}^{k-1} \lambda_j S^j + \mathcal{K}(\mathfrak{X}_k) \mapsto \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \cdots & \lambda_{k-1} \\ 0 & \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{k-2} \\ 0 & 0 & \lambda_0 & \lambda_1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_0 \end{pmatrix}$$

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- Norm closed ideals in $\mathcal{L}(\mathfrak{X}_k)$:

$$\mathcal{K}(\mathfrak{X}_k) \subsetneq \langle S^{k-1} \rangle \subsetneq \langle S^{k-2} \rangle \ldots \langle S \rangle \subsetneq \mathcal{L}(X_k).$$

Calkin algebras

The Calkin algebra of $\mathfrak{X}_\infty$ is (isometric) to $\ell_1(\mathbb{N}_0)$ under

$$\ell_1(\mathbb{N}_0) \ni (a_n)_{n=0}^{\infty} \mapsto \sum_{j=0}^{\infty} a_j S^j + \mathcal{K}(\mathfrak{X}_\infty)$$

Interesting properties of $\mathfrak{X}_\infty$

Lemma

If X is a Banach space for which the Calkin algebra is isomorphic (as a Banach algebra) to $\ell_1(\mathbb{N}_0)$ then X is indecomposable.

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- So P is compact or $P = I + K$. In either case, P is certainly a trivial projection.

$\square$

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- If $\mathbf{a} = (a_0, a_1, \dots) \in \ell_1(\mathbb{N}_0) \setminus \{0\}$, let k be minimal such that $a_k \neq 0$. Easy computation gives $\|\mathbf{a}^n\|_{\ell_1} \geq |a_k|^n$, so that $\lim_{n\to\infty} \|\mathbf{a}^n\|^{\frac{1}{n}} \geq |a_k|$.

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- Since Calkin algebra is $\ell_1(\mathbb{N}_0)$, we see T strictly singular implies $[T] = 0$.
- So, T strictly singular $\implies$ T compact.

$\square$

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Assuming I have a (special kind of) Schauder basis of ℓ_1, denote it by $(d_n^*)_{n=1}^\infty$. The biorthogonal vectors $(d_n)_{n=1}^\infty$ form a basic sequence in ℓ_∞. Taking the closed linear span of the d_n we obtain a Banach space X, with basis $(d_n)_{n=1}^\infty$.

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The embedding is onto precisely when (d_n^*) is boundedly complete basis for $\ell_1 \iff (d_n)$ is a shrinking basis for X.

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We say a sequence $(d_m^*)_{m=1}^{\infty}$ in ℓ_1 is *unit triangular* if $d_m^* = e_m^*$ for all $m \in \Delta_1$ and, for $m \in \Delta_n, n > 1$, we have

$$d_m^* = e_m^* - c_m^*$$

where $\operatorname{supp} c_m^* \subseteq \cup_{j=1}^{n-1} \Delta_j$. Refer to the c_m^* vectors as *BD functionals*.

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Any such sequence is an algebraic basis for c_{00}. We can thus define linear projections $P_m^* \colon c_{00} \to \ell_1^m \subseteq \ell_1$ by

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The important part of the BD construction is to construct the c_n^* in such a way that the projections P_n^* are uniformly bounded (and consequently $(d_n^*)_{n=1}^{\infty}$ is a basis for ℓ_1).

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Constructing interesting operators on BD spaces

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- Restrict T^*; get $T^*| : X \to \ell_\infty$.
- Look at behaviour of T on vectors d_n^* to determine if $T^*|$ actually maps into X.

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Begin by defining F on Δ_1 (which for the space $\mathfrak{X}_k$ of the theorem is the set $\{1, 2 \ldots k\}$) by $F(j) = j - 1$ for $j \geq 2$ and $F(1) = \text{undefined}$.

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Since for $m \in \Delta_1$ we have $d_m^* = e_m^*$ we also have the formula

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Notice also that when $F(m)$ is defined, the image of m under F is still in Δ_1. More generally, given $j \in \mathbb{N} \cap \Delta_n$ we will say *F preserves the rank of j* if either $F(j)$ is undefined, or, $F(j) \in \Delta_n$

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We suppose inductively that F has been defined on $\Gamma_n := \cup_{j \leq n} \Delta_j$ such that F preserves rank for all $m \in \Gamma_n$ and S^* is defined on $\ell_1(\Gamma_n)$ such that the previous formulae hold for the e^*'s and d^*'s.

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Consider an $m \in \Delta_{n+1}$ and recall we have $d_m^* = e_m^* - c_m^*$ where $\operatorname{supp} c_m^* \subseteq \Gamma_n$. Thus $S^* c_m^*$ is already defined. So in order to preserve linearity, we are only free to define one of $S^* d_m^*$ or $S^* e_m^*$.

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The c^*'s are carefully chosen so that either $S^* c^*_m = c^*_{m'}$ where $m' \in \Delta_{n+1}$, in which case define $F(m) = m'$, or $S^* c^*_m = 0$ in which case set $F(m) =$ undefined.

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The c^*'s are carefully chosen so that either $S^* c_m^* = c_{m'}^*$, where $m' \in \Delta_{n+1}$, in which case define $F(m) = m'$, or $S^* c_m^* = 0$ in which case set $F(m) = $ undefined.

Easily checked that we can extend definition of S^* to $\ell_1(\Gamma_{n+1})$ satisfying the required formulae.

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Showing that $S|$ is strictly singular is harder!

Further Research

Theorem (Daws, Haydon, Schlumprecht, White)

Let E be the space generated by the closed linear span in $\ell_\infty(\mathbb{Z})$ of the vector

$$x_0 = (\ldots, 0, 0, 1, 2^{-1}, 2^{-1}, 2^{-2}, 2^{-1}, 2^{-2}, 2^{-2}, 2^{-3}, 2^{-1}, \ldots)$$

and its bilateral shifts. The space E is is a (shift invariant) concrete predual of $\ell_1(\mathbb{Z})$ isomorphic to c_0 that induces a non-canonical weak topology on $\ell_1(\mathbb{Z})$.*

Here, x_0 is the vector with 1 in the zero'th component and for $n > 0$, the n'th component of x_0 is $2^{-b(n)}$ where $b(n)$ is the number of 1's in the binary expansion of n.

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- Easy to see $[d_n^* : n \in \mathbb{N}_0] = \ell_1(\mathbb{N}_0)$.
- In fact, this fits entirely into the framework of the BD construction so that, in particular (d_n^*) is a Schauder basis for $\ell_1(\mathbb{N}_0)$. The biorthogonal vectors $[d_n \colon n \in \mathbb{N}_0]$ generate a (BD) (sub)space of ℓ_∞.

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- Turns out $d_0 = (1, 2^{-1}, 2^{-1}, 2^{-2}, 2^{-1}, 2^{-2}, 2^{-2}, 2^{-3}, \dots)$. Also, $[d_n \colon n \in \mathbb{N}_0]$ is the same as the closed linear span of d_0 and all its right shifts.

Thank you.