Convergence to equilibrium for a thin film equation

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joint work with

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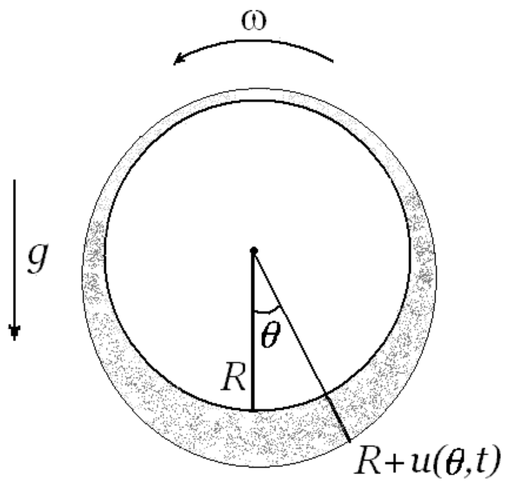
The physical model

$$u_t + \partial_x\big(u^3(u_{xxx} + u_x - \sin x)\big) + \omega u_x = 0\,.$$

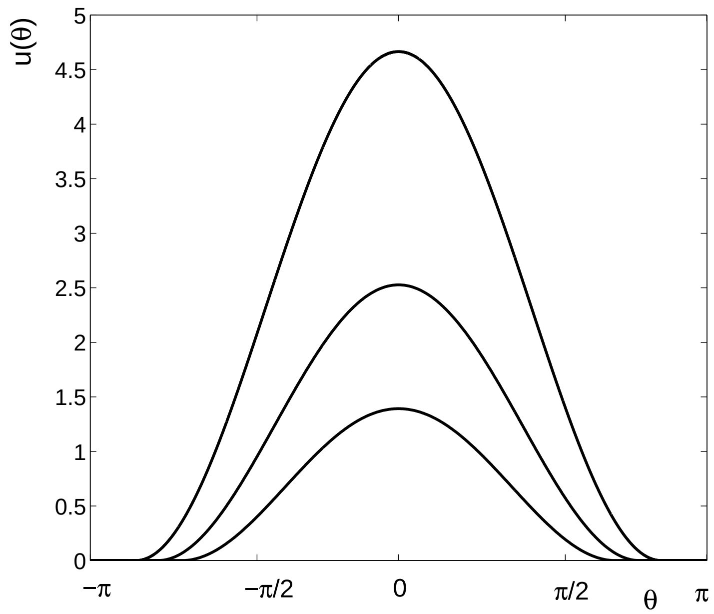


- $u \geq 0$, **periodic;**
- $\partial_x(u^3(u_{xxx} + u_x))$ **surface tension term;**
- $-\partial_x(u^3 \sin x)$ **gravitational drainage;**
- **rotation speed** ω**.**

[Moffatt 1976, Pukhnachev 1977, Benilov & al. 2000-date]

Summary of results (Pukhnachev's model with $\omega = 0$)

- **For every mass there is a unique energy minimizer u^*;**
- **u^* is globally attractive;**
- **(no better than) power-law decay $||u(\cdot,t),u^*||_{H^1} \geq Ct^{-\frac{2}{3}}$.**



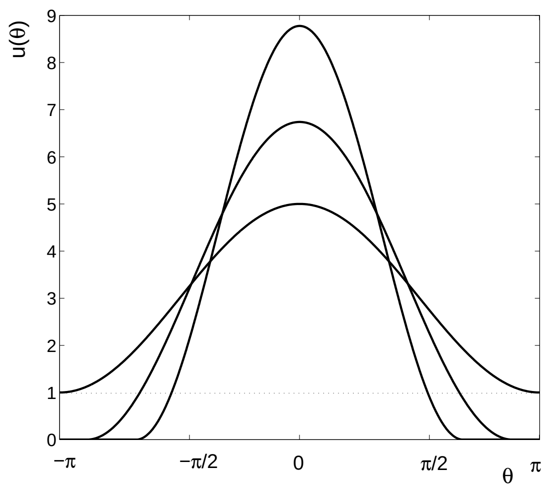
Energy minimizers for three values of the mass.

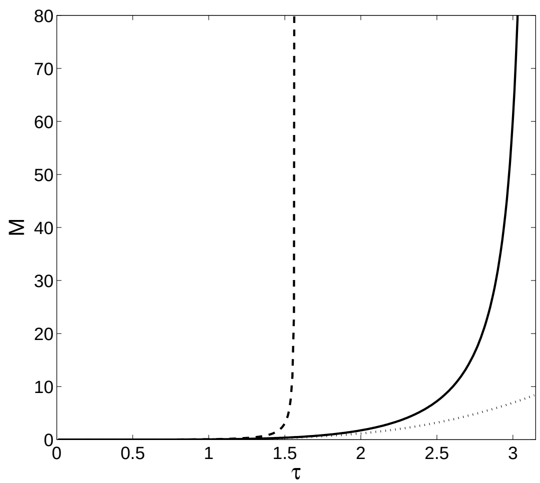
Energy minimizers

For every value of $\alpha, n, M > 0$, functional

$$E(u) = \frac{1}{2}\int_{-\pi}^{\pi} u_x^2 - \alpha^2 u^2 \, dx - \int_{-\pi}^{\pi} u \cos x \, dx \, .$$

has a unique global minimizer of mass M.



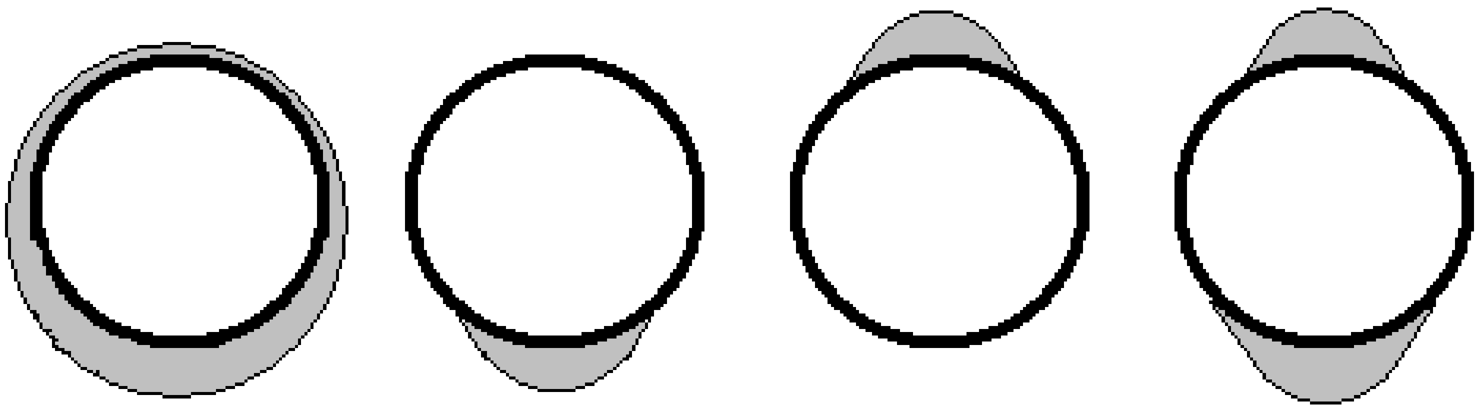


Minimizers for $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$.

(Note that E is not convex when $\alpha > 1$.)

Other critical points of the energy

Droplet-shaped critical points have zero contact angle.
For $\alpha \leq 1$, the global minimizer is the unique critical point.
For $\alpha > 1$, there may be others (depending on the mass):

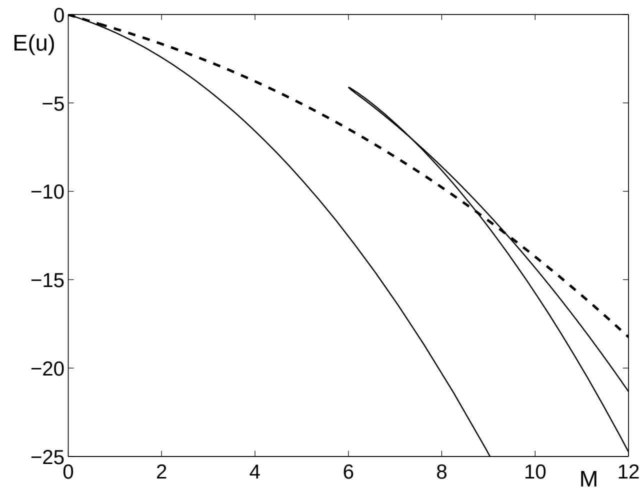


Are these critical points all the steady states?

Does Lyapunov's principle apply?

"The ω-limit set of an orbit under a gradient flow consists of critical points of the Lyapunov function."

Bifurcation diagram



Energy levels of critical points, depending on the mass.
$\alpha = 1$ **(dashed line),** $\alpha = \sqrt{2}$ **(solid line).**

Convergence to the energy minimizer

Let $u(\cdot, t)$ be a solution of finite entropy, and let u^* be the global energy minimizer of the same mass. (If $\alpha > 1$, assume also that no other critical points have energy below $E(u)$.) Then:

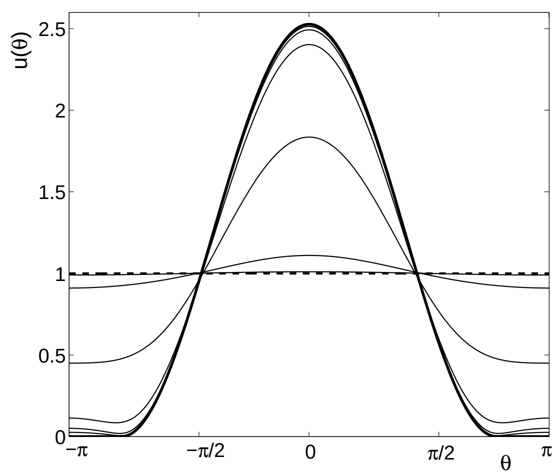
- **The solutio:n $u(t)$ converges to u^* as $t \to \infty$.**
  (Proof: An energy-entropy compactness argument.)

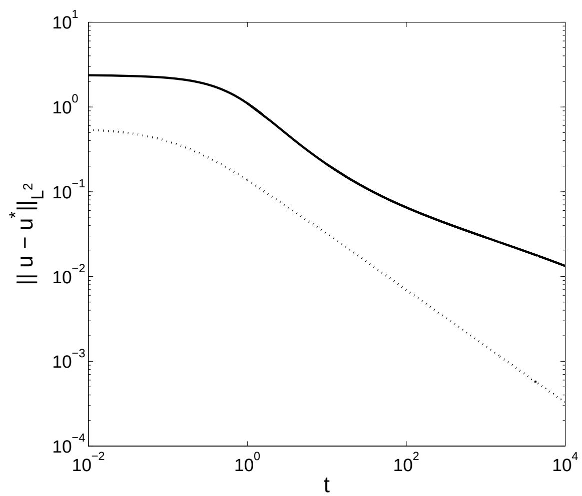
- **For $n > \frac{3}{2}$, the distance from a droplet cannot decay faster than a power law.**
  (Proof: Entropy grows at most linearly in t.)

- **If u^* is positive, then $u(\cdot, t)$ converges exponentially.**
  (Proof: Compare the dissipation with the energy.)

Open questions

- What is the rate of convergence really? (Perhaps $t^{-\frac{1}{3}}$?)
  How can we linearize around a droplet? ***[Slepčev 2008]***





- Do all solutions converge to equilibrium
  (even when there are many steady states?)
- How to take advantage of the gradient flow structure?
  [Otto 1998, . . . , Ambrosio-Gigli-Savaré (book), . . . , . . . , Kamalinejad 2012,. . .]